Averaging Methods for Weakly Nonlinear and Multi-Harmonic Oscillatory Systems

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Introduction

Method Of Averaging

Method of averaging is a powerful tool to get information regarding approach to the limit cycle.Here

$$\frac{d^2x}{dt^2} + \omega^2 x = \epsilon F(x, \dot{x}, t)$$

Here ϵ is a small term. Now we form anstaz.

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} a(t)cos(\omega t + \phi(t)) \\ -\omega a(t)sin(\omega t + \phi(t)) \end{pmatrix}$$

we chose this ansatz because ϵ is a very small term the dynamics will behave like harmonic oscillator as $\epsilon = 0$, we will find a constant amplitude a and constant phase differnece ϕ. This weak non linear oscillator will behave like a harmonic oscillator as we do the near identity transforms near the fixed points.

Now taking derrivative of the first part will make it

$$\dot{x} = -a(t)sin(\omega t + \phi(t))(\omega + \frac{d\phi}{dt}) + \frac{da}{dt}cos(\omega t + \phi(t))$$

$$-\omega a(t)sin(\omega t + \phi(t)) = -\omega a(t)sin(\omega t + \phi(t)) - \frac{d\phi}{dt}a(t)sin(\omega t + \phi(t)) + \frac{da}{dt}cos(\omega t + \phi(t)$$

$$-a(t)\sin(\omega t + \phi(t))\frac{d\phi}{dt} + \frac{da}{dt}cos(\omega t + \phi(t)) = 0 \tag{1}$$

diffrentiating the second part we'll get

$$\ddot{x} = -\omega a(t)cos(\omega t + \phi(t))(\omega + \frac{d\phi}{dt}) - \omega\frac{da}{dt}sin(\omega t + \phi)$$

$$\ddot{x} = -\omega^2 a(t)cos(\omega t + \phi(t)) - \omega a(t)cos(\omega t + \phi(t))\frac{d\phi}{dt} - \omega\frac{da}{dt}sin(\omega t + \phi)$$

Now substituting everything into the weak oscillator equation we'll get

$$\begin{aligned} -\omega^2 a(t)cos(\omega t + \phi(t)) - \omega a(t)cos(\omega t + \phi(t))\frac{d\phi}{dt} - \omega\frac{da}{dt}sin(\omega t + \phi) + \omega^2 a(t)cos(\omega t + \phi) \\ = \epsilon F(a(t)cos(\omega t + \phi(t)), -\omega a(t)sin(\omega t + \phi(t)), t) \end{aligned}$$

$$-\omega a(t)cos(\omega t + \phi(t))\frac{d\phi}{dt} - \omega\frac{da}{dt}sin(\omega t + \phi) = \epsilon F(a(t)cos(\omega t + \phi(t)), -\omega a(t)sin(\omega t + \phi(t)), t)$$

$$\begin{pmatrix} -a(t)\sin(\omega t + \phi(t)) & cos(\omega t + \phi(t)) \\ -\omega a(t)cos(\omega t + \phi(t)) & -\omega sin(\omega t + \phi) \end{pmatrix}\begin{pmatrix} \dot{\phi} \\ \dot{a} \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon F(a(t)cos(\omega t + \phi(t)), -\omega a(t)sin(\omega t + \phi(t)), t) \end{pmatrix}$$

$$\begin{pmatrix} -a(t)\sin(\Theta) & \cos(\Theta) \\ -\omega a(t)\cos(\Theta) & -\omega\sin(\Theta) \end{pmatrix}\begin{pmatrix} \dot{\phi} \\ \dot{a} \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon F(a(t)\cos\Theta,\ -\omega a(t)\sin\Theta,\ t) \end{pmatrix} \tag{2}$$

$$\Theta = \omega t + \phi(t).$$

$$\begin{aligned}
\det &= (-a\sin\Theta)(-\omega\sin\Theta) - (\cos\Theta)(-\omega a\cos\Theta) && (3)\\
&= \omega a(\sin^2\Theta + \cos^2\Theta) && (4)\\
&= \omega a && (5)
\end{aligned}$$

$$\begin{pmatrix} -a\sin\Theta & \cos\Theta \\ -\omega a\cos\Theta & -\omega\sin\Theta \end{pmatrix}^{-1} = \frac{1}{\omega a}\begin{pmatrix} -\omega\sin\Theta & -\cos\Theta \\ \omega a\cos\Theta & -a\sin\Theta \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} \dot\phi \\ \dot a \end{pmatrix} = \frac{1}{\omega a}\begin{pmatrix} -\omega\sin\Theta & -\cos\Theta \\ \omega a\cos\Theta & -a\sin\Theta \end{pmatrix}\begin{pmatrix} 0 \\ \epsilon F \end{pmatrix} \tag{7}$$

$$\begin{aligned}
\dot\phi &= \frac{1}{\omega a}\left[-\cos\Theta\cdot\epsilon F\right] && (8)\\
&= -\frac{\varepsilon}{\omega a}\cos(\omega t+\phi)\,F(a\cos(\omega t+\phi), -\omega a\sin(\omega t+\phi), t) && (9)
\end{aligned}$$

$$\begin{aligned}
\dot a &= \frac{1}{\omega a}\left[-a\sin\Theta\cdot\epsilon F\right] && (10)\\
&= -\frac{\epsilon}{\omega}\sin(\omega t+\phi)\,F(a\cos(\omega t+\phi), -\omega a\sin(\omega t+\phi), t) && (11)
\end{aligned}$$

$$\boxed{\begin{aligned}
\dot\phi &= -\frac{\epsilon}{\omega a}\cos(\omega t+\phi)\,F(a\cos(\omega t+\phi), -\omega a\sin(\omega t+\phi), t)\,,\\
\dot a &= -\frac{\epsilon}{\omega}\sin(\omega t+\phi)\,F(a\cos(\omega t+\phi), -\omega a\sin(\omega t+\phi), t)\,.
\end{aligned}} \tag{12}$$

Now we we'll do the near identity transforms to get the averaged equations around the fixed points

$$a = \bar a + \epsilon v_1 + O(\epsilon^2)$$

$$\phi = \bar\phi + \epsilon v_2 + O(\epsilon^2)$$

Substituting it into the eq 12 we'll get the se of equations

$$\dot{\bar a} = -\epsilon(\frac{\partial v_1}{\partial t} + \frac{1}{\omega \bar a}\sin(\omega t+\phi)\,F(\bar a\cos(\omega t+\phi), -\omega\sin(\omega t+\phi), t)$$

$$\dot{\bar\phi} = -\epsilon(\frac{\partial v_2}{\partial t} + \frac{1}{\omega \bar a}\cos(\omega t+\phi)\,F(\bar a\cos(\omega t+\phi), -\omega\bar a\sin(\omega t+\phi), t))$$

Now we'll choose v_1 and v_2 such that only the average of the equation is left inside both of the equation so

$$\dot{\bar a} = -\epsilon(\frac{1}{2\pi\omega}\int_0^{2\pi}\sin(\omega t+\phi)\,F(\bar a\cos(\omega t+\phi), -\omega\bar a\sin(\omega t+\phi), t)dt)$$

$$\dot{\bar\phi} = -\epsilon(\frac{1}{2\pi\omega\bar a}\int_0^{2\pi}\sin(\omega t+\phi)\,F(\bar a\sin(\omega t+\phi), -\omega\bar a\sin(\omega t+\phi), t)dt)$$

and derrivatives of v_1 and v_2 were choosen as

$$< f_1 > - f_1 = \frac{\partial v_1}{dt}$$

$$< f_2 > - f_2 = \frac{\partial v_2}{dt}$$

and after find all the values and integrating it back to the original near identity transforms we we'll get the averaged equations. This is how we generally perform the averaging using the near identity transforms , now we'll look at how it's done for the multiple harmonics.

Fourier Averaging of Multi-Harmonics

Now what do we mean by multiple harmonics , the system with fundamental frequency ω with the higher harmonics with linear frequency 2ω 3ω ... this can be written as

$$x(t) = \sum_n a_n \cos n\Omega t + b_n \sin n\Omega t$$

where the a_n and b_n and amplitude associated with the cosine and sine of nth harmonic.

For a general system :

$$m\ddot{x} + c\dot{x} + kx + f_{nl} = f_{ext}$$

For the first Harmonic

We can write the first harmonic as putting the values of n = 1.

$$x(t) = a_1 \cos \Omega t + b_1 \sin \Omega t$$

We can write the equations as:

$$x = u = \begin{pmatrix} C_{\Omega t} & S_\Omega \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$\dot{x} = v = \begin{pmatrix} C_\Omega & S_\Omega \end{pmatrix} \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

The matrix $\begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$ is the matrix orginates when we take the time derrivative of u. now we will use these equations to write the state space equations.

The state space equations are :

$$v = \dot{u}$$

$$m\dot{v} + cv + ku + f_{nl} = f_{ext}$$

Computing :

$$\begin{pmatrix} C_\Omega & S_\Omega \end{pmatrix} \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} C_\Omega & S_\Omega \end{pmatrix} \begin{pmatrix} \dot{a_1} \\ \dot{b_1} \end{pmatrix} = \begin{pmatrix} C_\Omega & S_\Omega \end{pmatrix} \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

$$\begin{pmatrix} C_\Omega & S_\Omega \end{pmatrix} \begin{pmatrix} \dot{a_1} \\ \dot{b_1} \end{pmatrix} = 0$$

We got our first state space eqaution , now we will use the general harmonic equation to find the other one :

$$\dot{v} = \begin{pmatrix} C_\Omega & S_\Omega \end{pmatrix} \begin{pmatrix} -\Omega^2 & 0 \\ 0 & -\Omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} C_\Omega & S_\Omega \end{pmatrix} \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} \begin{pmatrix} \dot{a}_1 \\ \dot{b}_1 \end{pmatrix}$$

substituting the general equation becomes :

$$\begin{pmatrix} C_\Omega & S_\Omega \end{pmatrix} \left[\begin{pmatrix} 0 & m\Omega \\ -m\Omega & 0 \end{pmatrix} \begin{pmatrix} \dot{a_1} \\ \dot{b_1} \end{pmatrix} + \begin{pmatrix} k - \Omega^2 m & c\Omega \\ c\Omega & k - \Omega^2 m \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} f_{nl,a_1} \\ f_{nl,b_1} \end{pmatrix} - \begin{pmatrix} f_{ext,a_1} \\ f_{ext,b_1} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we have both of our state space equations and now we can combine both of them to get

$$\begin{pmatrix} -m\Omega S_\Omega & m\Omega C_\Omega \\ C_\Omega & S_\Omega \end{pmatrix} \begin{pmatrix} \dot{a_1} \\ \dot{b_1} \end{pmatrix} + \begin{pmatrix} C_\Omega & S_\Omega \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} k - \Omega^2 m & c\Omega \\ c\Omega & k - \Omega^2 m \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} g_{a_1} \\ g_{b_1} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{a_1} \\ \dot{b_1} \end{pmatrix} = \frac{1}{m\Omega} \begin{pmatrix} S_\Omega & -m\Omega C_\Omega \\ -C_\Omega & -m\Omega S_\Omega \end{pmatrix} \begin{pmatrix} C_\Omega & S_\Omega \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{a_1} \\ R_{b_1} \end{pmatrix}$$

$$\begin{pmatrix} \dot{a_1} \\ \dot{b_1} \end{pmatrix} = \frac{1}{2m\Omega} \begin{pmatrix} S_{2\Omega} & 1 - C_{2\Omega} \\ -1 - C_{2\Omega} & -S_{2\Omega} \end{pmatrix} \begin{pmatrix} R_{a_1} \\ R_{b_1} \end{pmatrix}$$

Now we have our coefficients of the first harmonic coefficients now we will do our near identity transformations to get our averaged dynamics.We will introduce $\bar{a}_1$ $\bar{b}_1$ just like we did in our averaging part(refer above).

$$a_1 = \bar{a}_1 + \varepsilon u + O(\varepsilon^2)$$
$$b_1 = \bar{b}_1 + \varepsilon v + O(\varepsilon^2)$$

Taking the time derrivative in order to fit it into the equations:

$$\dot{a}_1 = \dot{\bar{a}}_1 + \varepsilon \dot{u} + O(\varepsilon^2)$$
$$\dot{b}_1 = \dot{\bar{b}}_1 + \varepsilon \dot{v} + O(\varepsilon^2)$$

substituting the set of equations into the state space equation we will get:

$$\begin{pmatrix} \dot{\bar{a}}_1 \\ \dot{\bar{b}}_1 \end{pmatrix} = \underbrace{-\varepsilon \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \frac{1}{2m\Omega} \begin{pmatrix} S_{2\Omega} & 1 - C_{2\Omega} \\ -1 - C_{2\Omega} & -S_{2\Omega} \end{pmatrix} \begin{pmatrix} R_{\bar{a}_1} \\ R_{\bar{b}_1} \end{pmatrix}}$$

As we have seen above we will choose $\dot{u}$ and $\dot{v}$ such that only the averaged term of the right side is left , the only averaging will be done the sine and the cosine part so hence they'll get averaged out and the term becomes:

$$\begin{pmatrix} \dot{\bar{a}}_1 \\ \dot{\bar{b}}_1 \end{pmatrix} = \frac{1}{2m\Omega\varepsilon} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} R_{\bar{a}_1} \\ R_{\bar{b}_1} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\bar{a}}_1 \\ \dot{\bar{b}}_1 \end{pmatrix} = \frac{1}{2m\Omega\varepsilon} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left[\begin{pmatrix} k - \Omega^2 m & c\Omega \\ c\Omega & k - \Omega^2 m \end{pmatrix} \begin{pmatrix} \bar{a}_1 \\ \bar{b}_1 \end{pmatrix} + \begin{pmatrix} f_{nl,\bar{a}_1} \\ f_{nl,\bar{b}_1} \end{pmatrix} - \begin{pmatrix} f_{ext,\bar{a}_1} \\ f_{ext,\bar{b}_1} \end{pmatrix} \right]$$

here we have a set of diffrential equations and solving this equation will give us $\begin{pmatrix} \dot{\bar{a}}_1 \\ \dot{\bar{b}}_1 \end{pmatrix}$ which we can substitute it back to the near identity transforms now we have to get u and v which we can get from using the method mentioned earlier , as we have substracted everything leaving only the averaged part.

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{2m\Omega\varepsilon} \begin{pmatrix} S_{2\Omega} & 1 - C_{2\Omega} \\ -1 - C_{2\Omega} & -S_{2\Omega} \end{pmatrix} \begin{pmatrix} R_{\bar{a}_1} \\ R_{\bar{b}_1} \end{pmatrix} - < \frac{1}{2m\Omega\varepsilon} \begin{pmatrix} S_{2\Omega} & 1 - C_{2\Omega} \\ -1 - C_{2\Omega} & -S_{2\Omega} \end{pmatrix} \begin{pmatrix} R_{\bar{a}_1} \\ R_{\bar{b}_1} \end{pmatrix} >$$

$$= \frac{1}{2m\Omega} \left[\begin{pmatrix} S_{2\Omega} & 1 - C_{2\Omega} \\ -1 - C_{2\Omega} & -S_{2\Omega} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \begin{pmatrix} R_{\bar{a}_1} \\ R_{\bar{b}_1} \end{pmatrix}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{2m\Omega} \begin{pmatrix} S_{2\Omega} & -C_{2\Omega} \\ -C_{2\Omega} & -S_{2\Omega} \end{pmatrix} \begin{pmatrix} R_{\bar{a}_1} \\ R_{\bar{b}_1} \end{pmatrix}$$

here solving this differential equation we'll get the u and v , the first order slowed dynamics and substituting the equations back to the near identity transformation we'll get our averaged equations and we can now look up to the jacobian of the linearized part to look up for the stability analysis.This is how we'll do the near identity transformation of any general harmonic to get our averaged dynamics but the multiple harmonic case don't give us the freedom of just taking out the inverse , we will use a trick , well properties of fourier series to gather our information.

Method of Fourier Based Averaging of Multiple Harmonics

We have already discussed the averaging method and how to apply it in one harmonic case which seems very straight forward to apply but in the multiple harmonic case we'll use the orthogonality of sine and cosines to get our desired harmonic because we just can't plug out any nth harmonic from the summation.I'll show how it can be done.

Our general Harmonic as we know will be:

$$x(t) = \sum_n a_n \cos n\Omega t + b_n \sin n\Omega t$$

We can do the tranforming we can write : We can write the equations as:

$$x = u = \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$\dot{x} = v = \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} 0 & n\Omega \\ -n\Omega & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$\dot{v} = \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} -n^2\Omega^2 & 0 \\ 0 & -n^2\Omega^2 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} + \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} 0 & n\Omega \\ -n\Omega & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

forming the state space equations:

$$v = \dot{u}$$

$$\sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} 0 & n\Omega \\ -n\Omega & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} + \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} = \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} 0 & n\Omega \\ -n\Omega & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$\sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} = 0$$

plugging the values into the general equation to form the other state space equation we will get:

$$\sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} 0 & n\Omega m \\ -n\Omega m & 0 \end{pmatrix} \begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} + \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} k - n^2\Omega^2 m & nc\Omega \\ nc\Omega & k - n^2 a\Omega^2 m \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} +$$

$$\sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} f_{nl,a_1} \\ f_{nl,b_1} \end{pmatrix} - \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} f_{ext,a_1} \\ f_{ext,b_1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} 0 & n\Omega m \\ -n\Omega m & 0 \end{pmatrix} \begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} + \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} R_{a_n} \\ R_{b_n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we have our both of the state space equations now it's time to combine and get to the result :

$$\sum_n \begin{pmatrix} -nm\Omega S_{n\Omega} & nm\Omega C_{n\Omega} \\ C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} + \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{a_n} \\ R_{b_n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

taking out inverse of the rth matrix of $\dot{a}_n$ $\dot{b}_n$ it will come out to be $\frac{-1}{rm\Omega} \begin{pmatrix} S_{r\Omega} & -rm\Omega C_{r\Omega} \\ -C_{r\Omega} & -rm\Omega S_{r\Omega} \end{pmatrix}$ So in order to separate the rth harmonic we will multiply the whole eqation by this inverse which will factor us out the rth harmonic.

$$\begin{pmatrix} \dot{a}_r \\ \dot{b}_r \end{pmatrix} - \frac{1}{rm\Omega} \sum_{n \neq r} \begin{pmatrix} S_{r\Omega} & -rm\Omega C_{r\Omega} \\ -C_{r\Omega} & -rm\Omega S_{r\Omega} \end{pmatrix} \begin{pmatrix} -nm\Omega S_{n\Omega} & nm\Omega C_{n\Omega} \\ C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} =$$

$$\frac{1}{rm\Omega} \begin{pmatrix} S_{r\Omega} & -rm\Omega C_{r\Omega} \\ -C_{r\Omega} & -rm\Omega S_{r\Omega} \end{pmatrix} \sum_n \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{a_n} \\ R_{b_n} \end{pmatrix}$$

$$\begin{pmatrix} \dot{a}_r \\ \dot{b}_r \end{pmatrix} - \frac{1}{rm\Omega} \sum_{n \neq r} \begin{pmatrix} S_{r\Omega} & -rm\Omega C_{r\Omega} \\ -C_{r\Omega} & -rm\Omega S_{r\Omega} \end{pmatrix} \begin{pmatrix} -nm\Omega S_{n\Omega} & nm\Omega C_{n\Omega} \\ C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} =$$

$$\frac{1}{rm\Omega} \sum_{n \neq r} \begin{pmatrix} S_{r\Omega} & -rm\Omega C_{r\Omega} \\ -C_{r\Omega} & -rm\Omega S_{r\Omega} \end{pmatrix} \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{a_n} \\ R_{b_n} \end{pmatrix} + \frac{1}{2rm\Omega} \begin{pmatrix} S_{2r\Omega} & 1 - C_{2r\Omega} \\ -1 - C_{2r\Omega} & -S_{2r\Omega} \end{pmatrix} \begin{pmatrix} R_{a_n} \\ R_{b_n} \end{pmatrix}$$

now we have our equations in term of the rth harmonic now here we can introduce the near identity transformation of the rth coefficients and get the averaged dynamics and here we will use the properties of the fourier series to factor out results.

$$a_r = \bar{a}_r + \varepsilon \hat{a}_r + O(\varepsilon^2)$$

$$b_r = \bar{b}_r + \varepsilon \hat{b}_r + O(\varepsilon^2)$$

after choosing $\hat{a}_r$ and $\hat{b}_r$ we'
get the averaged equations of that particular a_r and b_r as because all the other sine and the cosine terms are orthogonal they'll get cancelled out

$$\begin{pmatrix} \dot{\bar{a}}_r \\ \dot{\bar{b}}_r \end{pmatrix} = \frac{1}{2rm\Omega} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R_{\bar{a}_r} \\ R_{\bar{b}_r} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2rm\Omega \\ 2rm\Omega & 0 \end{pmatrix} \begin{pmatrix} \dot{\bar{a}}_r \\ \dot{\bar{b}}_r \end{pmatrix} = \begin{pmatrix} R_{\bar{a}_r} \\ R_{\bar{b}_r} \end{pmatrix}$$

now we have the averaged differential equation , here we can solve the differential equation to get our averaged equation. Now we have to get our first order slow dynamics:

$$\begin{pmatrix} \dot{\hat{a}}_r \\ \dot{\hat{b}}_r \end{pmatrix} = \frac{1}{rm\Omega} \sum_{n \neq r} \begin{pmatrix} C_{n\Omega} & S_{n\Omega} \end{pmatrix} \begin{pmatrix} H_{\bar{a}_r} \\ H_{\bar{b}_r} \end{pmatrix} - \begin{pmatrix} 0 & -2rm\Omega \\ 2rm\Omega & 0 \end{pmatrix} \begin{pmatrix} R_{\bar{a}_r} \\ R_{\bar{b}_r} \end{pmatrix}$$

now solving this differential equation we'll get the equations of the linear slowed dynamics and plugging in the values of the averaged and the slowed dynamics we'll get the near identity dynamics of our equations and we can do the stability and bifurcation analysis.