Fourier Methods for Bifurcations: Challenges and Outlook for Non-Smooth Flutter Problems

Aerospace Research Day 2026

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23 May 2026

Outline

1. Introduction

Aeroelastic flutter has existed as long as there have been aircrafts

Figure Source (NASA Dryden Flight Research Center, 2001)

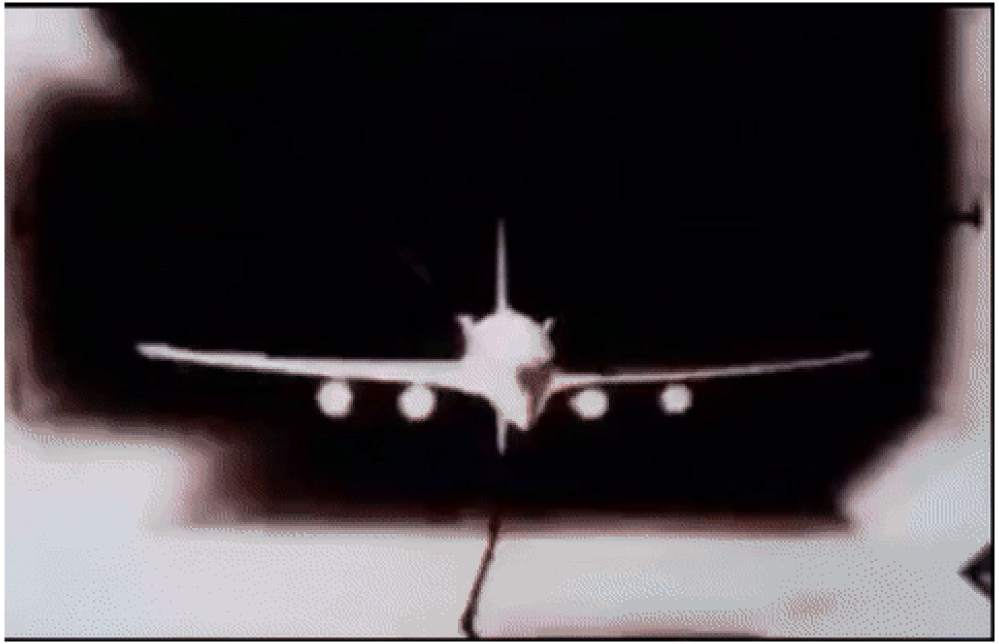
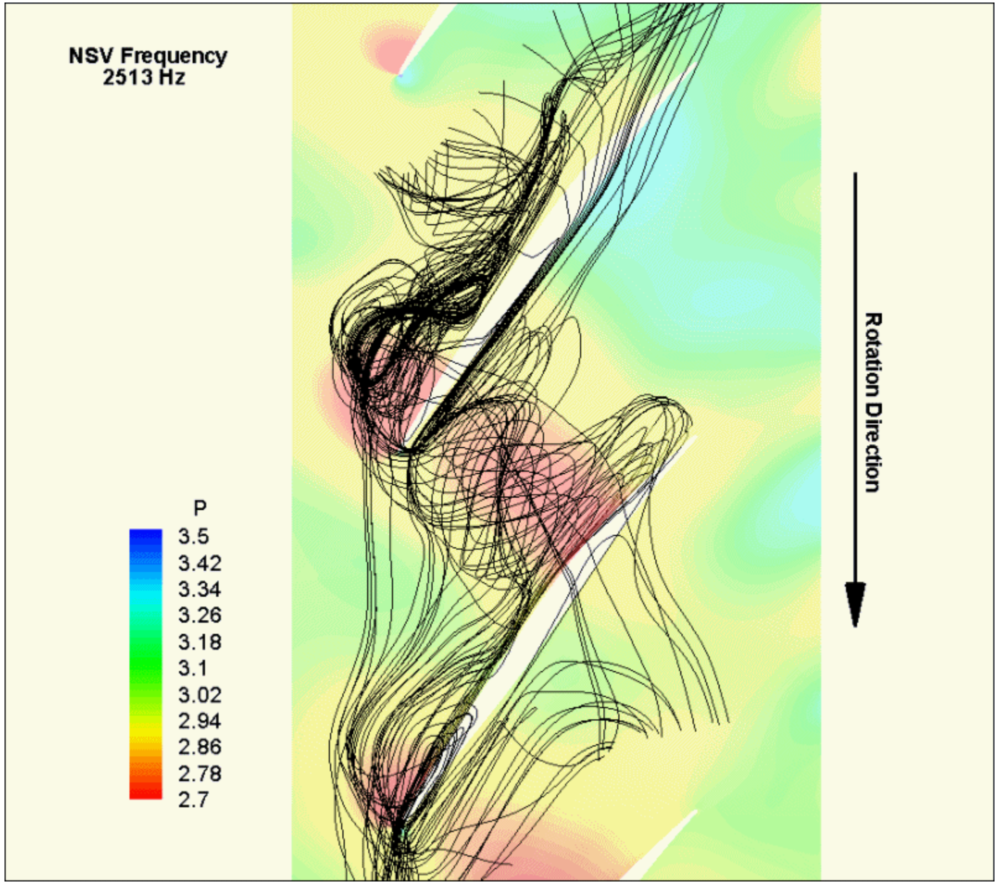
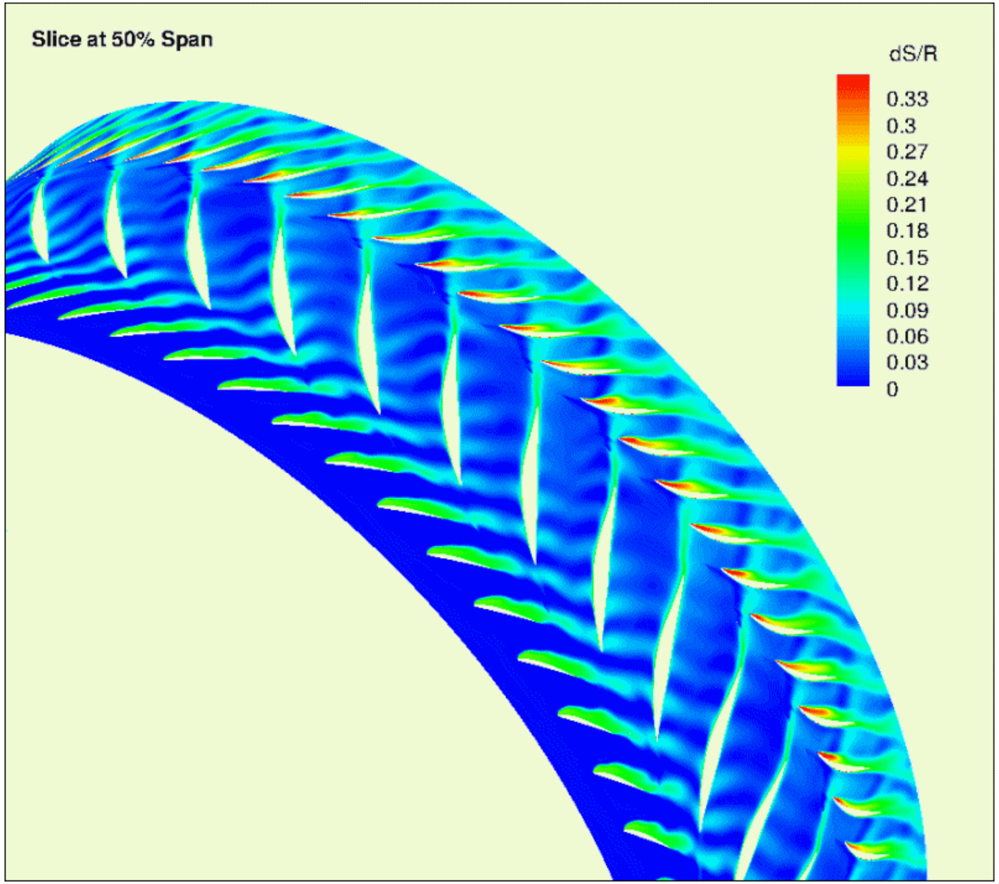


Figure Source (*Jason McDowell on Instagram*, 2001)

Torsional divergence was the first challenge encountered by aircraft manufacturers!

Jet Engine Blades Face a Very Rich Dynamical Environment





Non Synchronous Loads on a high speed axial compressor stage **CFD Results** from: (Zha, 2026)

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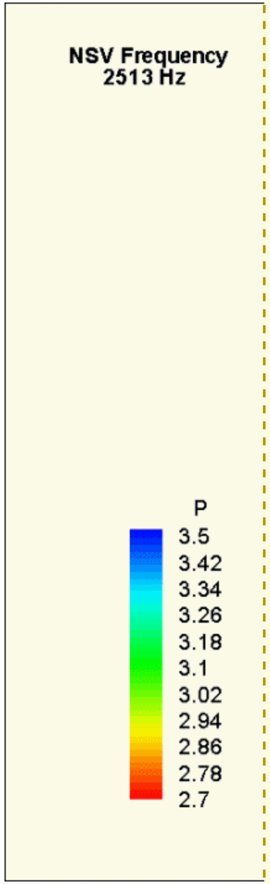

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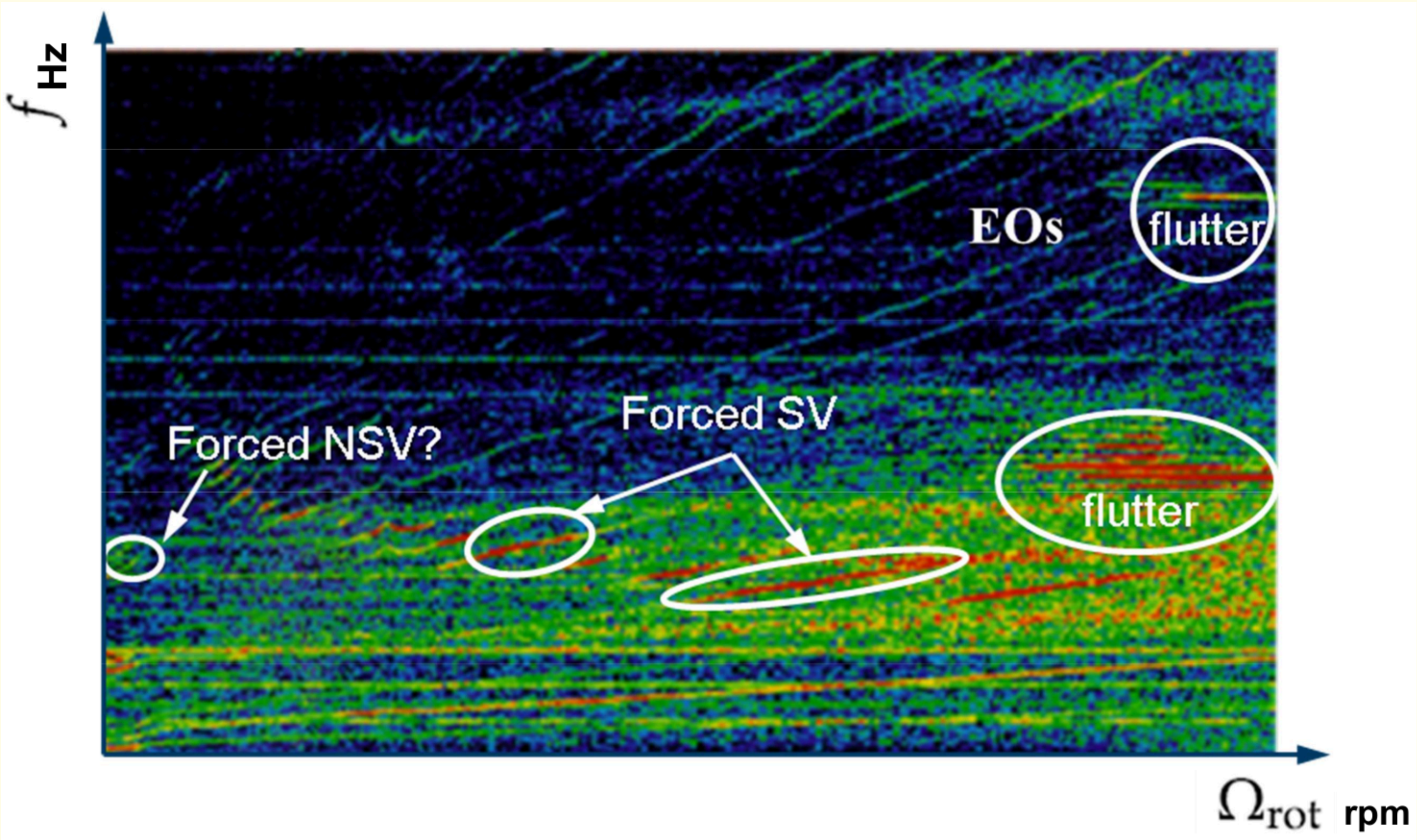
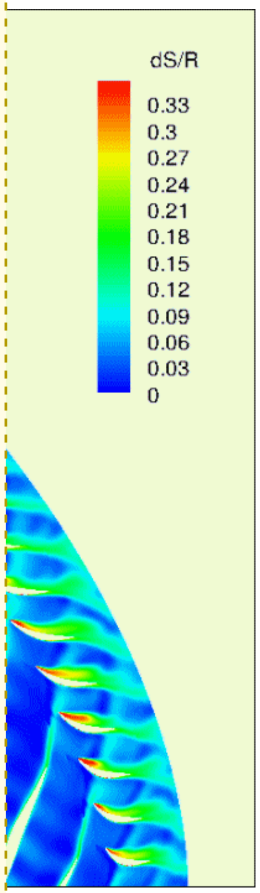


Experimental Campbell Diagram of a Blade Stage from a Test Engine source: (Hartung et al., 2017)

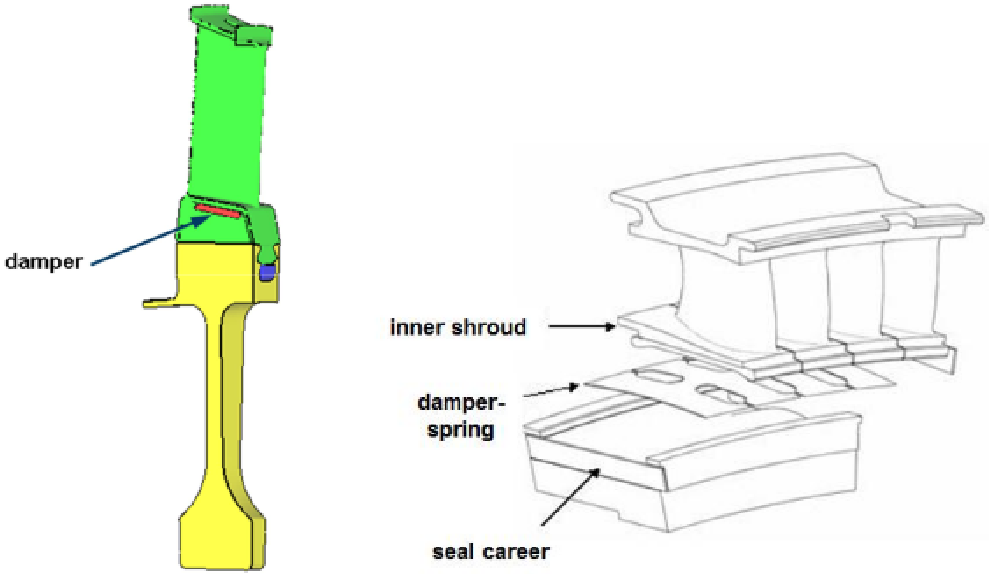


(Zha, 2026)

Research is under way for the Employment of Mechanical Damping Systems for Flutter Suppression

Traditional stall flutter involves aerodynamic saturation through periodic wake shedding.

- We may use **smooth models** to approximate these dynamics.



Mechanical Damping Systems for Flutter Suppression source: (Hartung et al., 2017)

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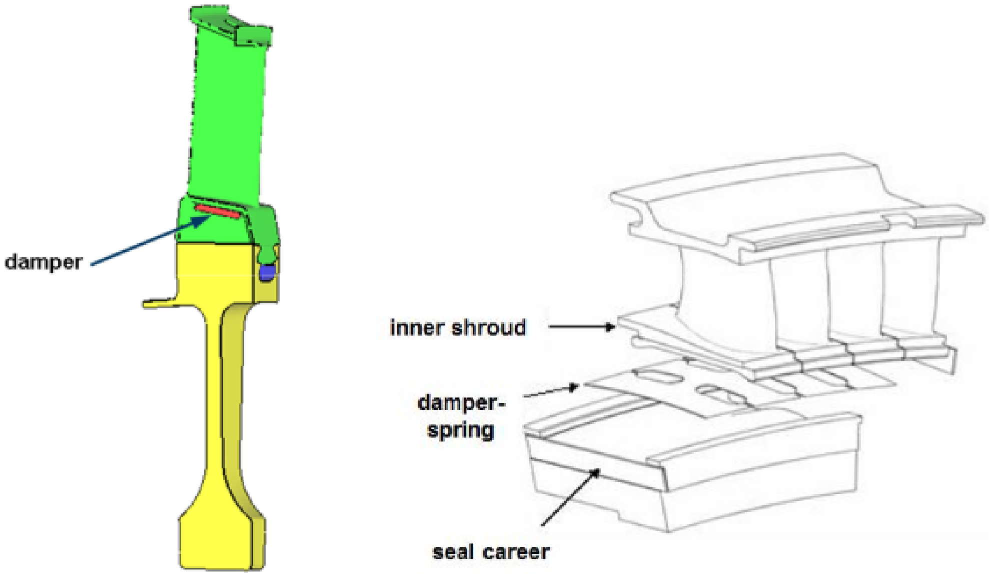
Traditional stall flutter involves aerodynamic saturation through periodic wake shedding.

- We may use **smooth models** to approximate these dynamics.

Friction is the primary mechanism for response suppression in damped configurations.

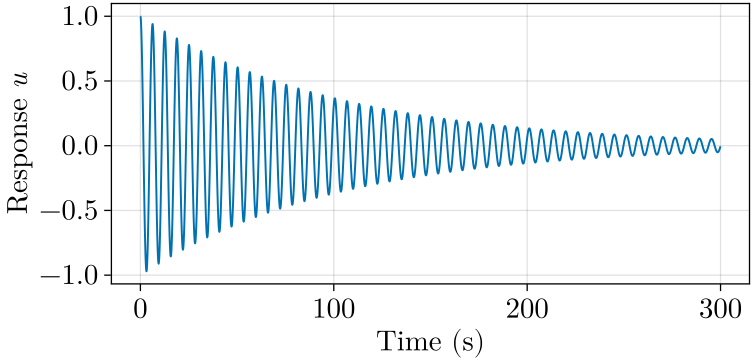
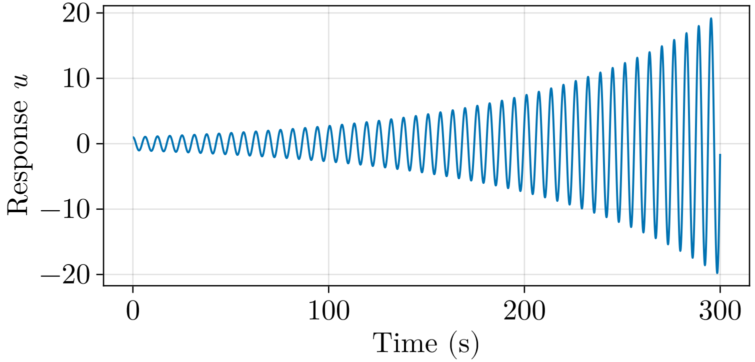
- Mechanical joints necessitate the consideration of **non-smooth dynamics**.



Mechanical Damping Systems for Flutter Suppression source: (Hartung et al., 2017)

1.1 Self-Excited Oscillations

Let's start with the simplest Single Degree-of-Freedom (SDoF) linear oscillator:

System	Response
$\ddot{u} + c\dot{u} + ku = 0$ $k = 1, c = 0.02$	
$\ddot{u} - c\dot{u} + ku = 0$ $k = 1, c = 0.02$	

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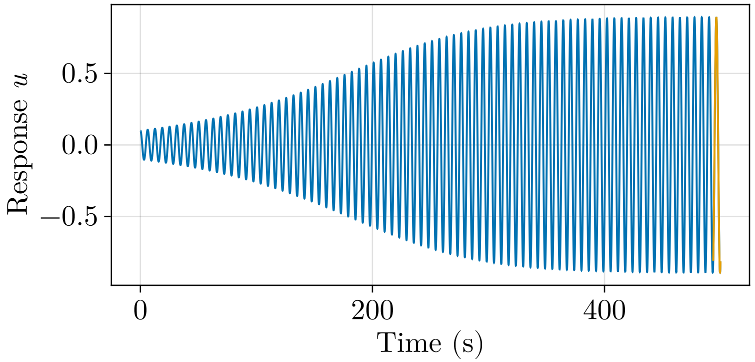
<tr><td>$$\ddot{u} - c\dot{u} + ku + \eta u^2\dot{u} = 0$$ $$k = 1,\ c = 0.02,\ \eta = 0.1$$</td><td>
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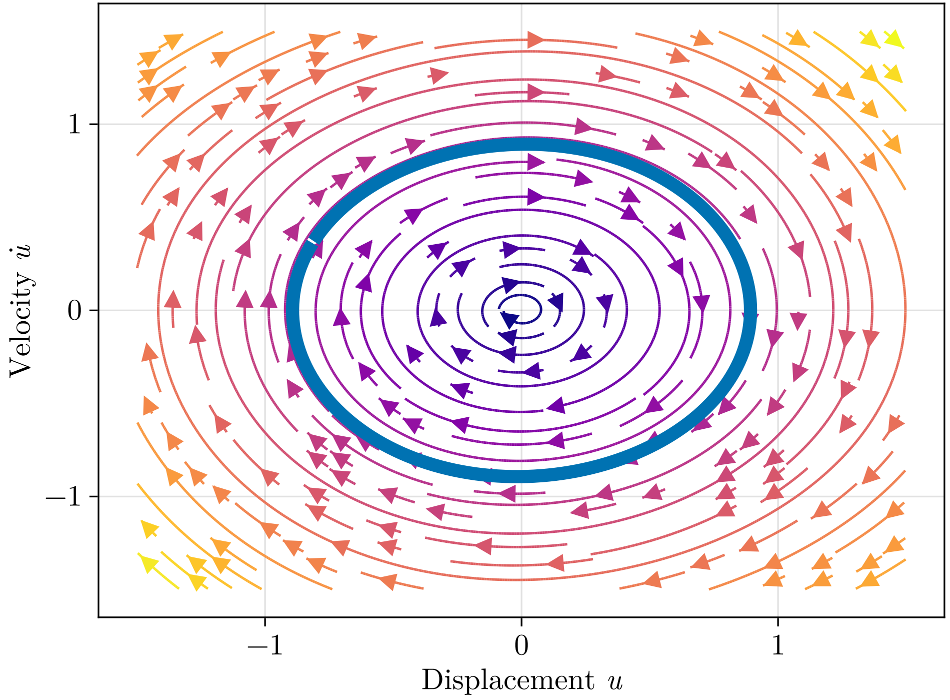
- In addition to the negative damping $-c\dot{u}$, we also have a **nonlinear cubic damping** term here, $\eta u^2 \dot{u}$: The "Van der Pol" Oscillator.

1.1 Self-Excited Oscillations

- This interplay between a **destabilizing term** ($-c\dot{u}$ here) and a **dissipative term** ($\eta u^2 \dot{u}$ here) characterizes self-excited oscillations.



The Phase Portrait of the Oscillator

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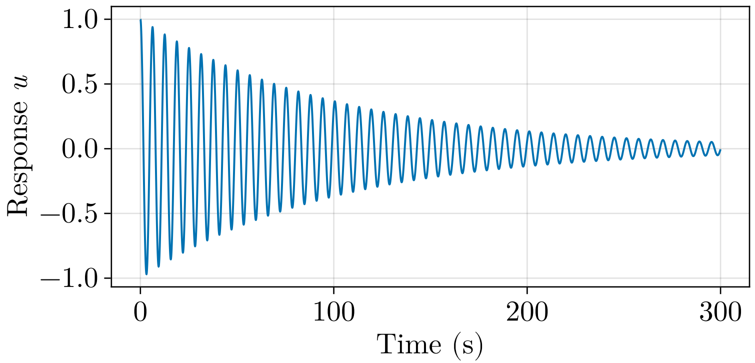
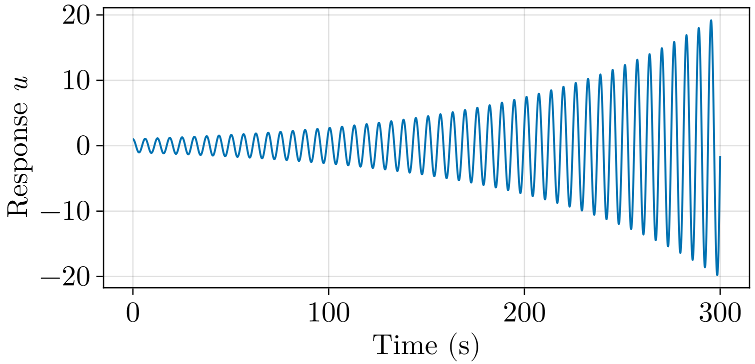
$\ddot{u} + c\dot{u} + ku = 0$ $k = 1, c = 0.02$	
$\ddot{u} - c\dot{u} + ku = 0$ $k = 1, c = 0.02$	

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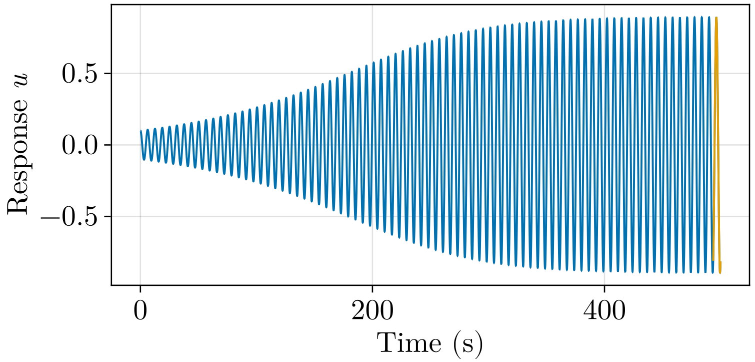
$\ddot{u} - c\dot{u} + ku + \eta u^2 \dot{u} = 0$ $k = 1, c = 0.02, \eta = 0.1$	

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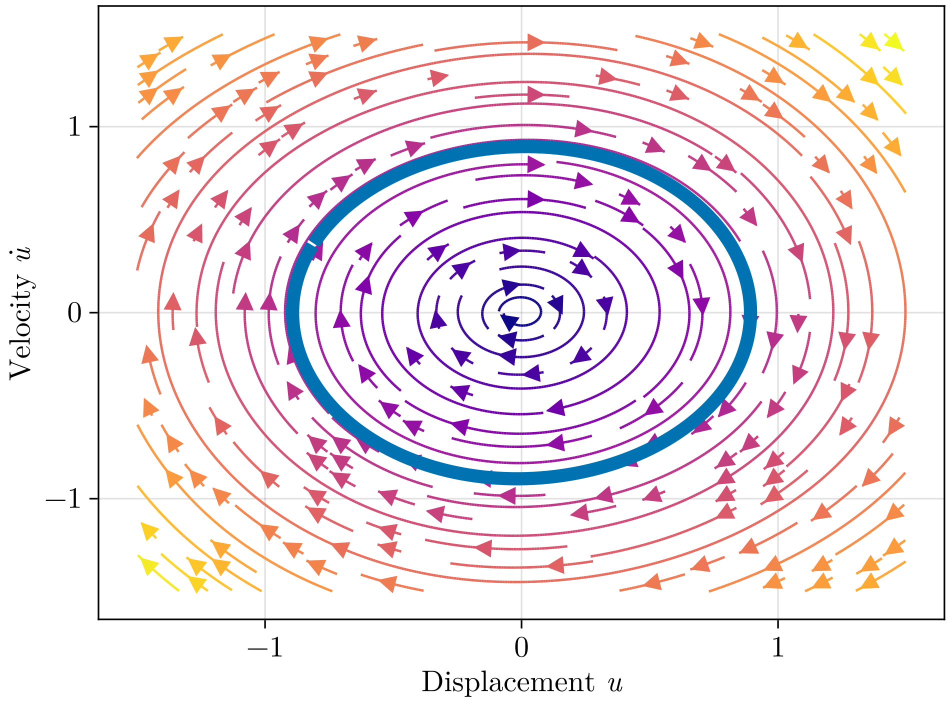
> Now what happens if an additional external excitation were present ?



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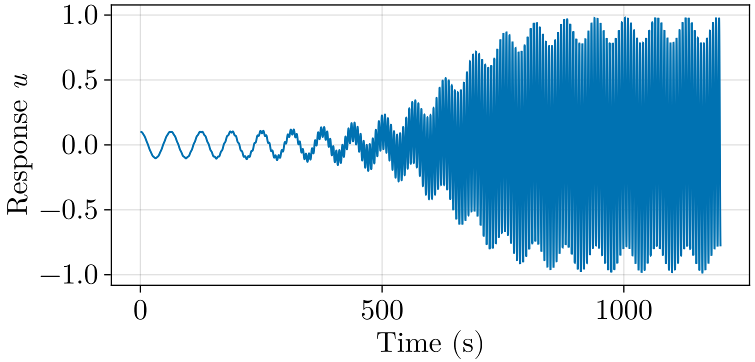
<tr><td>$$\ddot{u} - c\dot{u} + ku + \eta u^2 \dot{u} = F\cos\Omega t$$ $$k = 1,\ c = 0.02,\ \eta = 0.1,\ F = 0.1,\ \Omega = 0.1$$</td><td>
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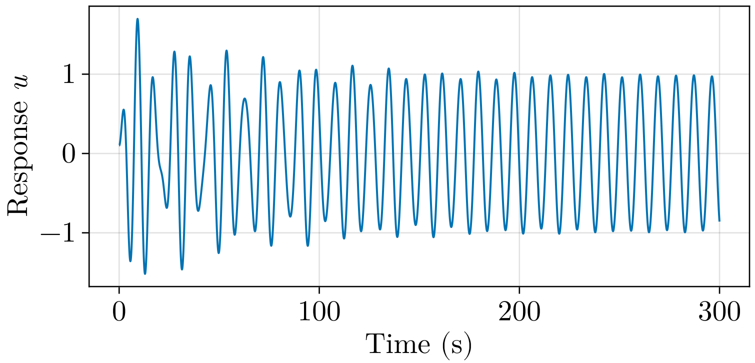
<tr><td>$$\ddot{u} - c\dot{u} + ku + \eta u^2 \dot{u} = F\cos\Omega t$$ $$k = 1,\ c = 0.02,\ \eta = 0.1,\ F = 0.5,\ \Omega = 0.7$$</td><td>
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2. Classical Bifurcation Theory

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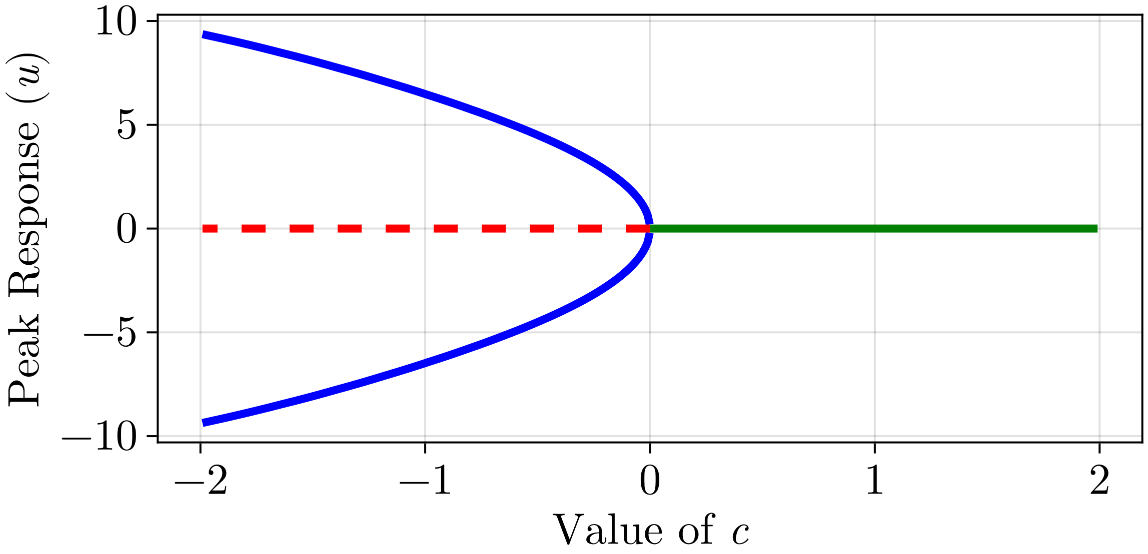
- Bifurcation is the qualitative change in the dynamics of a system as a parameter is varied.



- **Stability analysis** is used to detect the occurrence of a bifurcation.
- This is specifically called a **Hopf Bifurcation**.

2.1 Perturbation-Based Stability Analysis

- Perturbing a nonlinear system about a given solution leads to a linear system that governs the dynamics of the perturbation.

$$\dot{x} = f(x,t) \quad \text{with} \quad x = x^* + \delta x \Rightarrow \delta \dot{x} = \nabla_x f \bigg|_{x=x^*} \delta x$$

$$\Rightarrow \delta \dot{x} = A(t) \delta x$$

- For the oscillator at hand, the trivial solution $u^* = 0, \dot{u}^* = 0$ always exists and we have:

$$\frac{d}{dt}\begin{bmatrix} u \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k - 2\eta u \dot{u} & -c - \eta u^2 \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \end{bmatrix} \Rightarrow \frac{d}{dt}\begin{bmatrix} \delta u \\ \delta \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}.$$

- The eigenvalues of this matrix are

$$\lambda = -\frac{c}{2} \pm \frac{i}{2}\sqrt{4k - c^2}.$$

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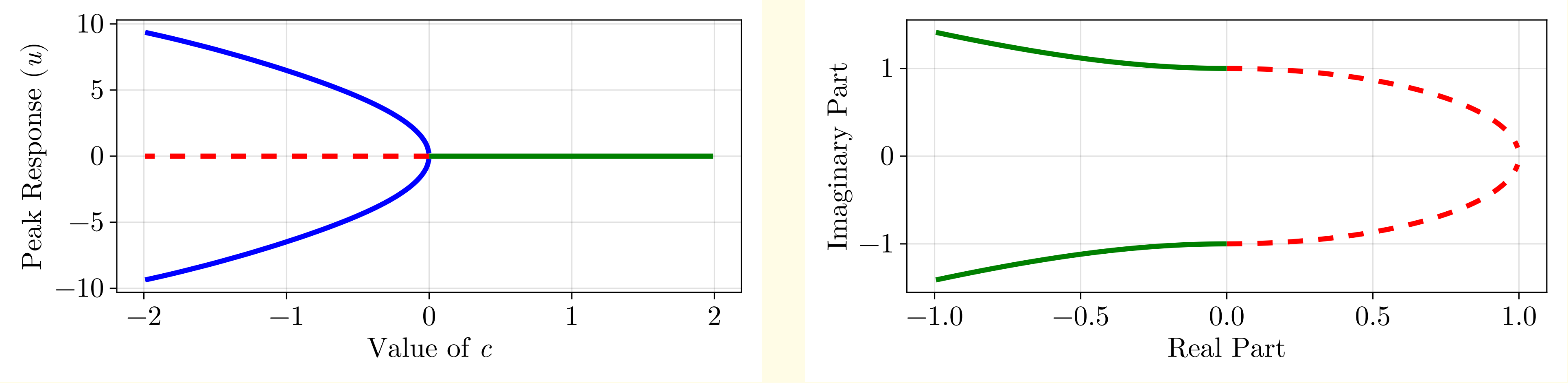

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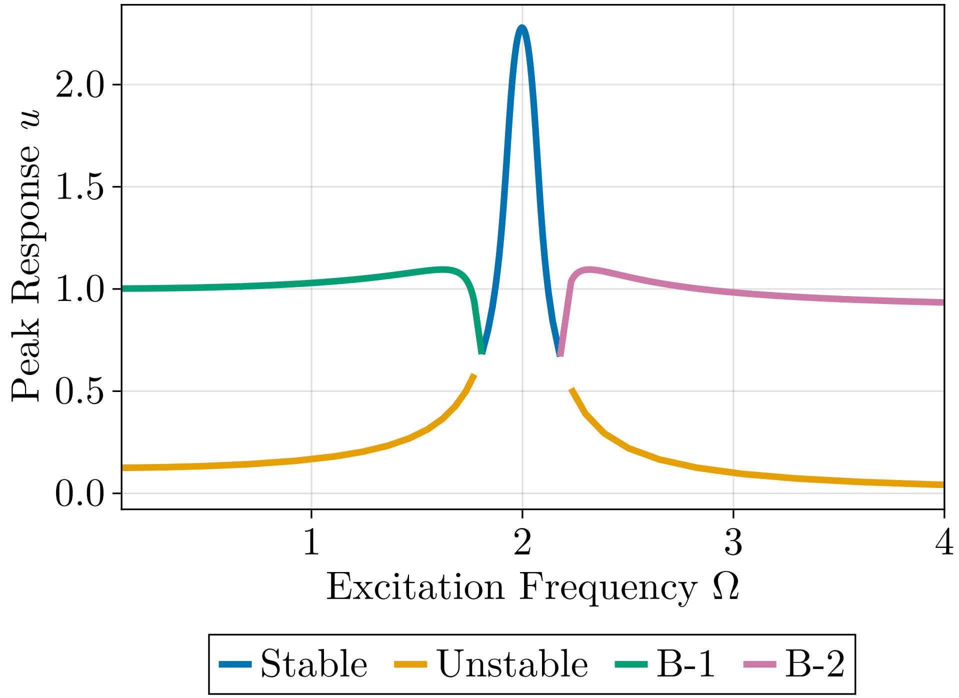
The Forced Case

- Considering the forced Van der Pol oscillator, we get a **Secondary Hopf Bifurcation**: Periodic solution bifurcates into a quasi-periodic response.
- This is the problem setting that we shall be concerned with.



$$\ddot{u} - c\dot{u} + ku + \eta u^2 \dot{u} = F \cos \Omega t$$

$$k = 4., \; c = 0.02, \; \eta = 0.1, \; F = 0.5$$

3. Non-Smooth Dynamical Systems

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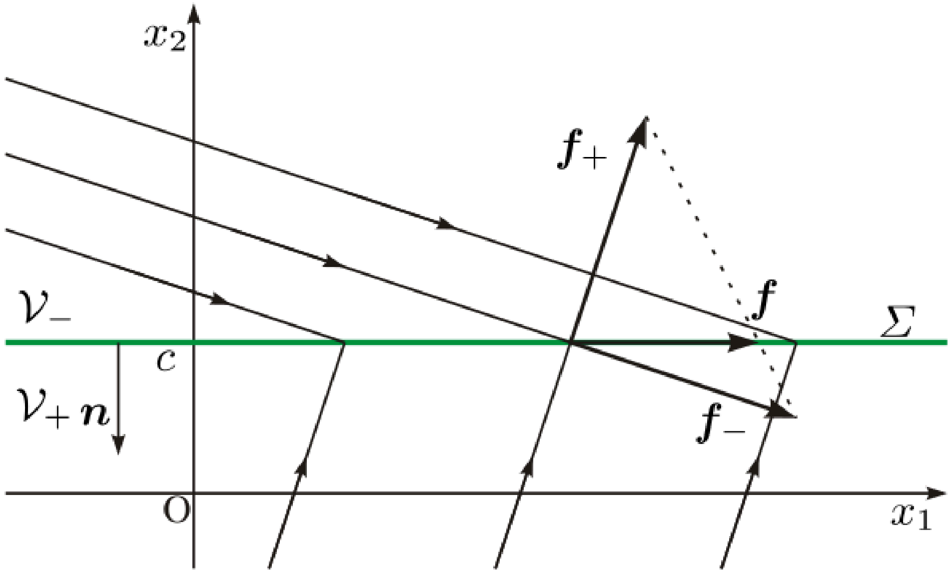
- A non-smooth continuous system may be written as

$$\dot{x} = \begin{cases} f_-(x,t) & x \in \mathcal{V}_- \\ f_+(x,t) & x \in \mathcal{V}_+ \end{cases}$$

- The intersection of $\Sigma = \mathcal{V}_- \cap \mathcal{V}_+$ is contentious so the above is **generalized by a differential inclusion**:

$$\dot{x} \in F(x,t) = \begin{cases} f_-(x,t) & x \in \mathcal{V}_- \\ \overline{\text{co}}\{f_-(x,t), f_+(x,t)\} & x \in \Sigma \\ f_+(x,t) & x \in \mathcal{V}_+ \end{cases}$$



Source: (Leine, 2006)

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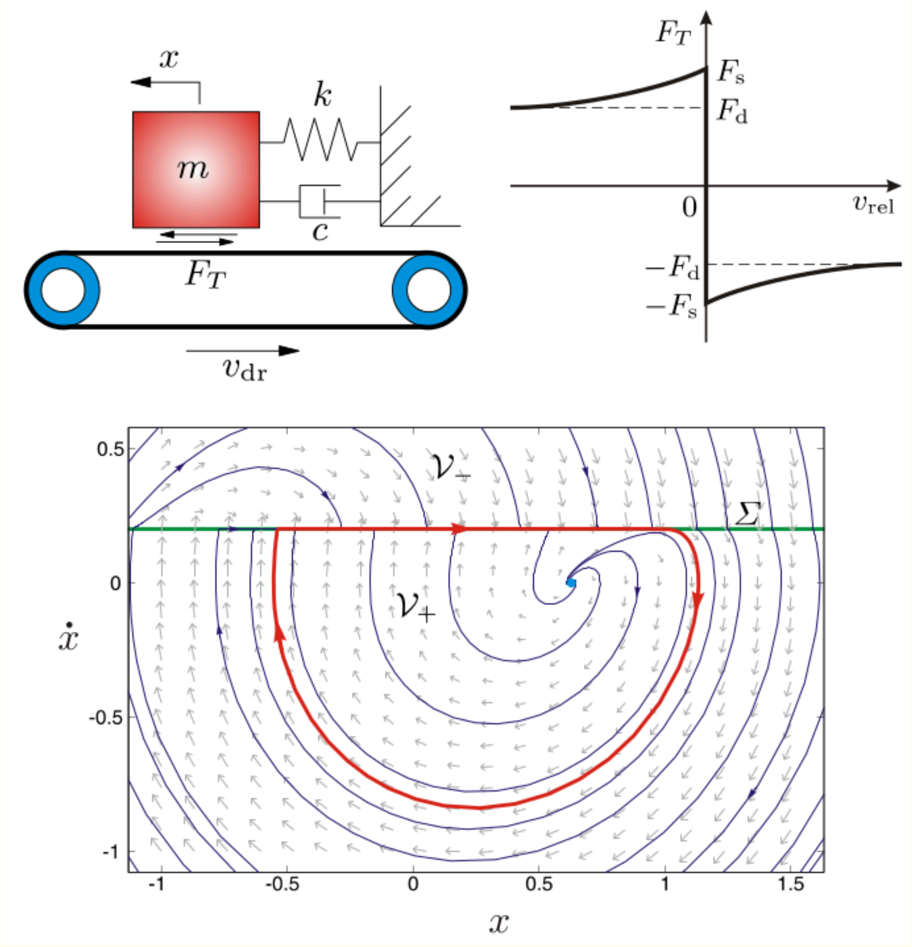
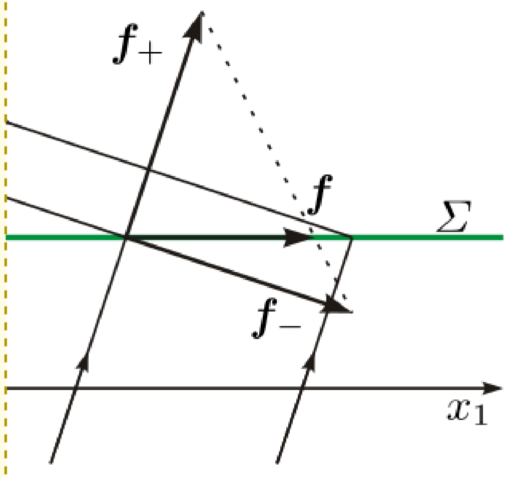


Responses of such systems can be very rich Source: (Leine, 2006)



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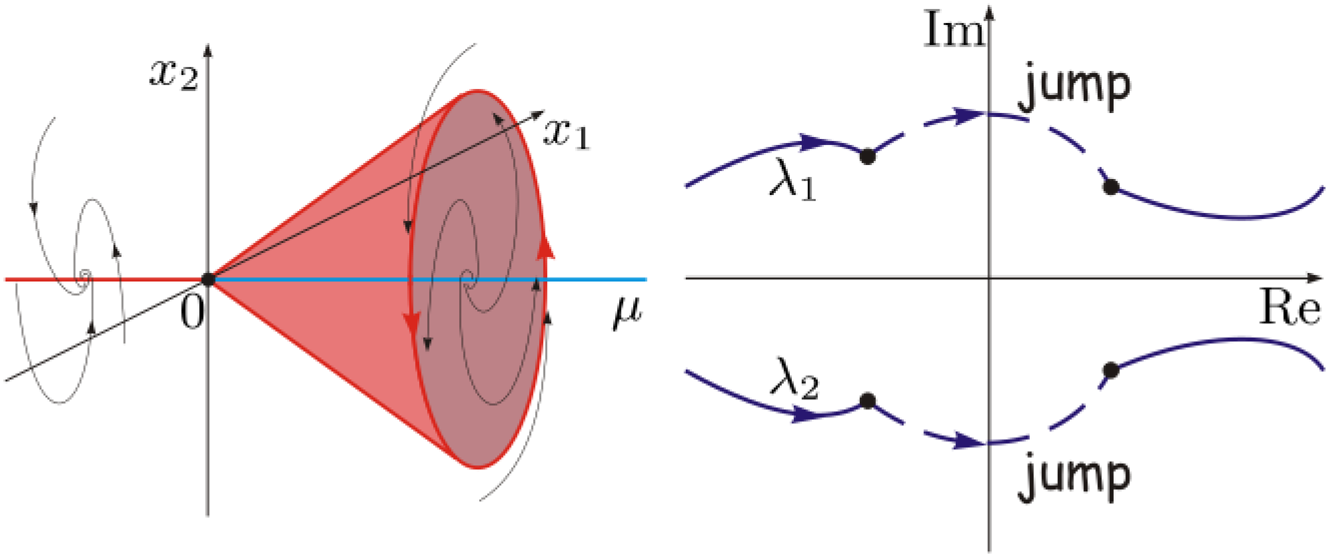
3.1 Smooth versus Non-Smooth Bifurcations



Source: (Leine, 2006)

- "Jump" bifurcations are extremely common in non-smooth systems, rendering perturbation approaches ineffective.
- In practice, this leads to very "abrupt" changes in responses of systems.

4. Averaging as a Framework for Analytical Dynamics

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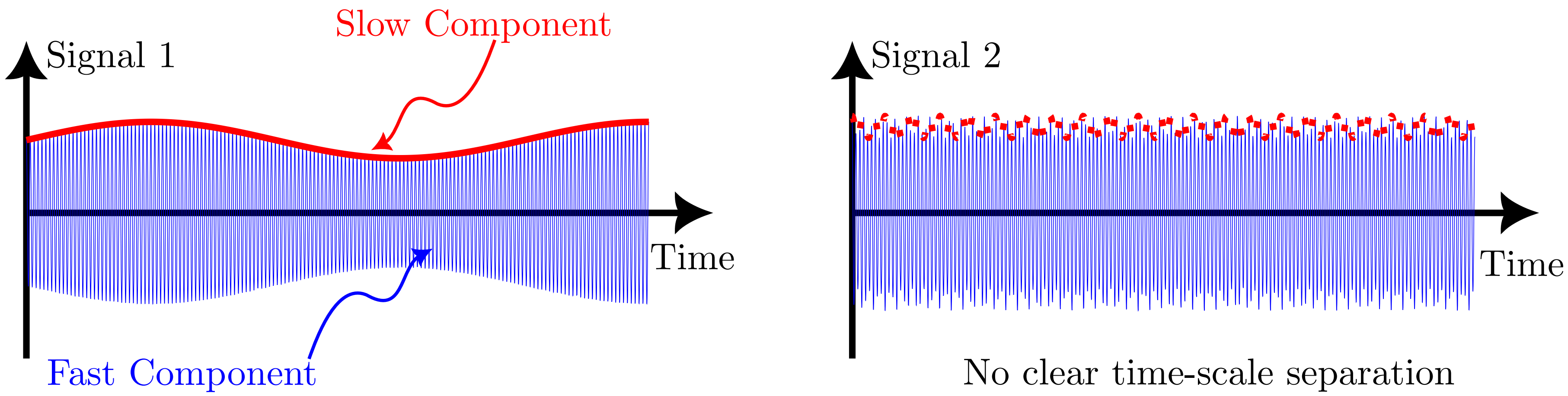
- **The Method of Averaging** is a classical analytical methodology in nonlinear dynamics (Manevitch, 1999).
- The idea is to decompose the response of a system in terms of **fastly varying** and **slowly varying components**:

$$x(t) = A(t)\cos(\Omega t + \varphi(t))$$



- Apart from the simplification, it offers newer ways of probing the stability of a system.

4.1 Averaging in Practice

- In practice, we use the ansatz $x(t) = A(t)\cos\Omega t + B(t)\sin\Omega t$ to derive governing equations for the amplitude $A(t), B(t)$.
- For an SDoF oscillator:

$$m\ddot{x} + c\dot{x} + kx + f_{\text{nl}} = F_c \cos\Omega t + F_s \sin\Omega t \quad (\mathcal{S})$$

$$\Rightarrow 2\Omega \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \begin{bmatrix} \dot{A} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} k-\Omega^2 m & \Omega c \\ -\Omega c & k-\Omega^2 m \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} F_{\text{nl,C}} \\ F_{\text{nl,S}} \end{bmatrix} - \begin{bmatrix} F_C \\ F_S \end{bmatrix} \quad (\mathcal{A}).$$

- The most obvious advantage is that periodic solutions of $(\mathcal{S})$ are fixed points of $(\mathcal{A})$.
- For smooth systems, it can be shown analytically that the **eigenvalues of perturbation of $(\mathcal{A})$ are identical to the system's Floquet multipliers**.

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Note that the accuracy of $(\mathcal{A})$ is closely related to the accuracy of the ansatz.

4.1 Averaging in Practice

- Although not completely well established (yet!), generalizations are possible.
- For a multi-harmonic Ansatz we can derive:

$$x(t) = \sum_{n=1}^{N} A_n(t) \cos n\Omega t + B_n(t) \sin n\Omega t,$$

$$2\Omega \begin{bmatrix} 0 & -m & & & \\ m & 0 & & & \\ & & 0 & -2m & \\ & & 2m & 0 & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} \dot{A}_1 \\ \dot{B}_1 \\ \dot{A}_2 \\ \dot{B}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} k-\Omega^2 m & \Omega c & & & \\ -\Omega c & k-\Omega^2 m & & & \\ & & k-n^2\Omega^2 m & n\Omega c & \\ & & -n\Omega c & k-n^2\Omega^2 & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} F_{\text{nl,C,1}} \\ F_{\text{nl,S,1}} \\ F_{\text{nl,C,2}} \\ F_{\text{nl,S,2}} \\ \vdots \end{bmatrix} - \begin{bmatrix} F_C \\ F_S \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

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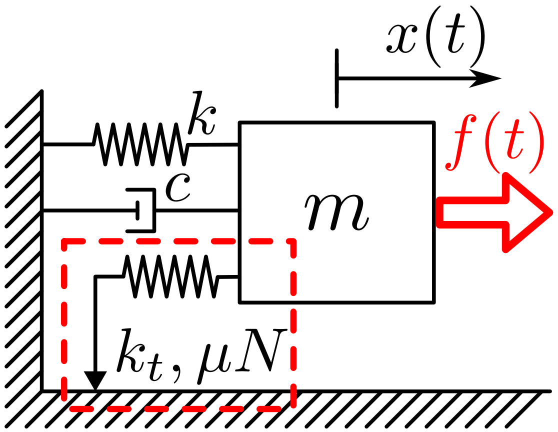
The technical difficulty in interpreting this stems from the fact that **slow-fast time-scale separation is no longer very intuitive**.

5. Application Examples

5.1 An SDoF Frictional Oscillator

Let's start with the simplest example of an **SDoF self-excited frictional oscillator**



$$\ddot{x} - 2\zeta\omega_n\dot{x} + \omega_n^2 x + f_{\text{fr}}(x) = F\cos\Omega t$$

$$f_{\text{fr}}(t_{\ell+1}) = \begin{cases} k_t(u_{\ell+1} - u_\ell) + f_{\text{fr}}(t_\ell) & \text{stick} \\ \text{sign}(f_{\text{sp}})\mu N & \text{slip} \end{cases}$$

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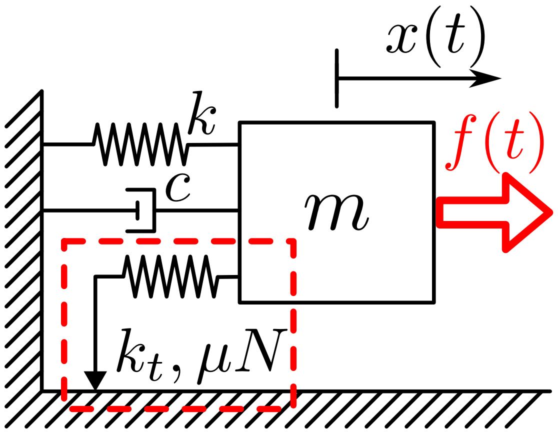

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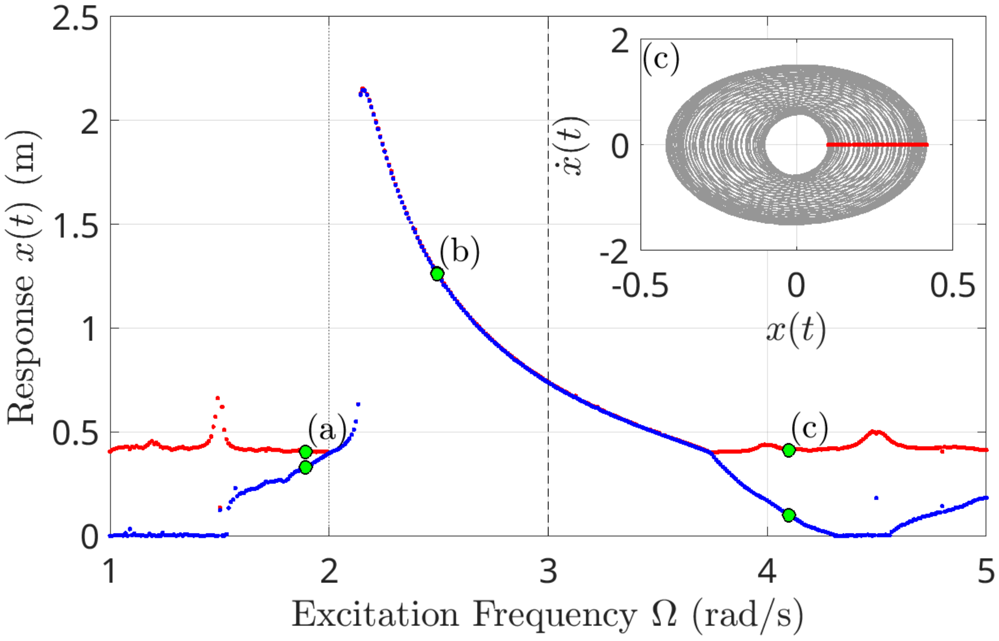
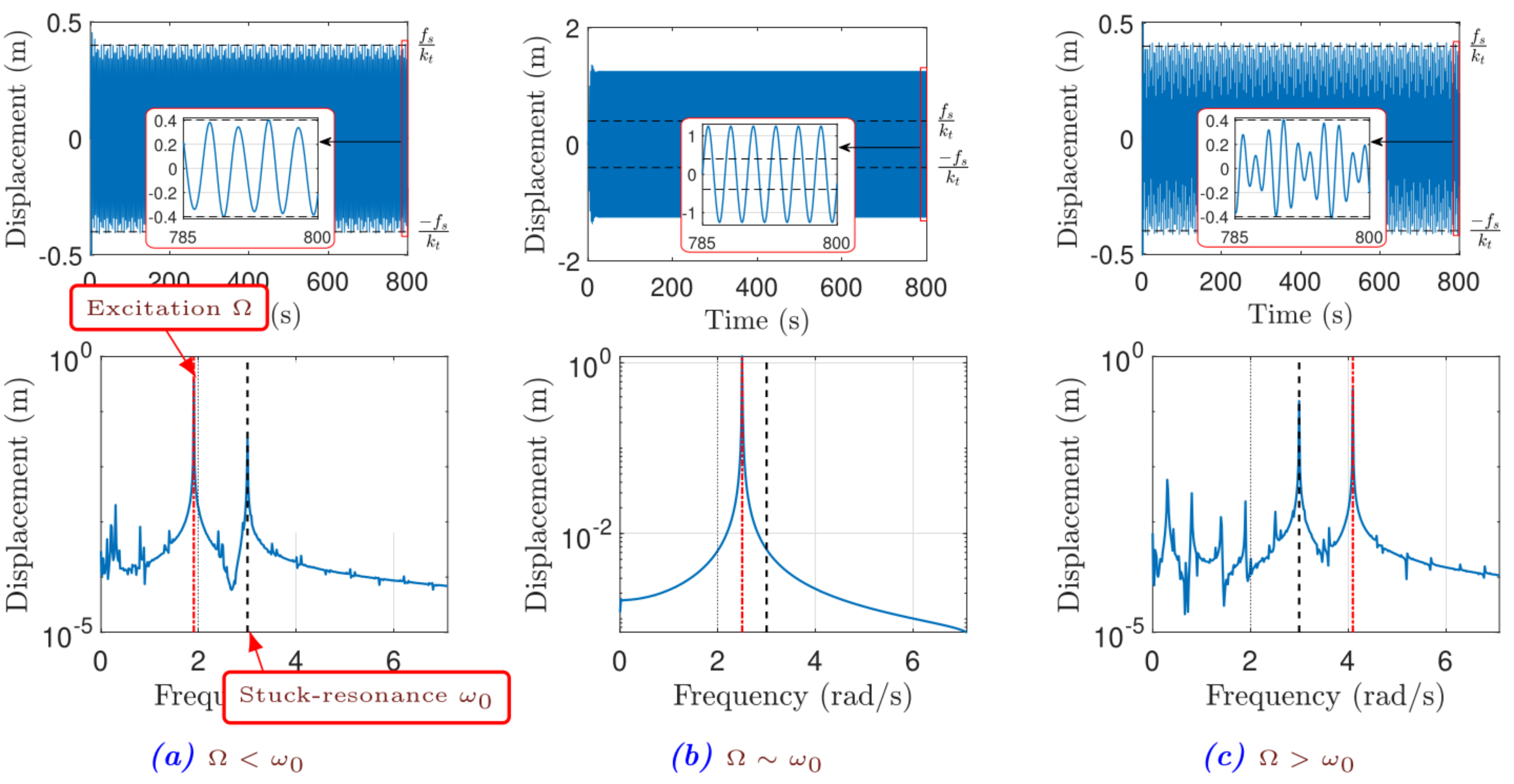


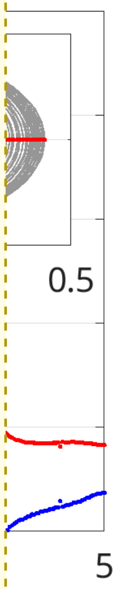
Transient Forced Response Results

Let's

SDo



Transient responses showing the synchronization-desynchronization behavior



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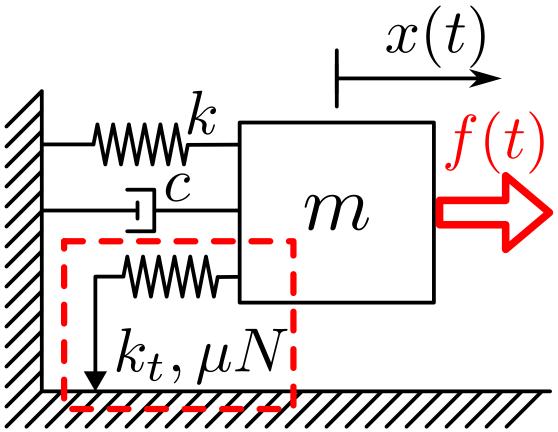

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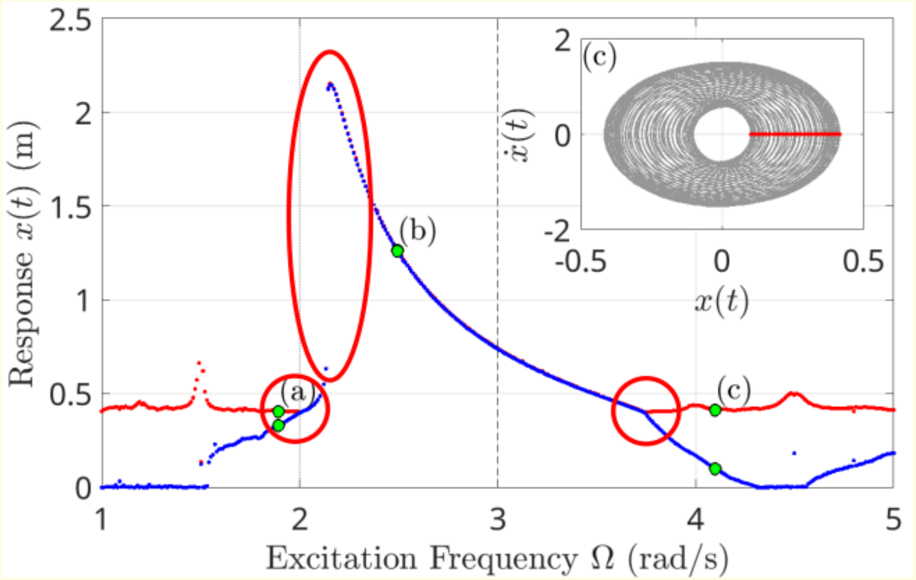
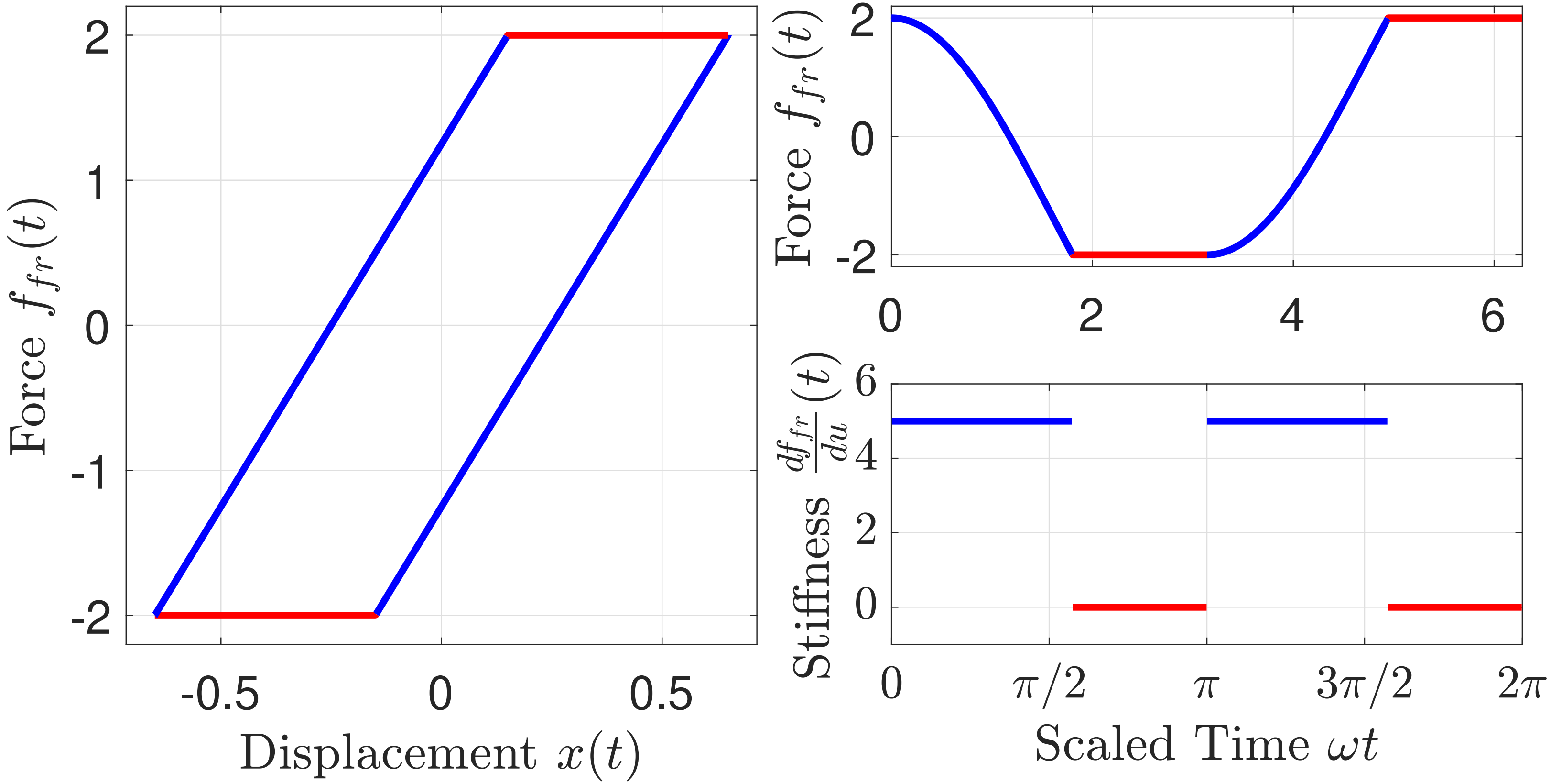


We can observe two types of bifurcations from this system:

- The **secondary Hopf bifurcation** at the synchronization points
- The **fold bifurcation** at the drop-offs

The Fourier approach guarantees the existence of at least a weak derivative



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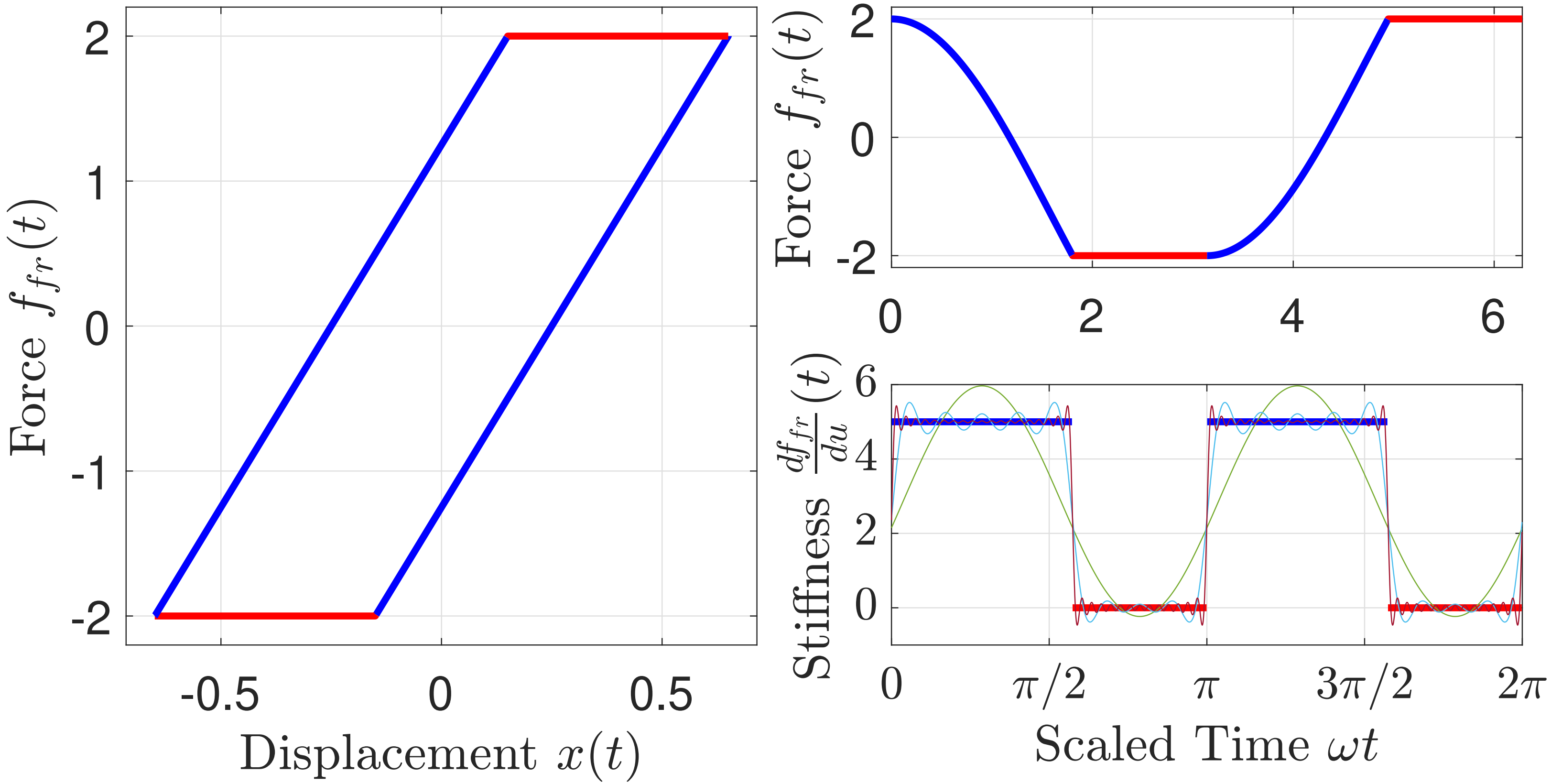
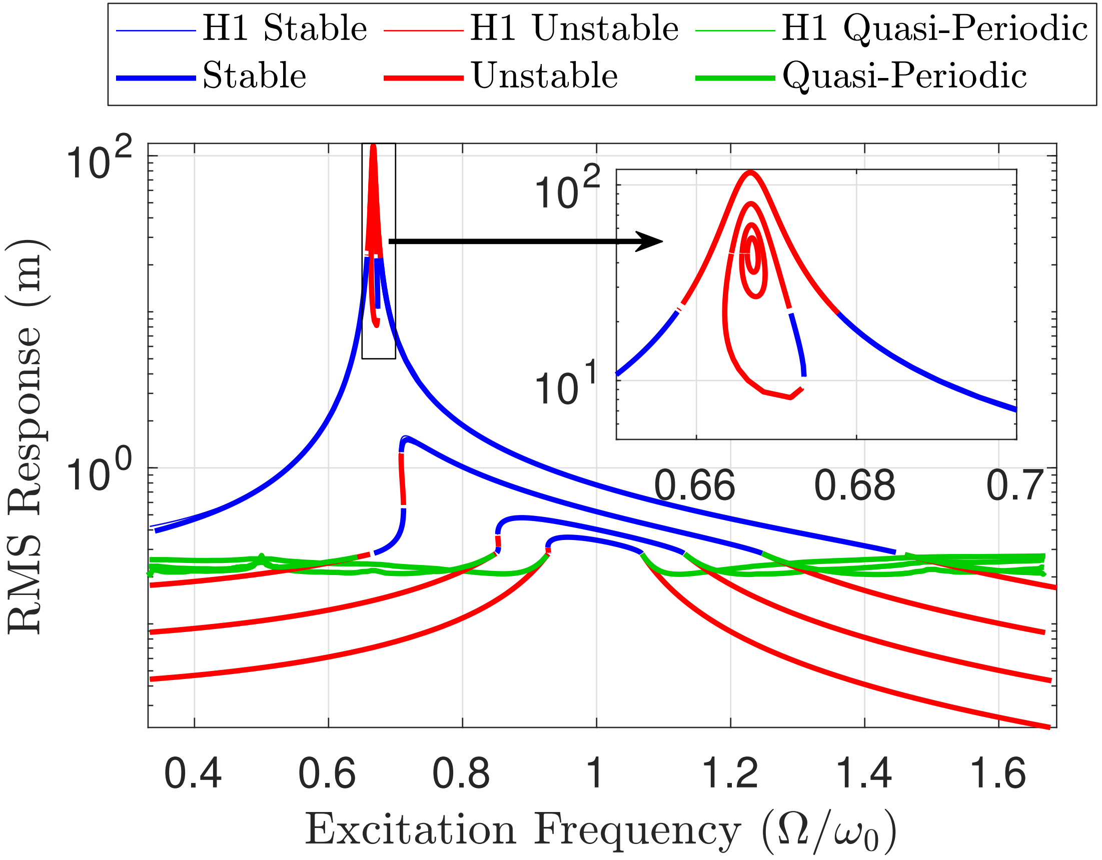


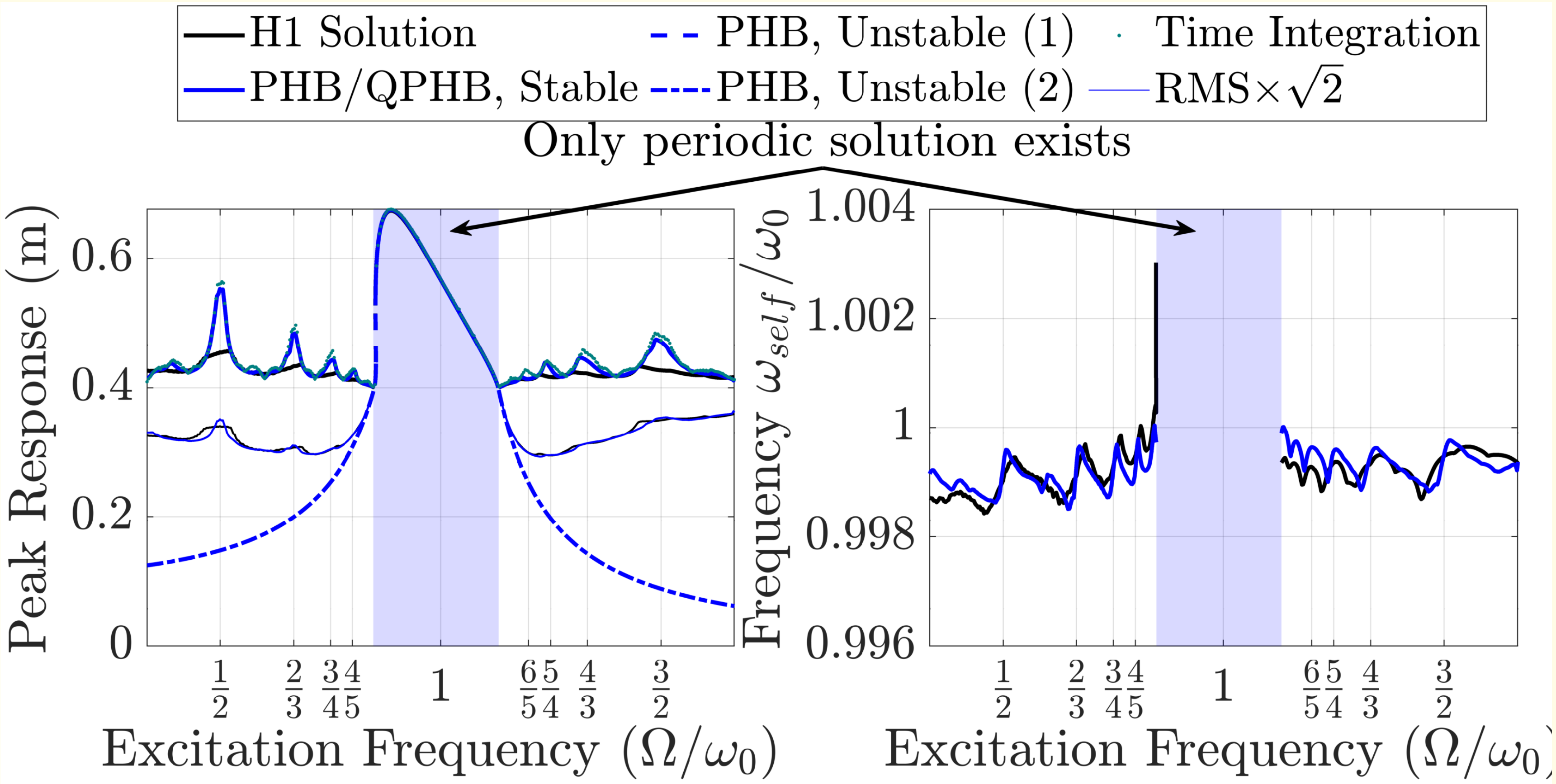
Although classical Floquet theorem is not applicable, local linearization for fixed point stability analysis is still justified as this only requires **piece-wise continuity** (Khalil, 2002).

The Single Harmonic Approach Already Yields Good Results

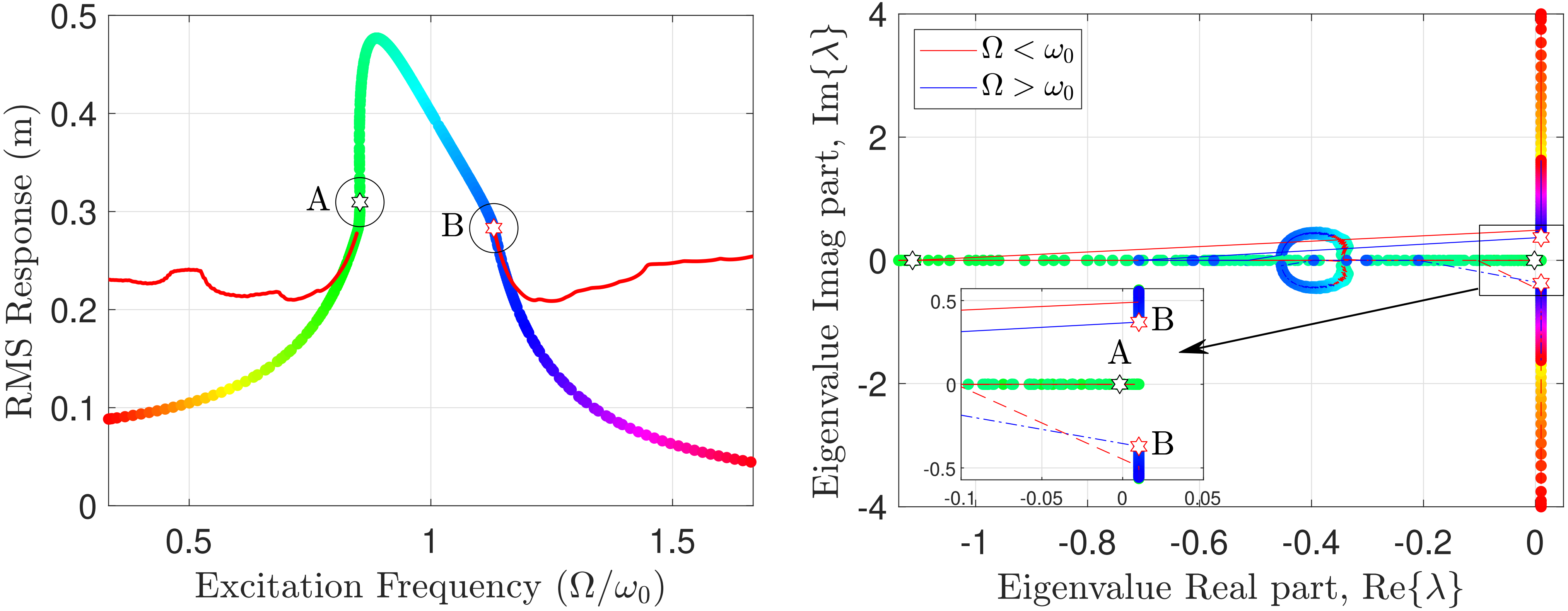


The complete forced response curve showing the different response branches

The Single Harmonic Approach Already Yields Good Results

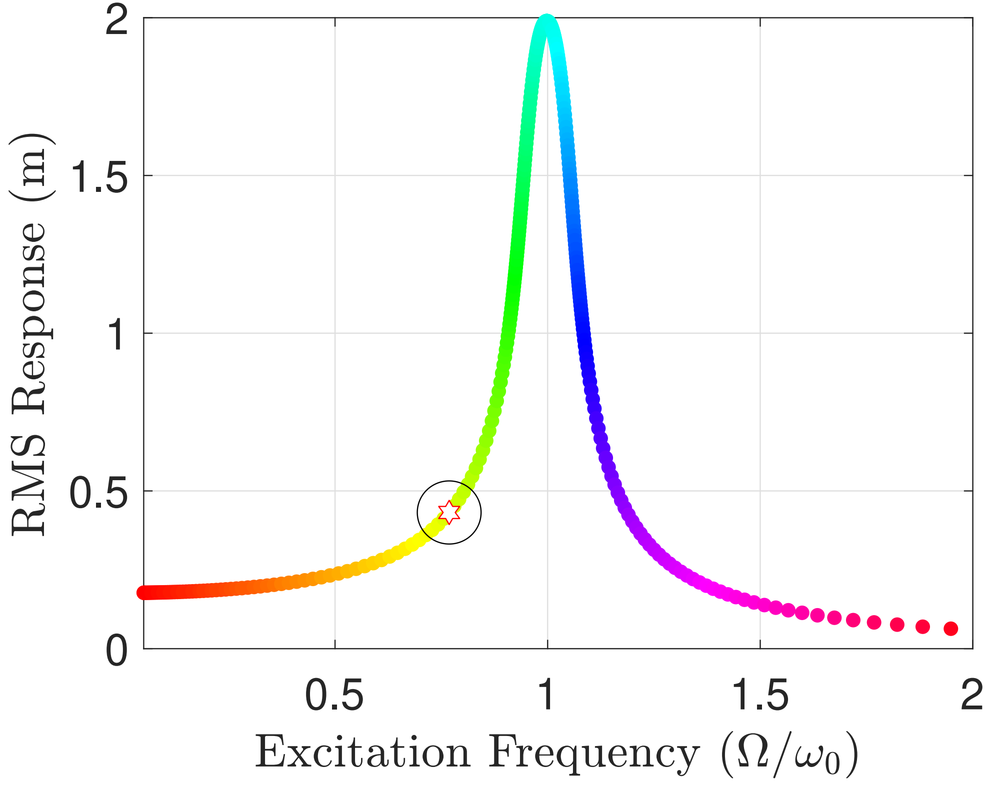


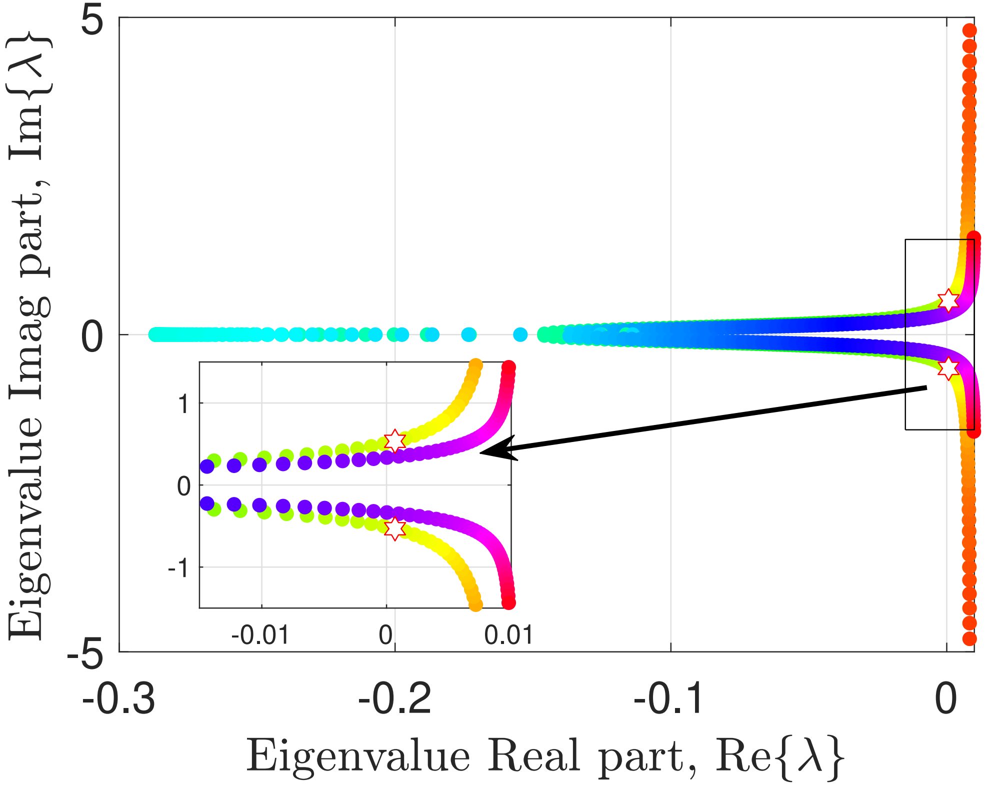
The "Jump" Bifurcation



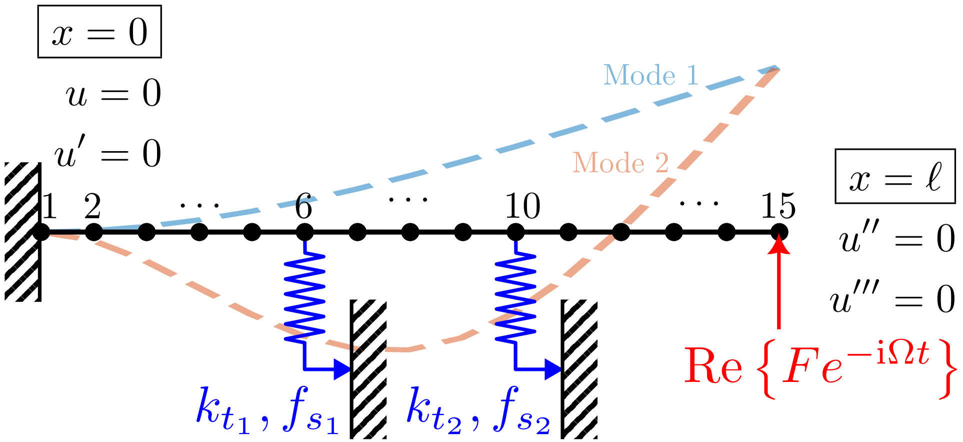
The Frictional Oscillator

The “Jump” Bifurcation

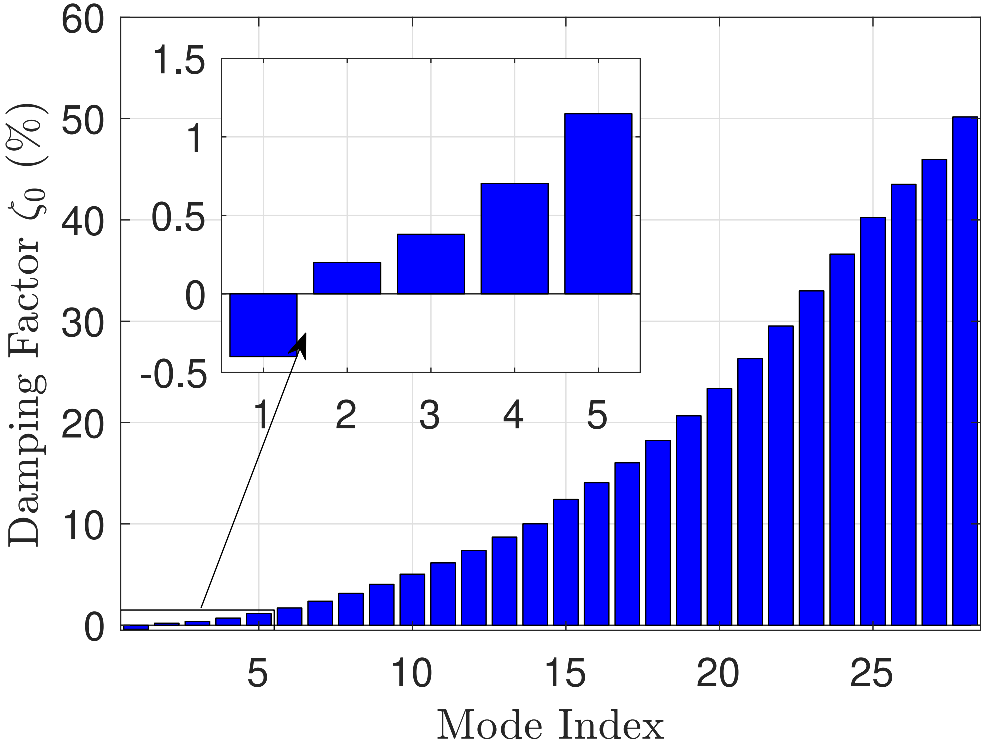




The Forced Van der Pol Oscillator

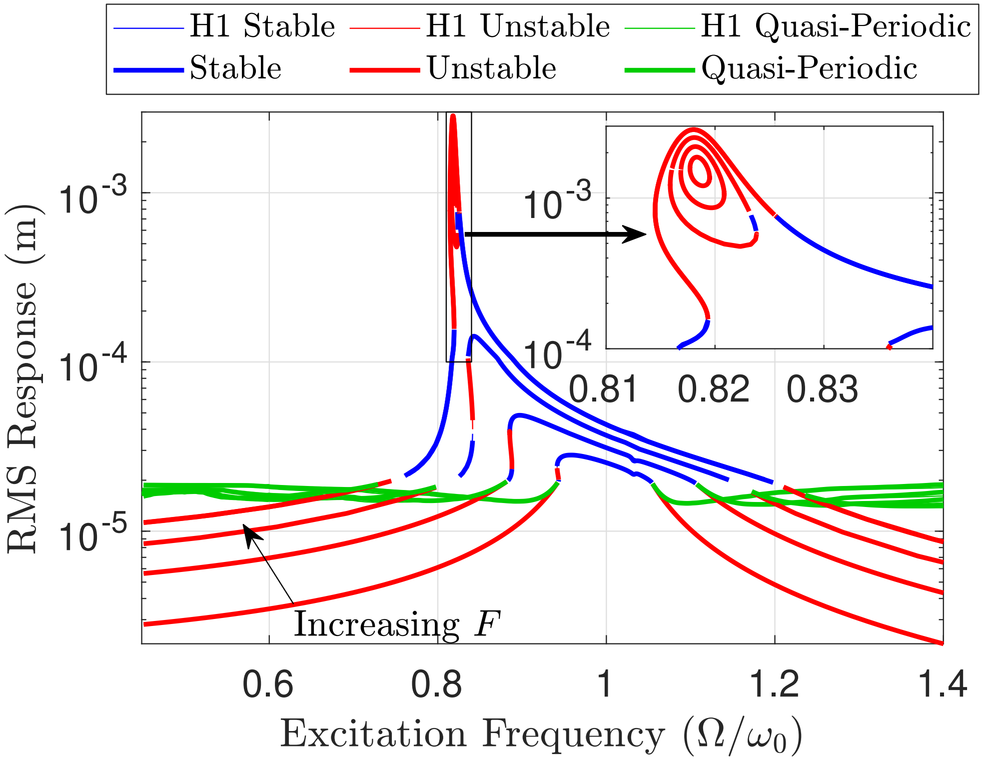


$$\ddot{u} + \mathcal{C}(\dot{u}) + \alpha^4 u'''' = 0, \quad x \in (0, \ell)$$



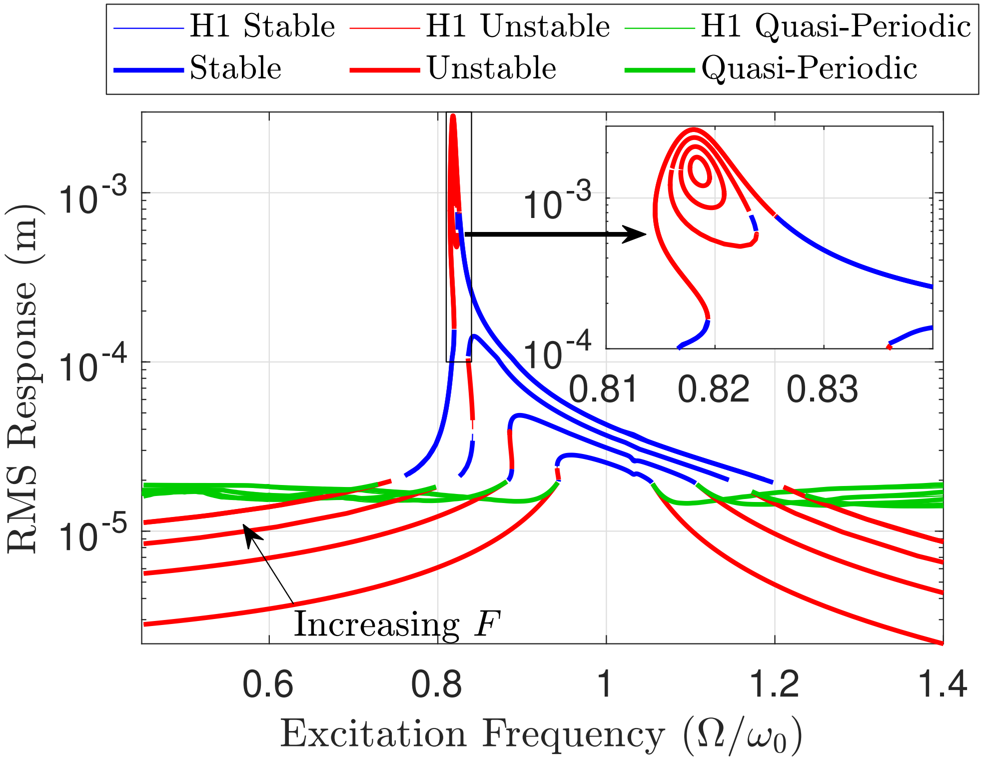
A tip-loaded cantilevered beam with frictional supports

5.2 An MDoF Example

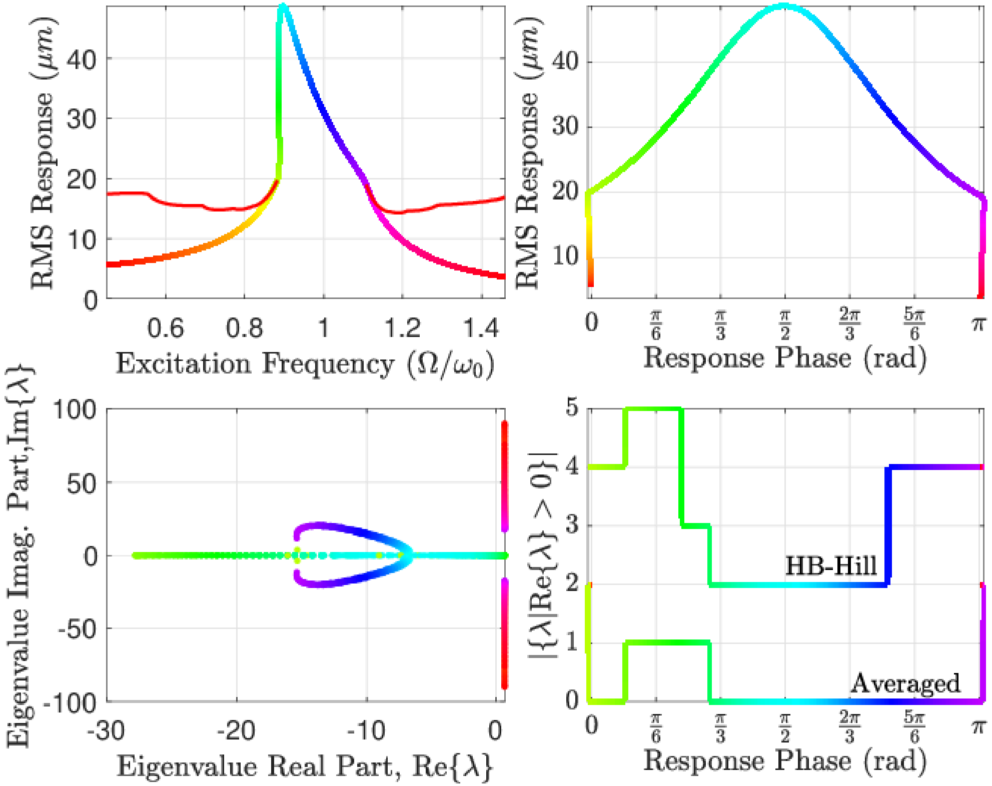


Forced response near the first resonance

5.2 An MDoF Example



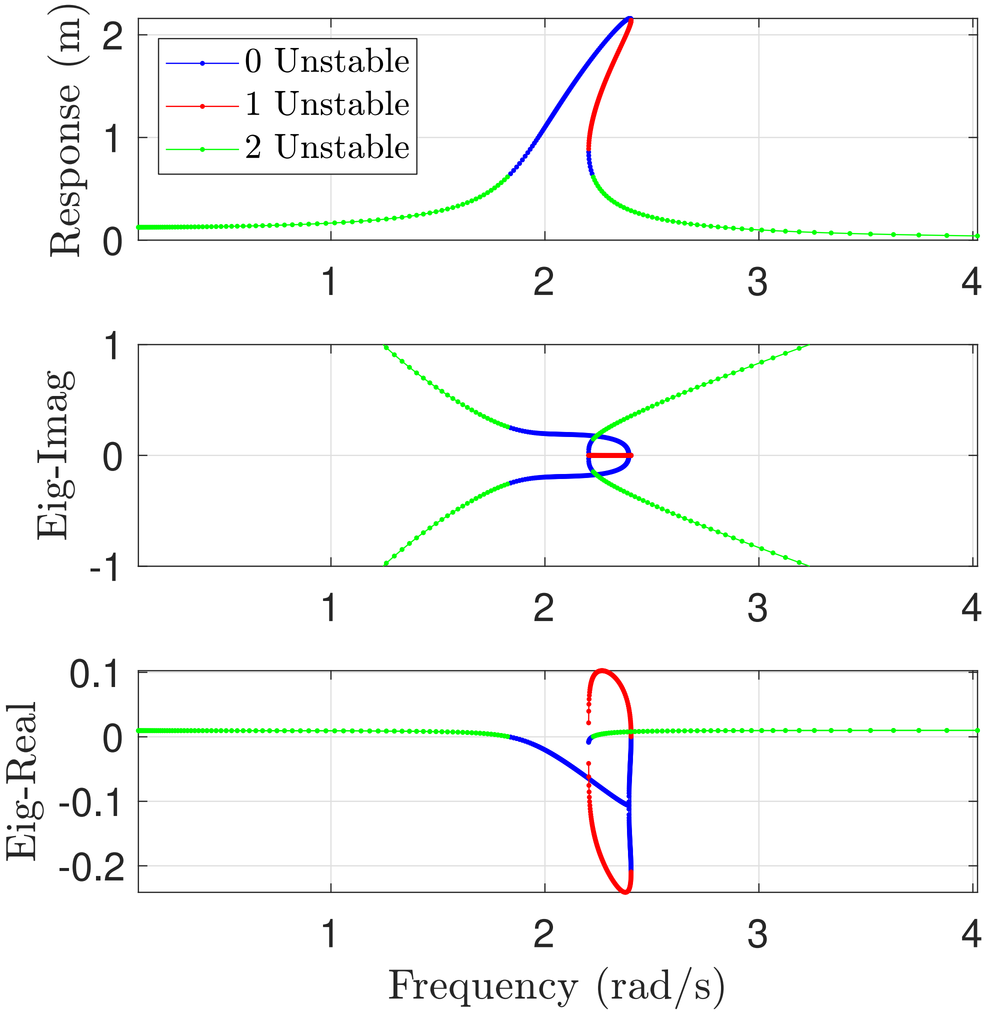
Forced response near the first resonance



The averaged approach compares very favourably against other frequency domain approaches in the literature (Von Groll & Ewins, 2001)

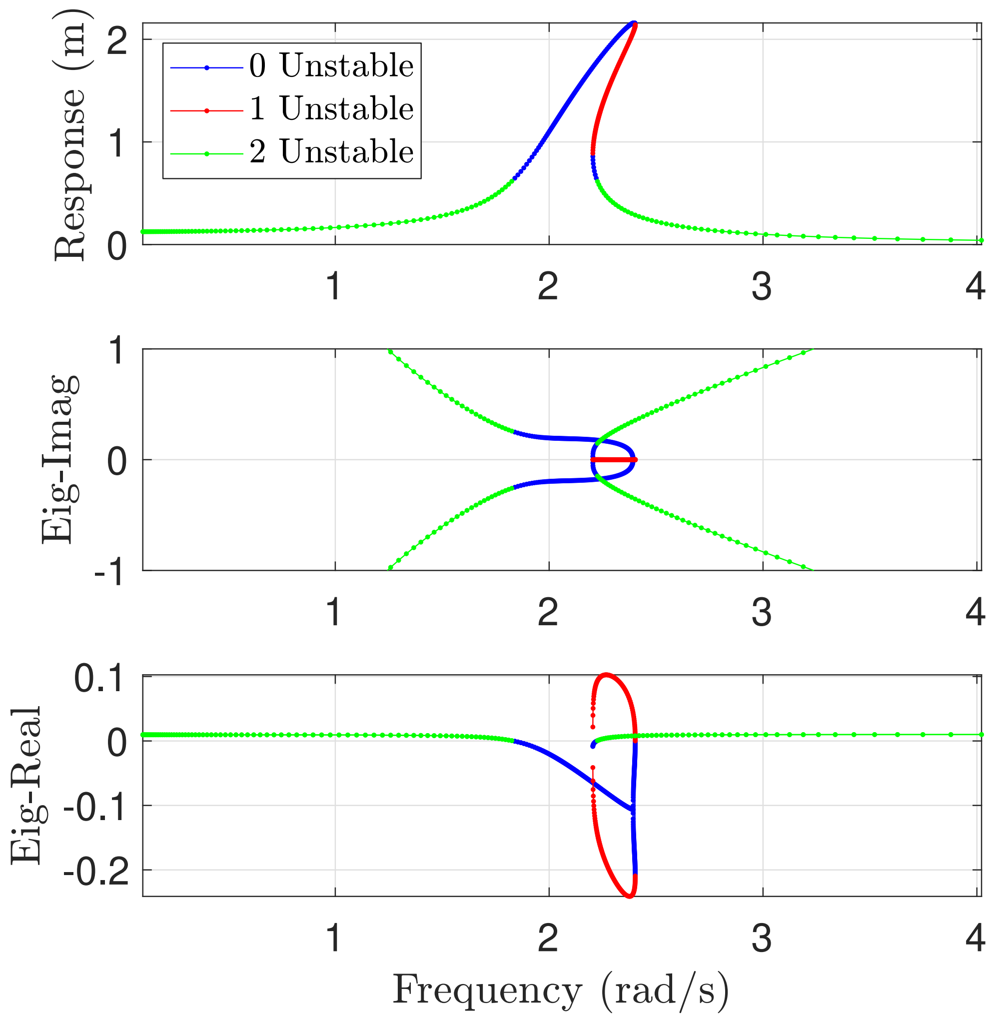
6. Some Unanswered Questions

6.1 Results from Multi-Harmonic Averaging

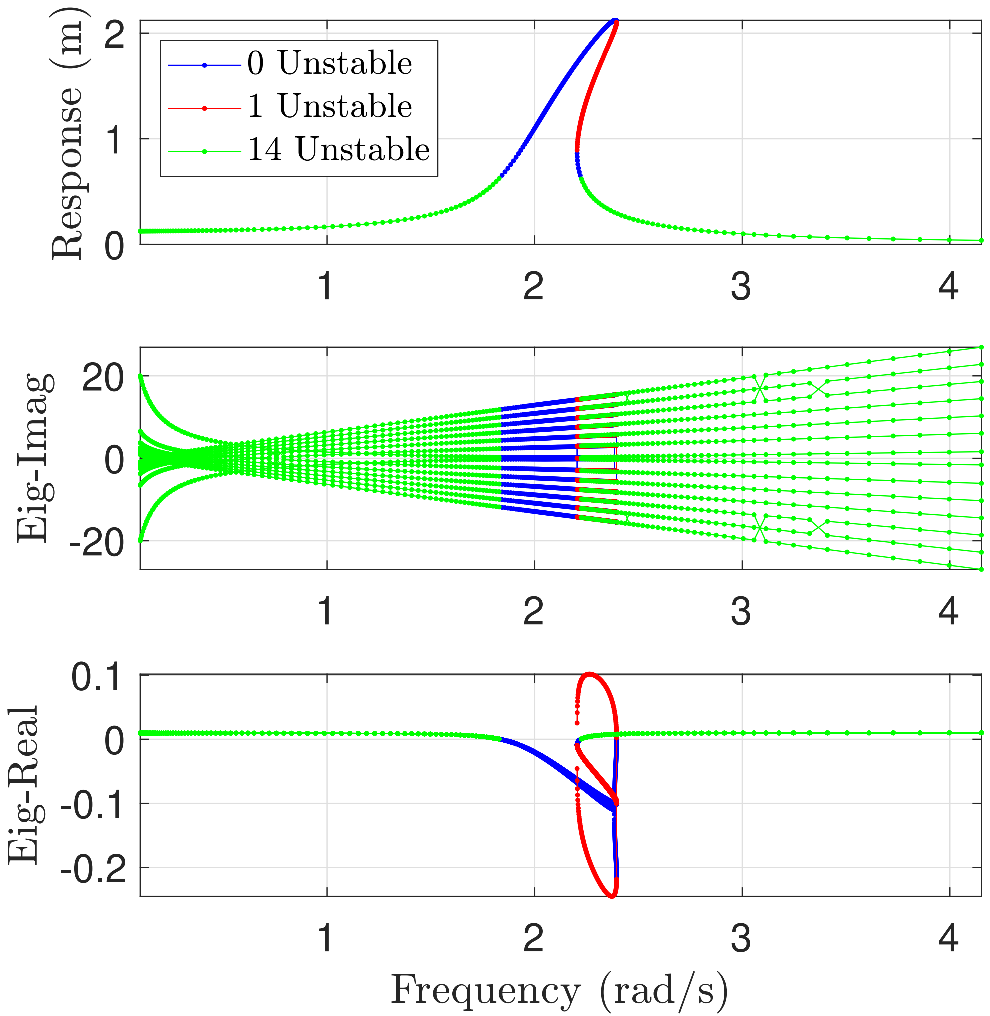


Single harmonic averaging

6.1 Results from Multi-Harmonic Averaging

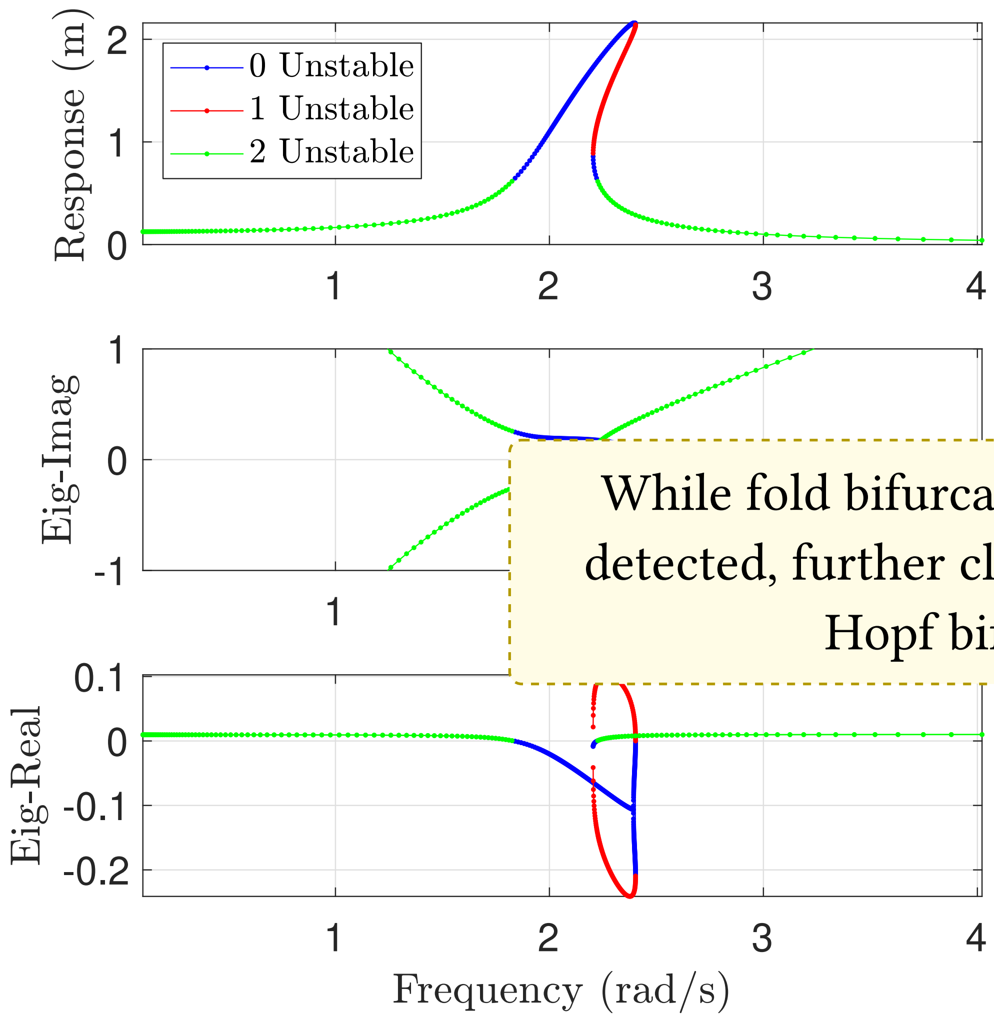


Single harmonic averaging

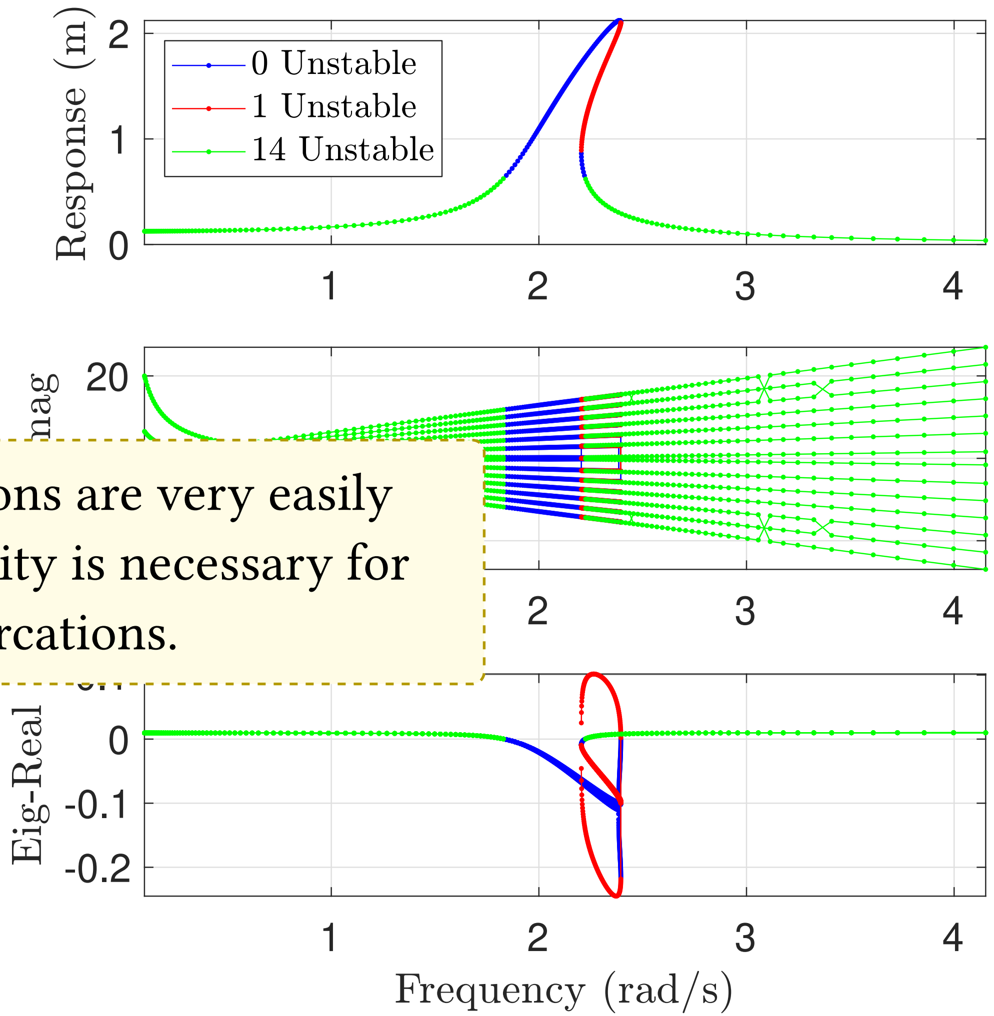


Seven harmonic averaging

6.1 Results from Multi-Harmonic Averaging



Single harmonic averaging



Seven harmonic averaging

While fold bifurcations are very easily detected, further clarity is necessary for Hopf bifurcations.

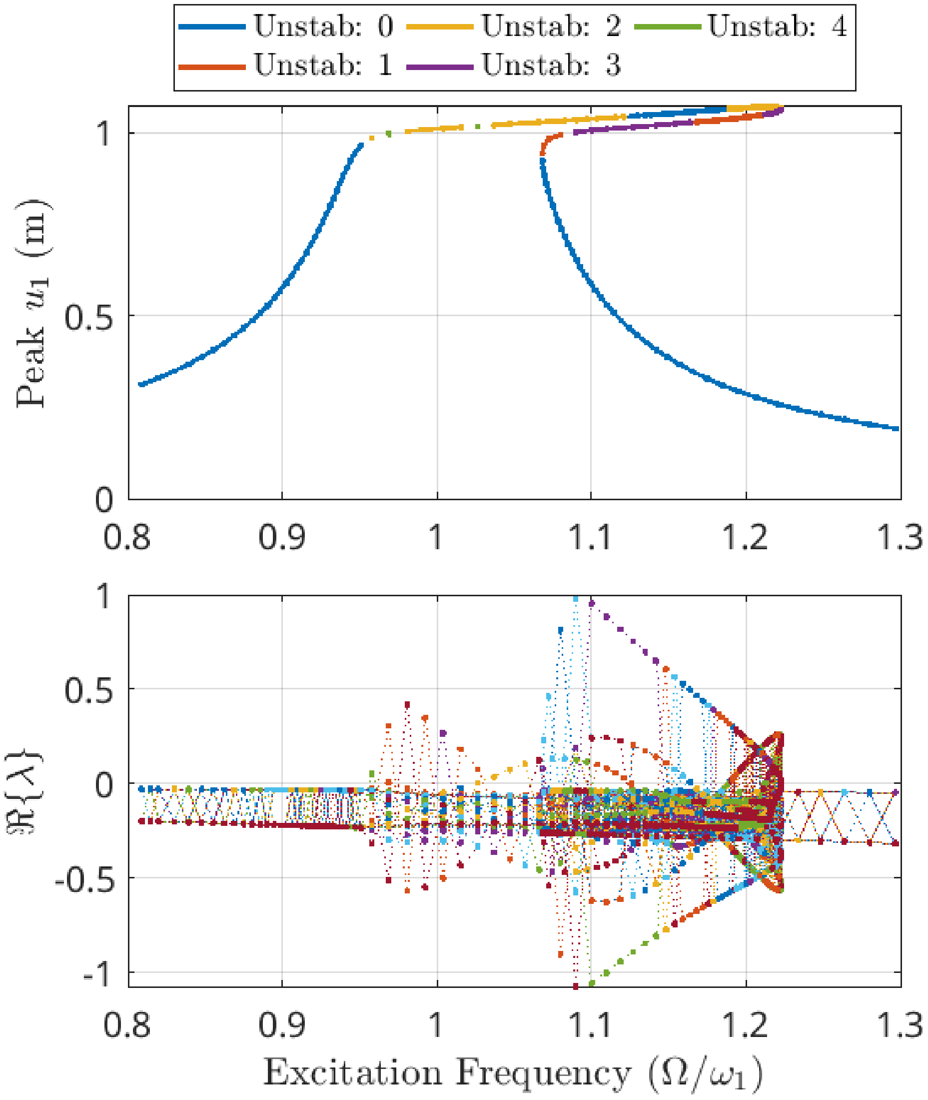
6.2 Highly Non-Smooth Problems

- Most stability classification approaches struggle even for very simple problems.
- The 2DoF example from (Woiwode & Krack, 2023):

$$\begin{bmatrix}\ddot{u}_1\\ \ddot{u}_2\end{bmatrix} + 0.03\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\end{bmatrix} + \begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + \begin{bmatrix}f_{\text{nl}}(u_1)\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0.1\end{bmatrix}\cos\Omega t$$

$$\text{with}\quad f_{\text{nl}}(u) = 100\frac{u-1}{2} + \sqrt{\left(100\frac{u-1}{2}\right)^2 + 0.2}$$

A single harmonic ansatz is clearly untenable in this case.



7. Outlook

7. Outlook

- The Fourier averaging approach provides a convenient framework for stability and bifurcation analysis .
- A single harmonic truncation, where appropriate, may be rigorously justified through the method of averaging .
  - For smooth problems this provides an **equivalent alternative to classical Floquet analysis**-
- The averaging formalism enables efficient characterization of periodic solution stability for non-smooth problems.

Theoretical justification for the multi-harmonic case needs work .

Bibliography

Bibliography

Hartung, A., Hackenberg, H.-P., & Retze, U. (2017). More Flexible Damping Systems for Blades and Vanes. *Technische Mechanik - European Journal of Engineering Mechanics*, *37*(2–5), 258–267. https://doi.org/10.24352/UB.OVGU-2017-102

Khalil, H. K. (2002). *Nonlinear Systems*. Prentice Hall.

Leine, R. (2006). *Bifurcation and Chaos in Non-smooth Mechanical Systems*. 61.

Manevitch, L. I. (1999). Complex Representation of Dynamics of Coupled Nonlinear Oscillators. In L. A. Uvarova, A. E. Arinstein, & A. V. Latyshev (Eds.), *Mathematical Models of Non-Linear Excitations, Transfer, Dynamics, and Control in Condensed Systems and Other Media: Mathematical Models of Non-Linear Excitations, Transfer, Dynamics, and Control in Condensed Systems and Other Media* (pp. 269–300). Springer US. https://doi.org/10.1007/978-1-4615-4799-0_24

NASA Dryden Flight Research Center. (2001, April). *Aerostructures Wing Test NASA Flight Test Video*. https://www.youtube.com/watch?v=TY3C36z3qao

Bibliography

NASA Langley Research Center. (2001,). https://www.instagram.com/cessnateur/reel/DGYGzsxOv0u/

Von Groll, G., & Ewins, D. (2001). The Harmonic Balance Method with Arc-Length Continuation in Rotor/Stator Contact Problems. *Journal of Sound and Vibration*, *241*(2), 223–233. https://doi.org/10.1006/jsvi.2000.3298

Woiwode, L., & Krack, M. (2023). Are Chebyshev-based Stability Analysis and Urabe's Error Bound Useful Features for Harmonic Balance?. *Mechanical Systems and Signal Processing*, *194*, 110265. https://doi.org/10.1016/j.ymssp.2023.110265

Zha, G. (2026,). *NSV*. https://acfdlab.miami.edu/projects/turbo-NSV.html