#18778: Response and Stability analysis of Self-Excited Systems with Non-Smooth Frictional Elements: A Fully Frequency-Domain Approach

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IMAC XLIII, Orlando, FL - February 10-13, 2025

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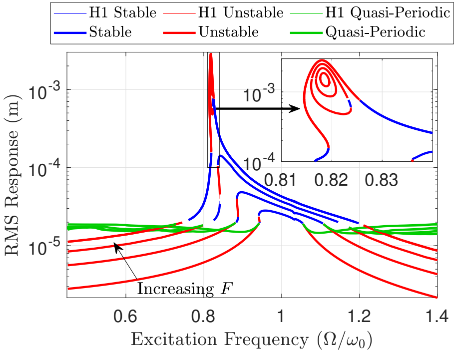
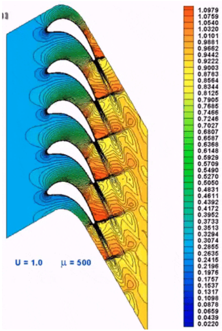


1. Introduction

- **Self-excited oscillators**: Dynamical systems with a negative source of damping

$$\ddot{x} - c\,\dot{x} + \omega_n^2 x = f_{ex}(t)$$

- **Aeroelastic flutter** is a commonly encountered example; negative damping influence from aerodynamic interations



FSIPRO2D Multible Turbine Blade Flutter (2 Way Coupled) - YouTube (n.d.)

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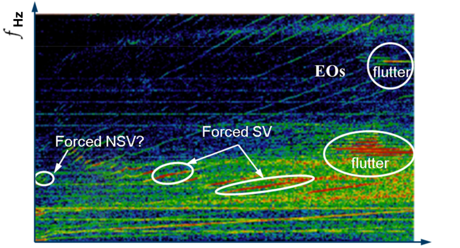
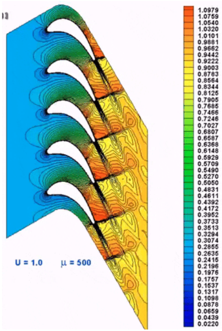


Hartung, Hackenberg, and Retze (2017)



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- Vibrations often saturated by frictional joints

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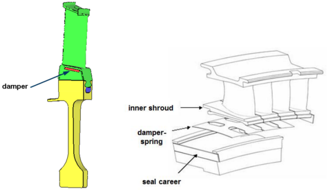

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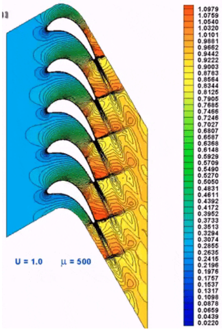

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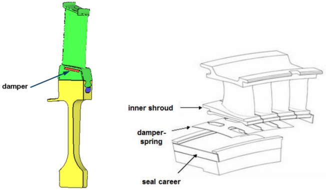

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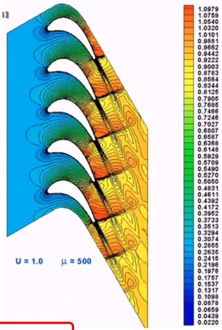
Periodic excitation often present



...2D Multible Turbine Blade ...(2 Way Coupled) - YouTube (n.d.)

1.1. Introduction

Problem Setting

We investigate the near-resonance forced self-excited oscillations of systems with frictional supports.

- Employing **The elastic dry-friction element**

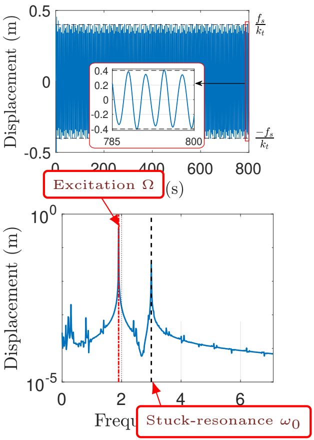
$$\ddot{x} - 2\zeta\omega_n\dot{x} + \omega_n^2 x + f_{nl}(x) = \frac{F}{2}e^{-j\Omega t} + \text{c.c.}$$

$$f_{nl}(t_{\ell+1}) - f_{nl}(t_\ell) = \begin{cases} k_t(x(t_{\ell+1}) - x(t_\ell)) + f_{nl}(t_\ell) & \text{stick} \\ \text{sgn}(f_{sp}(t_\ell))f_{sl} & \text{slip} \end{cases}.$$

1.1. Problem Setting

Introduction

The Synchronization Phenomenon (Lockin-Lockoff)



(a) $\Omega < \omega_0$

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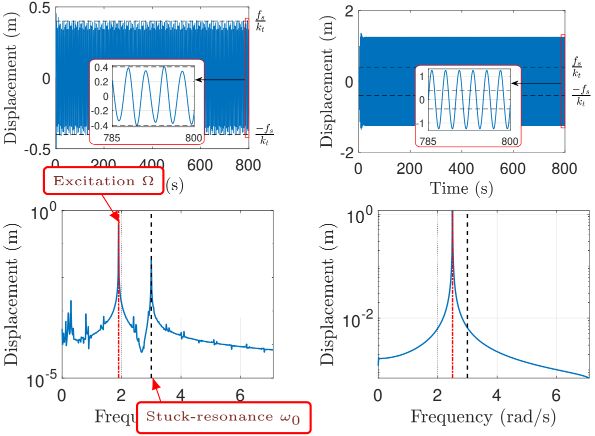

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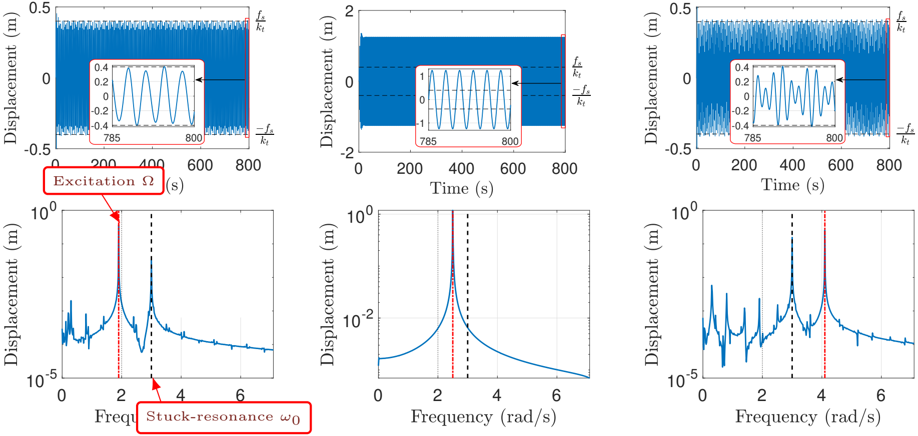

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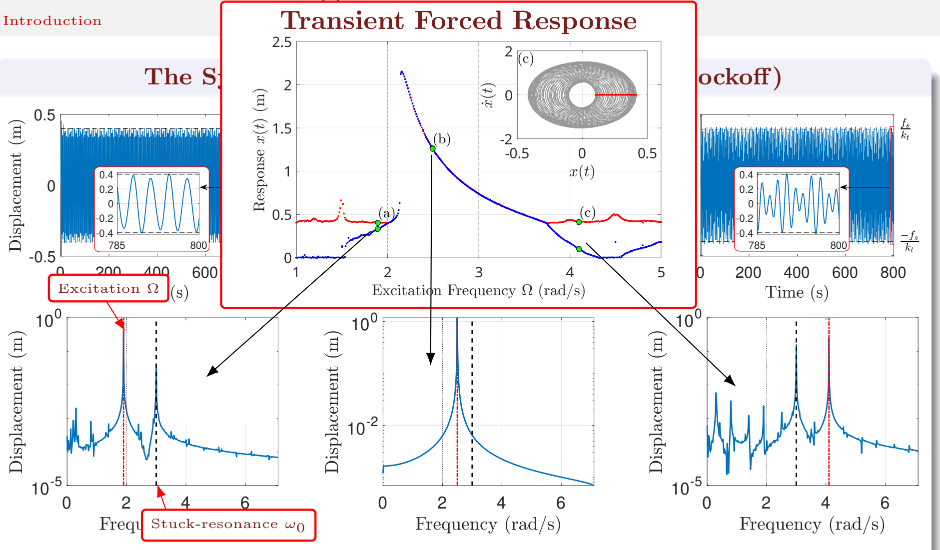

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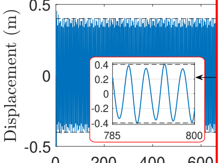
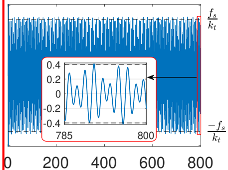
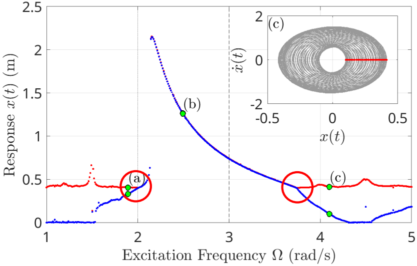
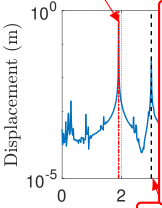




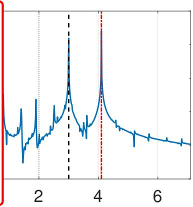
Transient Forced Response



Away from resonance, the periodic solution loses stability to give rise to quasi-periodic solutions: **Neimark-Sacker Bifurcations**



Frequency (rad/s)



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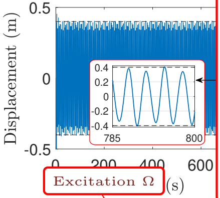

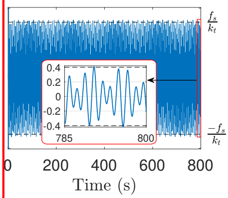

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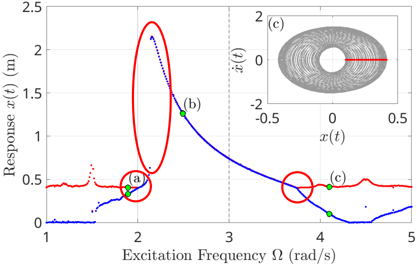
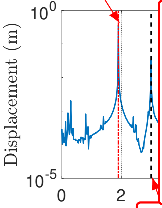


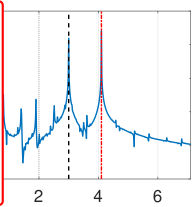
Excitation Ω



Stuck-resonance ω_0

Away from resonance, the periodic solution loses stability to give rise to quasi-periodic solutions: **Neimark-Sacker Bifurcations**

- There is also a "transient drop-off" region close to resonance, indicating a **Fold Bifurcation**.



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2. Stability of Periodic Solutions

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The Classical Approach

Stability $\rightarrow$ Perturbation Behavior $\rightarrow$ Linearization Methods

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The Classical Approach

Stability → Perturbation Behavior → Linearization Methods

- Linearizing about a periodic solution leads to a **parametrically excited system**

$$\delta\ddot{x} + c(t)\delta\dot{x} + k(t)\delta x = 0 \rightarrow \boxed{\underline{\dot{X}} = \underline{\underline{A}}(t)\underline{X}}. \tag{1}$$

The behavior of the solutions to this system are governed by **Floquet Theorem**.

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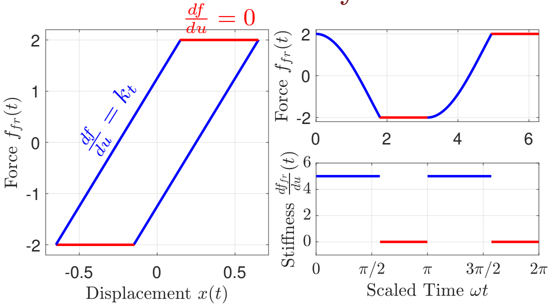
Floquet Theorem

Let $\underline{\underline{A}}(t)$ be a T-periodic continuous matrix function and denote by $\underline{\underline{\Phi}}$ a fundamental matrix solution of eq. (1). Then ..., there exists a real constant matrix $\underline{\underline{R}}$ and a real nonsingular, $2T$-periodic, $\mathcal{C}^1$ matrix function $\underline{\underline{Q}}(t)$ such that

$$\underline{\underline{\Phi}}(t) = \underline{\underline{Q}}(t)e^{\underline{\underline{R}}t}.$$

2. Stability of Periodic Solutions

Derivatives of the Elastic Dry-Friction Element



- L … excited s…

(1)

T … quet T…

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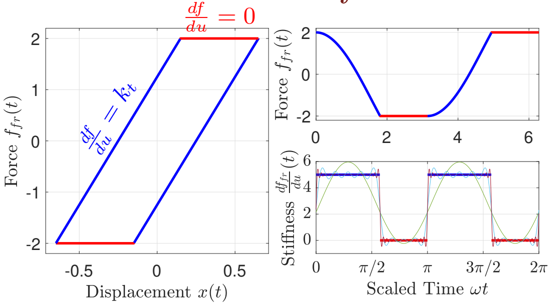

- Continuous derivatives exist only in a **weak sense**.

- L... **excited s...**

(1)

T... **...quet T...**

Let $\underline{\underline{A}}(t)$ be a T-periodic continuous matrix functi...

$v \in L^2 \quad \text{s.t.} \quad \oint g \frac{df}{du} = \oint g v.$

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2.1. The Non-Smooth Case

Stability of Periodic Solutions: An Averaging Approach

The perturbation of a frictional system is a non-smooth parametrically-excited oscillator, where **Floquet theorem does not hold**.

- Strictly speaking, since the Jacobian/linearized system only exists in a **weak sense**, we seek to handle the system in a weak form: **The Method of (Complexification) Averaging**[1]
- Under **CXA**, the response is written using

$$\underline{x}(t) := \underline{u}(t) = \boxed{\underline{\hat{q}}(t)} e^{-\mathrm{i}\Omega t} + \text{c.c.},$$

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$$\text{governed by } \boxed{\mathrm{i}2\Omega\underline{\underline{M}}\,\dot{\hat{\underline{q}}} = \underline{\underline{E}}\,\hat{\underline{q}} + \hat{\underline{f}}_{nl} - \hat{\underline{f}}}.$$

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- The differential equation governing $\hat{\underline{q}}(t)$ is **piece-wise continuous**: **Continuously differentiable *almost everywhere*.**

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2.2. A New Averaging Based Stability Certificate

Stability of Periodic Solutions

The Averaged System

$$\mathrm{i}2\Omega \underline{\underline{M}}\,\dot{\underline{\hat{q}}} = \underline{\underline{E}}\,\underline{\hat{q}} + \underline{\hat{f}}_{nl} - \underline{\hat{f}}$$

- Periodic solutions of the original system are **fixed points of the averaged system**.

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- **Lyapunov's Indirect Method**: Linearized stability analysis is applicable for piecewise continuous systems.

Lyapunov's Indirect Method (local asymptotic stability)

Let $\underline{x} = 0$ be an equilibrium point for the nonlinear system $\dot{\underline{x}} = \underline{f}(\underline{x})$ where $\underline{f} : \mathcal{D} \to \mathbb{R}^n$ is continuously differentiable and $\mathcal{D} \subset \mathbb{R}^n$ is a neighborhood of the origin. Let

$$\underline{\underline{A}} = \frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x})\Big|_{\underline{x}=\underline{0}}.\text{Then,}$$

1. The origin is asymptotically stable if $\lambda_i < 0$ for all eigenvalues of $\underline{\underline{A}}$.
2. The origin is unstable if $\lambda_i > 0$ for one or more of the eigenvalues of $\underline{\underline{A}}$.

2.2. Properties of the Averaged System

Stability of Periodic Solutions

The Averaged System

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Linearized Evolution of $\delta\hat{\underline{q}}(t)$

$$\mathrm{i}2\Omega \underline{\underline{M}}\,\delta\dot{\hat{\underline{q}}} = [\underline{\underline{E}} + \underline{\underline{J}}_{nl}]\delta\hat{\underline{q}}.$$

1. The RHS is the same the single harmonic HB residue:
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2. The linearized system yields exactly $2d$ eigenpairs for a d-DoF model:
   The Eigenvalues are the Floquet exponents.
   - No need for filtering!

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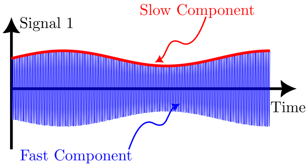
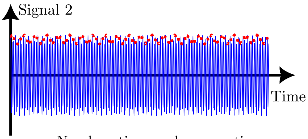
2. The linearized system yields exactly $2d$ eigenpairs for a d-DoF model: **The Eigenvalues are the Floquet exponents.**
   - No need for filtering!
3. The averaged system represents a slow-fast decomposition of the dynamics.

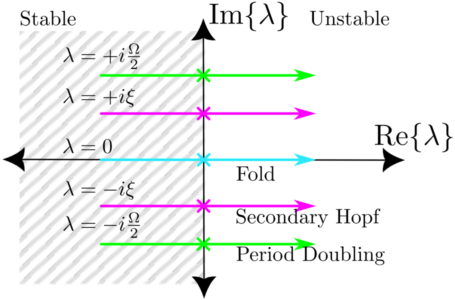




No clear time-scale separation

2.3. Bifurcation Treatment

Stability of Periodic Solutions



Fold Bifurcations Exponential blow-up; Always applicable ✓.

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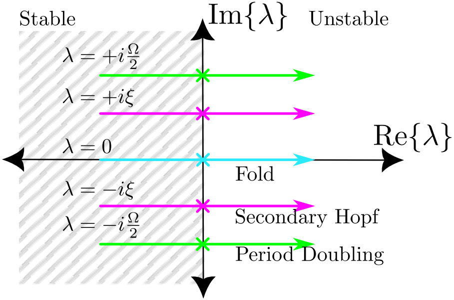

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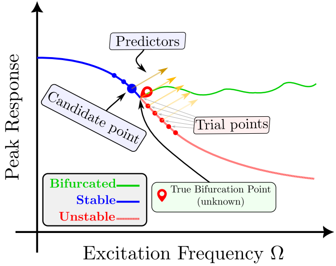
Hopf Bifurcations Conditionally applicable; $\Omega_1 \sim \Omega_2$ necessary.

2.3. Bifurcation Treatment

Stability of Periodic Solutions

Post-Bifurcation Analysis

- The eigenvectors associated with the instability can be used for branch-switching.



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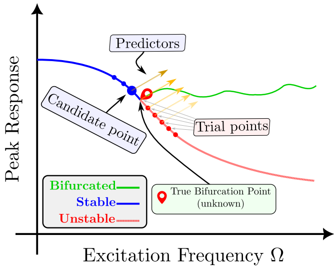
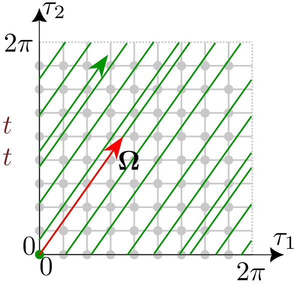


$$\tau_1 = \Omega_1 t$$
$$\tau_2 = \Omega_2 t$$



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- Note that the bifurcated branch is **quasi-periodic**, requiring special marching methods[a].

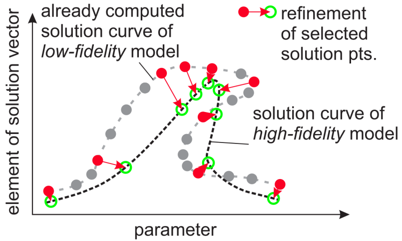
[a] Balaji, N. N., Gross, J., and Krack, M. "Harmonic Balance for Quasi-Periodic Vibrations under Nonlinear Hysteresis". **(2024)**.

3. Solution Refinement

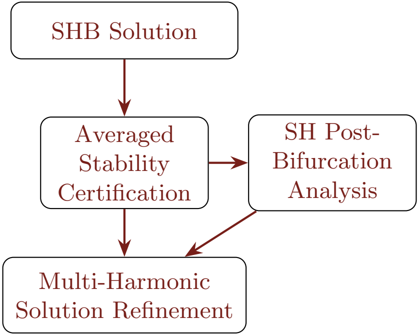
While the **single harmonic ansatz** has been critical for the averaged certification, this can be relaxed for the actual solution curve.

Parallelized Solution Refinement[a]

[a] Gross, J. et al. "A New Paradigm for Multi-Fidelity Continuation Using Parallel Model Refinement". **(2024)**.



Overall Procedure



4.1. SDoF Oscillator

Numerical Results

$$\boxed{\ddot{x} - c\dot{x} + kx + f_{nl}(x) = F\cos(\Omega t)}$$

$$c = 0.02\,\mathrm{Ns/m} \qquad k = 4\,\mathrm{N/m}$$
$$k_t = 5\,\mathrm{N/m} \qquad f_{sl} = 2\,\mathrm{N}$$
$$F \in [0.5\,\mathrm{N},\ 4\,\mathrm{N}] \qquad \Omega \in [1\,\mathrm{rad/s},\ 5\,\mathrm{rad/s}]$$

Energy Balance: $\frac{\Omega}{\pi}\oint (EOM)\,\dot{x}dt$

$$-c\oint \dot{x}^2 dt + \oint f_{nl}\dot{x}dt = F\oint \cos(\Omega t)\dot{x}dt$$
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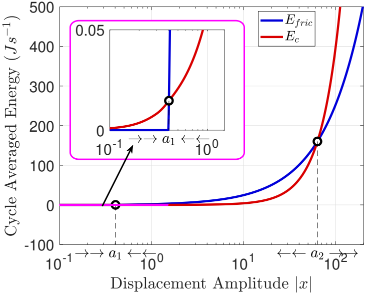

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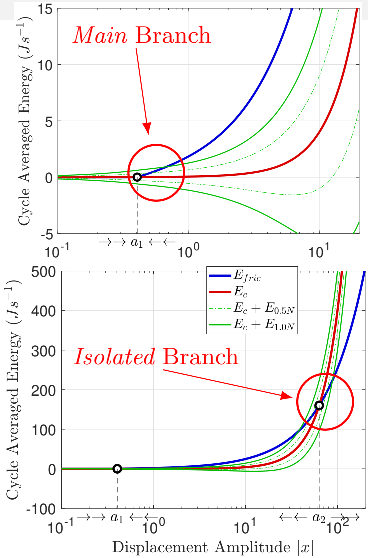
$$E_{fric} = \quad E_c \quad + \quad E_F$$

Influence of Excitation



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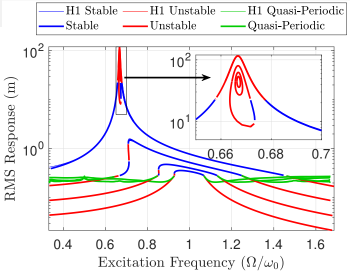
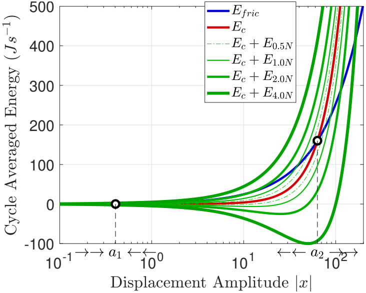
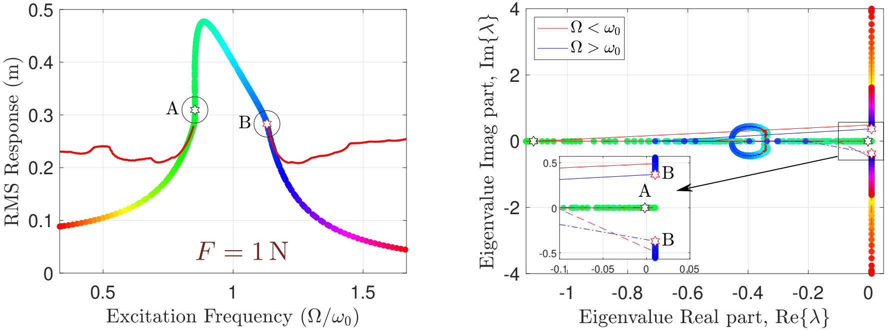
Forced Response





4.1. SDoF Oscillator: Forced Response with ASC

Numerical Results



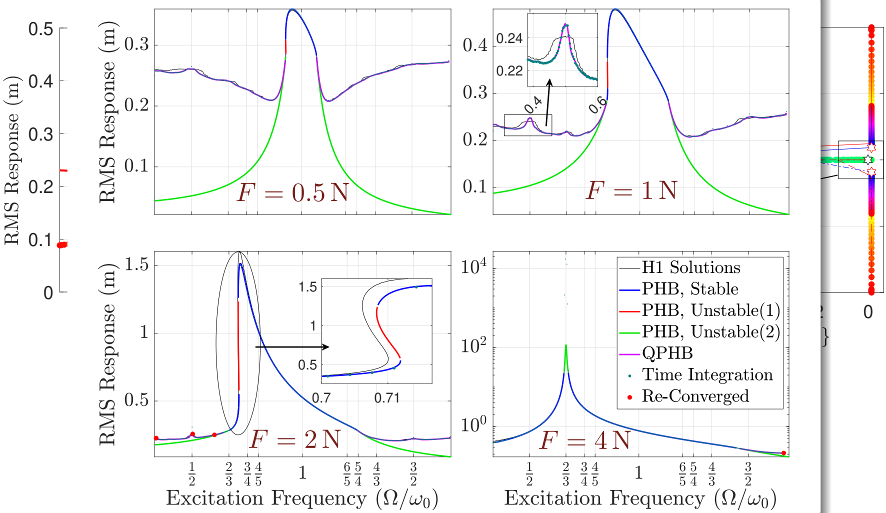
Point A: Fold Bifurcation

Point B: Neimark-Sacker Bifurcation

4.1. SDoF Oscillator: Forced Response with ASC

Numerical Results

Responses with Harmonic Refinement ($H = 9$)

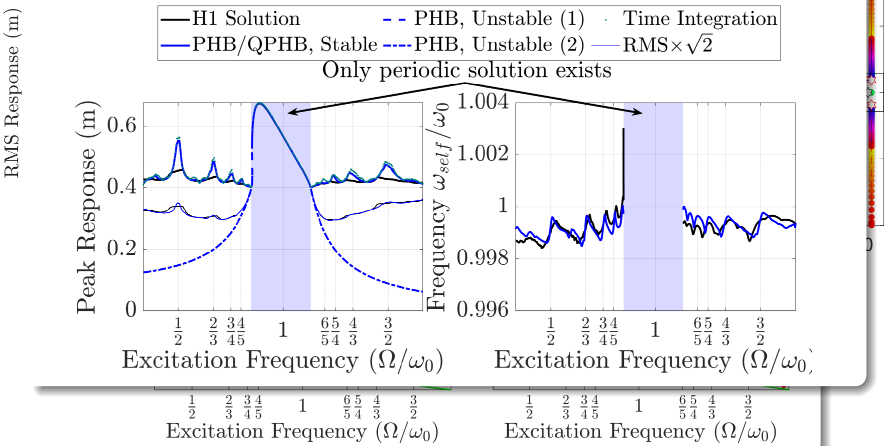


4.1. SDoF Oscillator: Forced Response with ASC

Numerical Results

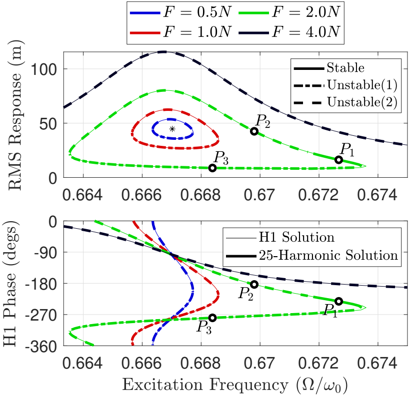
Responses with Harmonic Refinement ($H = 9$)

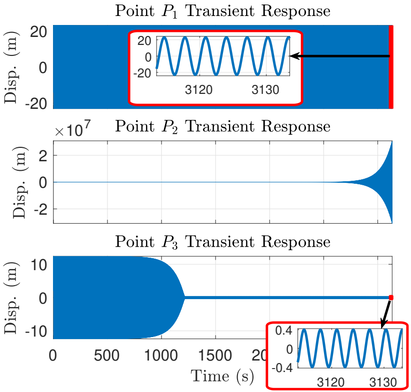
Secondary Resonance Features



4.1. SDoF Oscillator: Stability Verification

Numerical Results

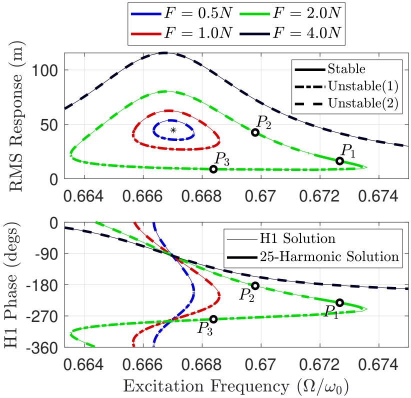


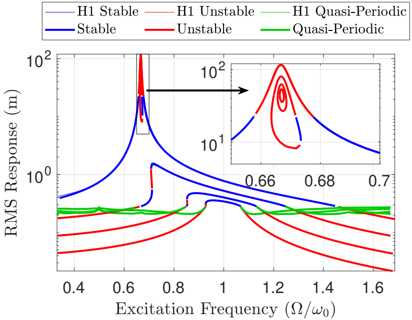


- Two types of instability encountered on the isolated branch
- *Small* stable region also detected

4.1. SDoF Oscillator: Stability Verification

Numerical Results





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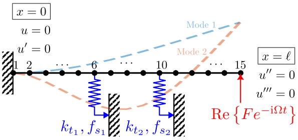
- *Small* stable region also detected

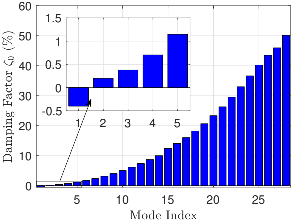
4.2. MDoF System

Numerical Results

A Fixed-Free beam with Frictional Supports



$$\ddot{u} + \mathcal{C}(\dot{u}) + \alpha^4 u'''' = 0, \quad x \in (0, \ell)$$



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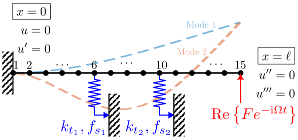

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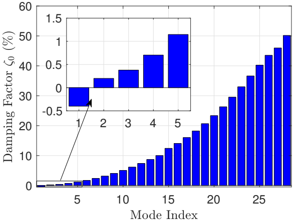
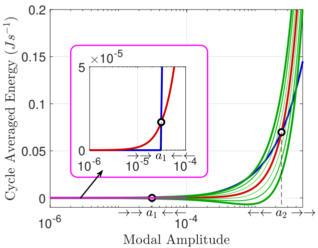
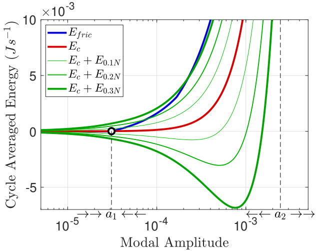


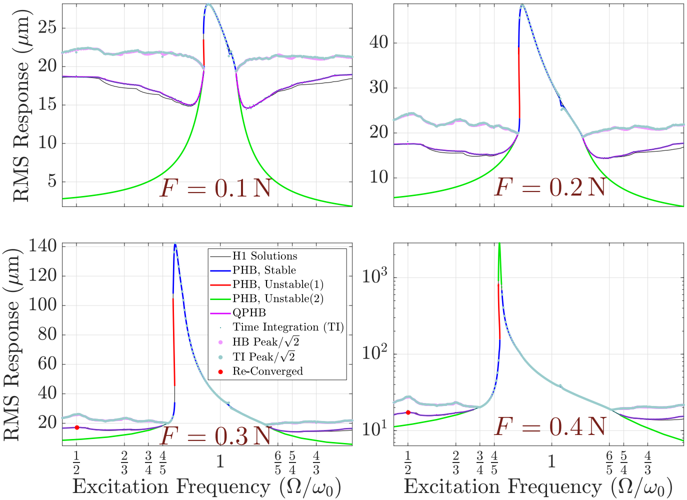
The (Near-Resonant) Energy Balance Diagram





4.2. MDoF System: Forced Response Results

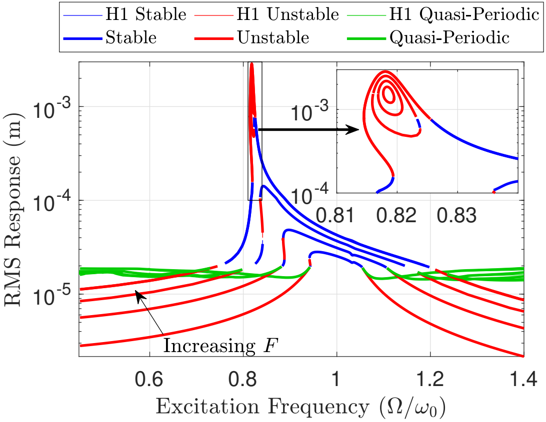
Numerical Results



4.2. MDoF System: Forced Response Results

Numerical Results

Complete Forced Response

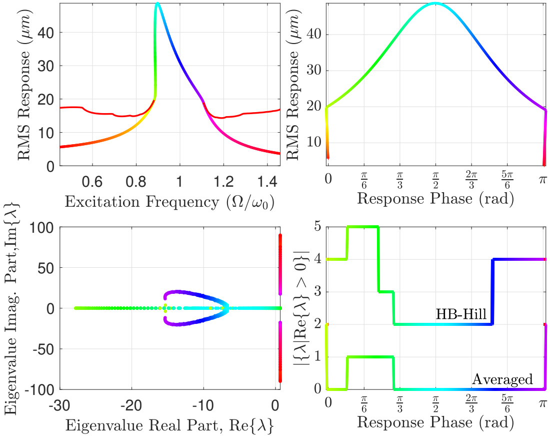


4.2. Stability Certification: Comparisons Against Frequency Domain Hill's Coefficients

MDoF System

- HB-Hill[a] generates $d(2H+1)$ eigenpairs. Sorting usually unreliable
- The averaging approach generates exactly $2d$ pairs and is reliable for the considered examples.

[a] Von Groll, G. and Ewins, D. "The Harmonic Balance Method with Arc-Length Continuation in Rotor/Stator Contact Problems". **(2001)**.



5. Conclusions and Future Work

- A novel **fully frequency-domain stability certification methodology** developed through averaging.
- The methodology is used to study the **friction-saturated forced responses of self-excited oscillators**.
- Reliability of the methodology is established through
  - Comparison with the current alternative;
  - Transient validation;
  - Exhaustive post-bifurcation analysis.
- Several salient features of forced self-excited dynamics have also been highlighted.

Avenues for Future Work

- The averaging methodology is fundamentally single-harmonic. Multi-harmonic generalizations?
- Further investigations into post-bifurcation behavior of friction-supported self-excited systems.

References I

[1] *FSIPRO2D Multible Turbine Blade Flutter (2 Way Coupled) - YouTube.* URL: https://www.youtube.com/ (visited on 12/17/2024) (cit. on pp. 3–6).

[2] A. Hartung, H.-P. Hackenberg, and U. Retze. "More Flexible Damping Systems for Blades and Vanes". Technische Mechanik - European Journal of Engineering Mechanics, **37**,2-5 (2017), pp. 258–267. ISSN: 2199-9244. DOI: 10.24352/UB.OVGU-2017-102. URL: https://journals.ub.uni-magdeburg.de/index.php/techmech/article/view/615 (visited on 02/01/2023) (cit. on pp. 3–6).

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[4] N. N. Balaji, J. Gross, and M. Krack. "Harmonic Balance for Quasi-Periodic Vibrations under Nonlinear Hysteresis". Journal of Sound and Vibration, **590**, (Nov. 2024), pp. 118570. ISSN: 0022-460X. DOI: 10.1016/j.jsv.2024.118570. URL: https://www.sciencedirect.com/science/article/pii/S0022460X2400333X (visited on 07/09/2024) (cit. on pp. 30–33, 62, 63).

[5] J. Gross et al. "A New Paradigm for Multi-Fidelity Continuation Using Parallel Model Refinement". Computer Methods in Applied Mechanics and Engineering, **423**, (Apr. 2024), pp. 116860. ISSN: 00457825. DOI: 10.1016/j.cma.2024.116860. URL: https://linkinghub.elsevier.com/retrieve/pii/S0045782524001166 (visited on 05/15/2024) (cit. on p. 34).

[6] G. Von Groll and D. Ewins. "The Harmonic Balance Method with Arc-Length Continuation in Rotor/Stator Contact Problems". Journal of Sound and Vibration, **241**,2 (Mar. 2001), pp. 223–233. ISSN: 0022460X. DOI: 10.1006/jsvi.2000.3298. URL: https://linkinghub.elsevier.com/retrieve/pii/S0022460X0093298X (visited on 12/17/2021) (cit. on p. 50).

[7] R. R. Leine. *Bifurcations in Discontinuous Mechanical Systems of the Fillippov-type.* 2000. URL: https://research.tue.nl/en/publications/bifurcations-in-discontinuous-mechanical-systems-of-the-fillippovtype(1b99d74b-4471-4363-a58f-c2450486fe18).html (visited on 09/21/2024) (cit. on pp. 54–56).

7. Backup Slides

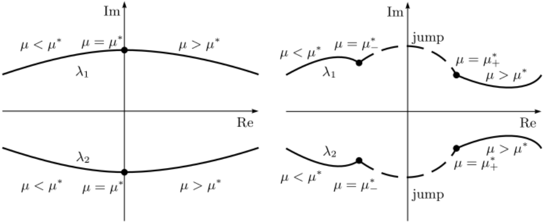
7.1. Non-Smooth Dynamical Systems

Backup Slides

- For analysis purposes, non-smooth systems have been generalized through **differential inclusions** and formalized in **Filippov Dynamical Systems**[2].

$$\underline{\dot{x}} = \underline{f}(\underline{x}) \quad \rightarrow \quad \underline{\dot{x}} \in \underline{F}(\underline{x})$$

  - Solution is continuous although the *system is set-valued.*
  - The fundamental solution matrix is expected to show discontinuous *jumps*, and representation through a Floquet normal form is not justified.
- Eigenvalues of the mapping matrix $\mathcal{M}$ relating perturbations across time-periods are referred to as **Floquet Multipliers**.



(a) continuous bifurcation (b) discontinuous bifurcation

[2] Leine, R. R. *Bifurcations in Discontinuous Mechanical Systems of the Fillippov-type.* **(2000)**.

7.1. Non-Smooth Dynamical Systems

Backup Slides

- Fo ... hrough
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  ... *jumps*,
- Ei ... s
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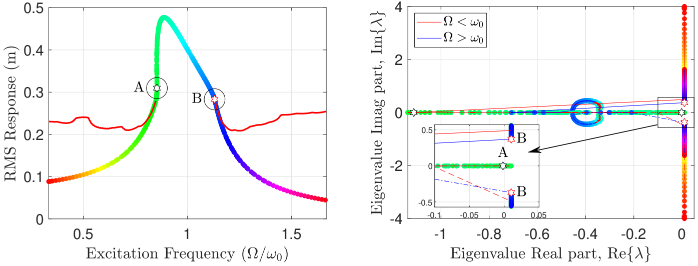
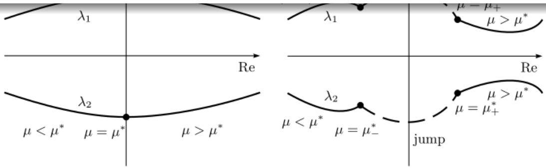
Frictional Oscillator





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Backup Slides

- Fo … hrough **di** … **Sy** … *jumps*,

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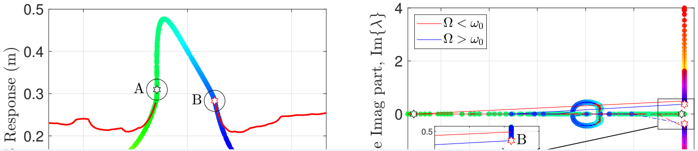
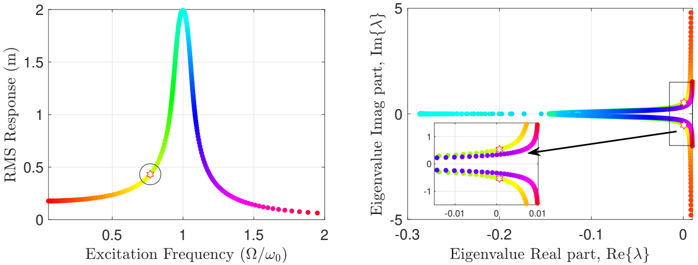


- Ei … s tir

Forced Van-der Pol Oscillator

$$\ddot{x} - c\dot{x} + kx + \mu x^2 \dot{x} = F \cos \Omega t$$



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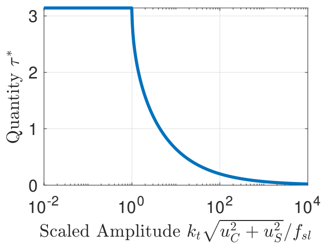
7.2. Continuity of the Fourier Coefficients For the Elastic Dry Friction Element

Backup Slides

Under harmonic displacement $u(t) = u_C \cos\tau + u_S \sin\tau$, the Fourier coefficients of the reaction force for $f_{fr}(t) = F_C \cos\tau + F_S \sin\tau$ are

$$\begin{bmatrix} F_C \\ F_S \end{bmatrix} = \frac{k_t}{2\pi} \begin{bmatrix} 2\tau^* - \sin 2\tau^* & 1 - \cos 2\tau^* \\ -(1-\cos\tau^*) & 2\tau^* - \sin 2\tau^* \end{bmatrix} \begin{bmatrix} u_C \\ u_S \end{bmatrix},$$

$$\text{with } \tau^* = \begin{cases} \cos^{-1}\left(1 - \frac{2f_s}{k_t\sqrt{u_C^2+u_S^2}}\right) & \sqrt{u_C^2+u_S^2} > \frac{f_{sl}}{k_t} \\ \pi & \text{otherwise} \end{cases}.$$



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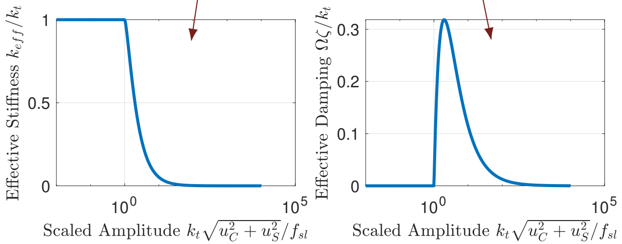
Effective Stiffness and Damping



Scaled Amplitude $k_t\sqrt{u_C^2 + u_S^2}/f_{sl}$

7.3. Consistency of Averaged Exponents and Floquet Exponents

Backup Slides

Displacement-Velocity Coordinates

$$\delta \underline{q}(t) = \Re\{\delta \hat{\underline{q}} e^{-\mathrm{i}\tau}\}$$

$$\delta \underline{v}(t) = \Re\{-\mathrm{i}\Omega \delta \hat{\underline{q}} e^{-\mathrm{i}\tau}\}$$

→

Eigen Perturbation

$$(\lambda, \underline{\phi})$$

$$\begin{bmatrix} \delta \underline{q}(t) \\ \delta \underline{v}(t) \end{bmatrix} = \eta e^{\lambda_{\Re} t} \left(\begin{bmatrix} \cos((\Omega - \lambda_{\Im})t) & \sin((\Omega - \lambda_{\Im})t) \\ -\Omega \sin((\Omega - \lambda_{\Im})t) & \Omega \cos((\Omega - \lambda_{\Im})t) \end{bmatrix} \otimes \underline{\underline{I}}_d \right) \begin{bmatrix} \underline{\phi}_{\Re} \\ \underline{\phi}_{\Im} \end{bmatrix}$$

↓

Monodromy Matrix, Multipliers

$$\underline{\underline{\mathcal{M}}} = e^{\lambda_{\Re} t} \left(\begin{bmatrix} \cos(\lambda_{\Im} T) & -\frac{1}{\Omega} \sin(\lambda_{\Im} T) \\ \Omega \sin(\lambda_{\Im} T) & \cos(\lambda_{\Im} T) \end{bmatrix} \otimes \underline{\underline{I_d}} \right)$$

$$\mu = e^{(\lambda_{\Re} \pm \lambda_{\Im})T}$$

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Eigen Perturbation

$$(\lambda, \underline{\phi})$$

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Monodromy Matrix, Multipliers

$$\underline{\underline{\mathcal{M}}} = e^{\lambda_{\Re} t} \left(\begin{bmatrix} \cos(\lambda_{\Im} T) & -\frac{1}{\Omega} \sin(\lambda_{\Im} T) \\ \Omega \sin(\lambda_{\Im} T) & \cos(\lambda_{\Im} T) \end{bmatrix} \otimes \underline{\underline{I}}_d \right)$$

$$\mu = e^{(\lambda_{\Re} \pm \lambda_{\Im})T}$$

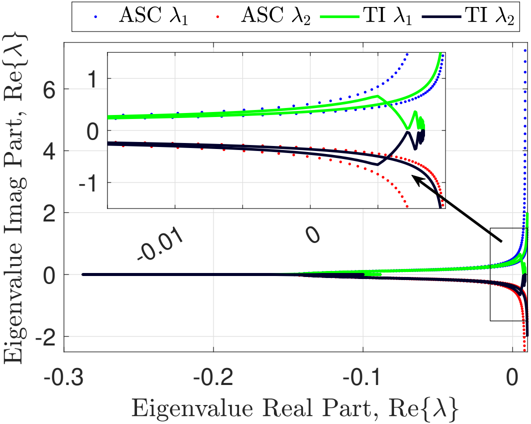
- No reference to Floquet normal form
- Mapping behavior represented *in average*.

7.3. Consistency of Averaged Exponents and Floquet Exponents

Backup Slides

Numerical Comparison for the Forced Van der Pol Oscillator

$$\ddot{x} - c\dot{x} + kx + \mu x^2 \dot{x} = F \cos \Omega t$$



Displ Ve Coo

$\delta \underline{q}(t) =$

$\delta \underline{v}(t) =$

- No refer Floquet
- Mappin represer

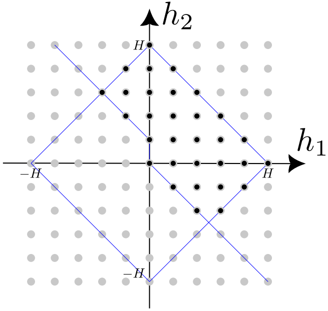
tipliers

7.4. Quasi-Periodic Numerics and Nonlinear Hysteresis

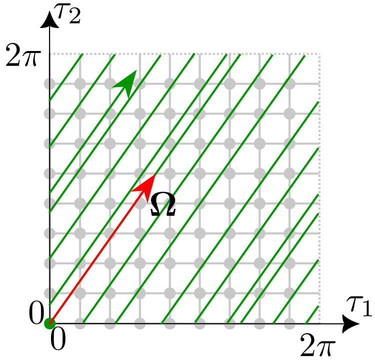
Backup Slides

- Time coordinates scaled as $\tau_1 = \Omega_1 t$, $\tau_2 = \Omega_2 t$, ...
- Fourier series written as

$$u(t) = \sum_{h \in \mathcal{H}} \hat{q}(\underline{h}) \exp(h_1\tau_1 + h_2\tau_2 + \dots).$$



- Physical time flows along the vector $\begin{bmatrix}\Omega_1 & \Omega_2\end{bmatrix}^T$ in torus space.
- Hysteretic marching must also be along this.



7.4. Quasi-Periodic Numerics and Nonlinear Hysteresis

Backup Slides

- Time coordinates scaled as $\tau_1 =$ …
- Fou…

$u(t)$…

- Physical time flows along the … space.
- … also be

Marching in 2-D[a]

[a] Balaji, N. N., Gross, J., and Krack, M. "Harmonic Balance for Quasi-Periodic Vibrations under Nonlinear Hysteresis". (2024).