Fokker–Planck Analysis of a Stochastic Hopf Oscillator: Steady States, Probability Currents, and Characteristic Curves

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Abstract:

Deterministic nonlinear oscillators provide a clear description of asymptotic behavior through trajectories in phase space; however, in realistic physical systems, intrinsic or external noise renders such descriptions incomplete. In the presence of stochastic forcing, individual trajectories lose predictive meaning, and the system must instead be characterized in terms of probability density functions (PDFs). This shift in perspective motivates the use of stochastic differential equations and their corresponding Fokker–Planck formulations to study oscillator dynamics.

In this work, we analyze stochastic oscillators by deriving the Fokker–Planck equation in polar coordinates, which naturally separates amplitude and phase dynamics. Stationary solutions are obtained by exploiting the periodicity of the angular variable and expressing the probability density as a Fourier series, leading to reduced equations for the radial components. These solutions reveal nontrivial steady-state distributions and associated probability currents in phase space. For the general, time-dependent case, the evolution of the probability density is studied using characterization curves, providing insight into the transport of probability beyond the stationary regime.

1 Introduction:

Nonlinear oscillators play a central role in many areas of physics, ranging from classical mechanics and electronics to biological and chemical systems. In deterministic settings, such systems are typically analyzed using phase-space trajectories, fixed points, and limit cycles, which provide a clear description of long-time behavior [5]. However, real physical systems are inevitably influenced by noise arising from thermal fluctuations, environmental coupling, or internal degrees of freedom. In such cases, a purely deterministic description becomes inadequate.

The Hopf oscillator serves as a canonical model for studying the onset of self-sustained oscillations through a bifurcation mechanism. While its deterministic dynamics are well understood, the inclusion of stochastic forcing leads to qualitatively new phenomena, such as fluctuations in amplitude and phase, noise-induced shifts in stability, and persistent probability currents. These effects cannot be fully captured by individual trajectories and instead require a statistical description in terms of probability density functions [4].

To address this, stochastic Hopf oscillators are naturally studied using the Fokker–Planck equation, which governs the time evolution of the probability density associated with the underlying stochastic dynamics; in this work, attention is restricted to the drift-dominated limit. Expressing the Fokker–Planck equation in polar coordinates allows for a clear separation between radial and angular dynamics, making it particularly suitable for oscillatory systems. This formulation provides direct access to steady-state distributions and probability currents in phase space, which are key indicators of nonequilibrium behavior.

2 Background Concepts:

2.1 Hopf Bifurcation:

A fundamental mechanism for the emergence of limit cycles is the Hopf bifurcation, where a stable fixed point loses stability as a control parameter is varied. Near the bifurcation, the dynamics can be reduced to a universal normal form[5]. In Cartesian coordinates, this describes the evolution of a complex amplitude, while in polar coordinates the dynamics naturally separates into amplitude and phase equations,

$$\dot{r} = \lambda r + \alpha r^3, \qquad \dot{\theta} = \omega + \beta r^2. \tag{1}$$

Here, λ controls the growth or decay of oscillations, with $\lambda > 0$ leading to a stable limit cycle when $\alpha < 0$. The parameter ω sets the base oscillation frequency, while β accounts for amplitude dependent frequency shifts. This form highlights how nonlinearities stabilize the oscillation amplitude while preserving phase dynamics.

2.2 Smooth and Non-smooth systems

Classical normal forms assume smooth vector fields and analyticity near the bifurcation point. However, many physical systems involve non-smooth effects such as switching, impacts, or threshold dynamics. In such cases, deterministic descriptions may be insufficient, particularly in the presence of noise, motivating a stochastic formulation that remains meaningful beyond idealized smooth systems.

2.3 Stochastic Dynamics

To account for fluctuations, deterministic equations are extended to stochastic differential equations (SDEs) by introducing noise [1]. These are commonly written in the Langevin form,

$$\dot{x}_i = A_i(\mathbf{x}) + \eta_i(t), \tag{2}$$

where A_i represents the deterministic drift and $\eta_i(t)$ is a stochastic force, typically modeled as Gaussian white noise derived from a Wiener process. While different interpretations of stochastic calculus exist—most notably Itô and Stratonovich—their differences are not essential for the present discussion and are treated briefly.

A key feature of SDEs is their Markov property: the future evolution of the system depends only on its present state, not on its history. This property allows the dynamics to be reformulated in terms of probability densities rather than individual trajectories.

3 Fokker–Planck Equation: Derivation and Interpretation

The evolution of stochastic dynamical systems can be described either at the level of individual trajectories or, more generally, in terms of probability density functions. For Markovian processes, the latter approach leads naturally to the Fokker–Planck equation. In this section, we briefly outline its derivation starting from the Chapman–Kolmogorov equation and the Kramers–Moyal expansion[4, 3].

3.1 Chapman–Kolmogorov Equation

Let $p(\mathbf{x}, t)$ be the probability of finding the system at state $\mathbf{x}$ at time t. For a Markov process, the transition probabilities satisfy the Chapman–Kolmogorov equation,

$$p(\mathbf{x}, t + \Delta t) = \int d\mathbf{x}'\, P(\mathbf{x}, t + \Delta t \mid \mathbf{x}', t)\, p(\mathbf{x}', t), \tag{3}$$

where $P(\mathbf{x}, t+\Delta t|\mathbf{x'}, t)$ is the conditional transition probability. This equation expresses the conservation of probability and the absence of memory beyond the present state.

3.2 Kramers–Moyal Expansion

To obtain a differential equation for $p(\mathbf{x}, t)$, the transition probability is expanded for small Δt. Introducing the increment $\xi = \mathbf{x} - \mathbf{x'}$, the Chapman–Kolmogorov equation can be expanded in moments of ξ, leading to the Kramers–Moyal expansion,

$$\partial_t p(\mathbf{x}, t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1 \cdots i_n} \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} \left[M^{(n)}_{i_1 \cdots i_n}(\mathbf{x})\, p(\mathbf{x}, t) \right], \tag{4}$$

where the Kramers–Moyal coefficients are defined as

$$M^{(n)}_{i_1 \cdots i_n}(\mathbf{x}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle \xi_{i_1} \cdots \xi_{i_n} \rangle\,. \tag{5}$$

These coefficients encode the statistical properties of the stochastic increments and determine the structure of the evolution equation.

3.3 Liouville Equation (Deterministic Limit)

In the absence of noise, only the first-order moment survives, and higher-order terms vanish. Retaining only the first term of the expansion yields the Liouville equation,

$$\partial_t p(\mathbf{x}, t) = -\nabla \cdot (\mathbf{A}(\mathbf{x})\, p(\mathbf{x}, t))\,, \tag{6}$$

where $A(\mathbf{x})$ is the deterministic drift field. This equation describes the transport of probability along deterministic trajectories and reflects strict conservation of probability in phase space.

3.4 Classical Fokker-Planck Equation

For stochastic systems driven by Gaussian noise, the Kramers–Moyal expansion truncates at second order, in accordance with Pawula's theorem. Retaining terms up to $n = 2$ leads to the Fokker–Planck equation,

$$\partial_t p(\mathbf{x}, t) = -\nabla \cdot (\mathbf{A}(\mathbf{x})\, p(\mathbf{x}, t)) + \nabla\nabla : (\mathbf{D}(\mathbf{x})\, p(\mathbf{x}, t))\,, \tag{7}$$

where D is the diffusion tensor associated with the second-order moments of the noise. The first term represents deterministic drift, while the second accounts for the spreading of probability due to stochastic fluctuations.

The Fokker–Planck equation can be written in the form of a continuity equation,

$$\partial_t p + \nabla \cdot \mathbf{J} = 0, \tag{8}$$

with probability current

$$\mathbf{J} = \mathbf{A}p - \nabla \cdot (\mathbf{D}p). \tag{9}$$

This formulation emphasizes probability conservation and provides a natural framework for studying steady states, probability currents, and non-equilibrium behavior in stochastic oscillatory systems.

4 Transformation into polar coordinates:

Consider the Liouville/drift part of Fokker-Planck equation in Cartesian coordinates,[4]

$$\frac{\partial p}{\partial t} = -\left[\frac{\partial}{\partial x}(A_x p) + \frac{\partial}{\partial y}(A_y p)\right] = -\nabla \cdot (\mathbf{A}p), \tag{10}$$

where,

$$\mathbf{A} = (A_x, A_y). \tag{11}$$

In polar coordinates (r, θ), the gradient operator is

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta}. \tag{12}$$

The drift vector is decomposed as

$$\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}}, \tag{13}$$

The polar unit vectors are related to Cartesian ones by

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}, \tag{14}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}. \tag{15}$$

Their derivatives satisfy

$$\frac{\partial \hat{\mathbf{r}}}{\partial r} = 0, \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial r} = 0, \tag{16}$$

$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}, \qquad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}}. \tag{17}$$

Using these relations, the divergence term becomes

$$\nabla \cdot (\mathbf{A}p) = \frac{\partial}{\partial r}(A_r p) + \frac{1}{r}A_r p + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta p). \tag{18}$$

Hence, the evolution equation in polar coordinates is,

$$\frac{\partial p}{\partial t} = -\left[\frac{\partial}{\partial r}(A_r p) + \frac{1}{r}A_r p + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta p)\right]. \tag{19}$$

$$\boxed{\frac{\partial p}{\partial t} = -\frac{1}{r}\frac{\partial (rA_r p)}{\partial r} - \frac{1}{r}\frac{\partial A_\theta p}{\partial \theta}} \tag{20}$$

5 Fourier Reduction in Polar Coordinates:

Having transformed the Fokker–Planck equation into polar coordinates, we now exploit the periodicity of the angular variable θ to reduce the problem using a Fourier expansion. This method allows a systematic separation of angular and radial dynamics and plays a central role in analyzing stationary and near-stationary solutions.

5.1 Angular Fourier Expansion

Since the probability density $p(r, \theta, t)$ is 2π periodicity in θ, it admits a Fourier Series representation,

$$p(r, \theta, t) = a_0(r, t) + \sum_{m=1}^{\infty} \left[a_m(r, t)\, e^{im\theta} + a_m^*(r, t)\, e^{-im\theta} \right]. \tag{21}$$

To obtain a reduced description, we retain only the lowest nontrivial angular modes,

$$p(r, \theta, t) = a_0(r, t) + a_1(r, t)\, e^{i\theta} + a_1^*(r, t)\, e^{-i\theta}. \tag{22}$$

From the continuity form of the Fokker–Planck equation,

$$\partial_t p = -\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r p) + \frac{\partial}{\partial \theta}(A_\theta p) \right], \tag{23}$$

we define the residual operator

$$R(r, \theta, t) \equiv \partial_t p + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r p) + \frac{\partial}{\partial \theta}(A_\theta p) \right]. \tag{24}$$

For the Hopf oscillator, the deterministic drift components are

$$A_r(r) = \lambda r + \alpha r^3, \tag{25}$$
$$A_\theta(r) = \omega + \beta r^2. \tag{26}$$

Importantly, both drift terms are independent of θ, which greatly simplifies the angular projections.

5.2 Zeroth Fourier Mode

To extract the equation governing the radial density $a_0(r, t)$, we project R onto the zeroth Fourier mode,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} R(r, \theta, t)\, d\theta = 0. \tag{27}$$

Using orthogonality of the Fourier modes,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im\theta}\,d\theta = \delta_{m0}, \tag{28}$$

all angular derivative terms vanish upon integration. This yields

$$\partial_t a_0 + \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r a_0\right) = 0. \tag{29}$$

Substituting the Hopf drift $A_r = \lambda r + \alpha r^3$,

$$\boxed{\partial_t a_0 = -\frac{1}{r}\frac{\partial}{\partial r}\left[r(\lambda r + \alpha r^3)\,a_0\right].} \tag{30}$$

This equation governs the evolution of the radial probability density.

5.3 First Fourier Mode

To obtain the equation for the first angular mode, we project onto $e^{i\theta}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} R(r,\theta,t)\,e^{-i\theta}\,d\theta = 0. \tag{31}$$

Evaluating each term yields

$$\partial_t a_1 + \frac{i}{r}A_\theta a_1 + \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r a_1\right) = 0. \tag{32}$$

Substituting the Hopf normal form,

$$\boxed{\partial_t a_1 = -\frac{i}{r}(\omega + \beta r^2)a_1 - \frac{1}{r}\frac{\partial}{\partial r}\left[r(\lambda r + \alpha r^3)\,a_1\right].} \tag{33}$$

6 Steady-State Probability Density of the Hopf Oscillator

6.1 Zeroth Fourier Mode

In the steady state, $\partial_t a_0 = 0$, and the equation reduces to

$$\frac{1}{r}\frac{d}{dr}\left[r(\lambda r + \alpha r^3)a_0(r)\right] = 0. \tag{34}$$

Integrating once with respect to r yields

$$r(\lambda r + \alpha r^3)a_0(r) = C_0, \tag{35}$$

where C_0 is a constant of integration. Solving for $a_0(r)$, we obtain

$$\boxed{a_0(r) = \frac{C_0}{r^2|\lambda + \alpha r^2|}.} \tag{36}$$

This expression represents the radial probability density in the steady state and exhibits a divergence at the deterministic limit-cycle radius $r_0 = \sqrt{-\lambda/\alpha}$, as expected for the deterministic Fokker–Planck equation.

6.2 First Fourier Mode

In the steady state, $\partial_t a_1 = 0$, giving

$$\frac{1}{r}\frac{d}{dr}\left[r(\lambda r + \alpha r^3)a_1(r)\right] + \frac{i}{r}(\omega + \beta r^2)a_1(r) = 0. \tag{37}$$

Multiplying through by r and expanding the derivative leads to

$$(\lambda r^2 + \alpha r^4)\frac{da_1}{dr} + (2\lambda r + 4\alpha r^3)a_1 = -i(\omega + \beta r^2)a_1. \tag{38}$$

Dividing by $\lambda r^2 + \alpha r^4$ and separating variables, we obtain

$$\frac{1}{a_1}\frac{da_1}{dr} = -\frac{2\lambda r + 4\alpha r^3}{\lambda r^2 + \alpha r^4} - i\frac{\omega + \beta r^2}{\lambda r^2 + \alpha r^4}. \tag{39}$$

Integrating, the real part determines the amplitude,

$$a_1(r) = \frac{C_1}{r^2|\lambda + \alpha r^2|}, \tag{40}$$

while the imaginary part determines the phase. For the supercritical Hopf case ($\alpha < 0$), the phase can be written in a real-valued form using a hyperbolic arctangent,

$$\phi(r) = \frac{\omega}{\lambda r} - 2\sqrt{\frac{-\alpha}{\lambda}}\,\tanh^{-1}\left(r\sqrt{\frac{-\alpha}{\lambda}}\right)\left(\frac{\omega}{2\lambda} - \frac{\beta}{2\alpha}\right) \tag{41}$$

Combining amplitude and phase, the steady-state first Fourier mode is

$$\boxed{a_1(r) = \frac{C_1}{r^2|\lambda + \alpha r^2|}\exp[i\phi(r)]\,.} \tag{42}$$

The phase is defined up to an additive constant, reflecting the freedom associated with angular probability currents in the steady state.

6.3 Reconstruction of the Probability Density

Having obtained the steady-state solutions for the zeroth and first angular Fourier modes, the full probability density can be reconstructed by truncating the Fourier expansion at first order. The angular Fourier representation reads

$$p(r,\theta) = a_0(r) + a_1(r)e^{i\theta} + a_1^*(r)e^{-i\theta}. \tag{43}$$

This expression is real-valued by construction. Writing the first mode in polar form,

$$a_1(r) = |a_1(r)|\,e^{i\phi(r)}, \tag{44}$$

the probability density takes the explicit form

$$\boxed{p(r,\theta) = a_0(r) + 2|a_1(r)|\cos[\theta + \phi(r)]\,.} \tag{45}$$

Here, $a_0(r)$ determines the isotropic radial distribution, while the first Fourier mode introduces angular modulation through a phase-shifted cosine term. This form makes the presence of angular probability currents explicit and is particularly convenient for visualization in polar or Cartesian coordinates.

Substituting the steady-state expressions for the radial amplitudes,

$$a_0(r) = \frac{C_0}{r^2|\lambda + \alpha r^2|}, \tag{46}$$

$$a_1(r) = \frac{C_1}{r^2|\lambda + \alpha r^2|}, \tag{47}$$

the reconstructed probability density becomes

$$\boxed{p(r,\theta) = \frac{1}{r^2|\lambda + \alpha r^2|}\left[C_0 + 2C_1\cos(\theta + \phi(r))\right].} \tag{48}$$

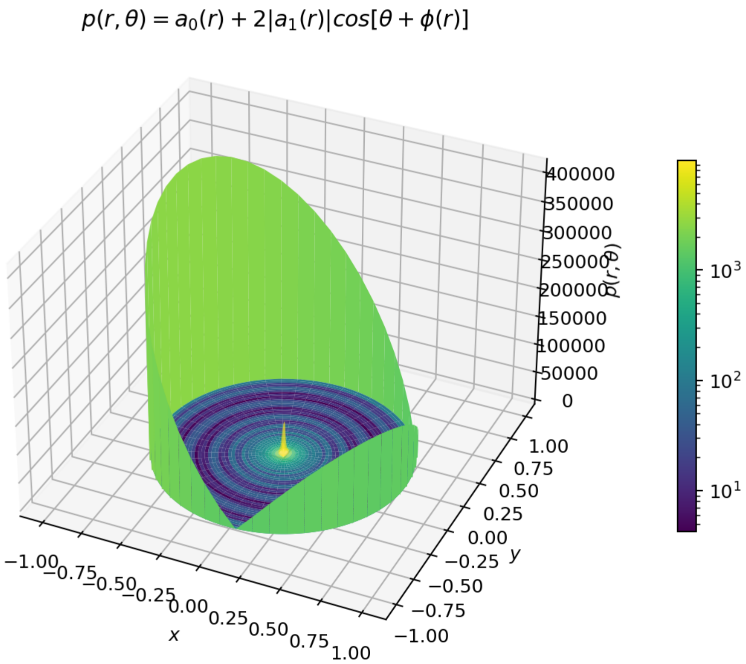


Figure 1: Steady State Probability Density
$\alpha = -0.8, \beta = 0.5, \omega = 2.8, \lambda = 0.5$

7 Stream Function and Probability Currents

In the steady state, the Fokker–Planck equation assumes the form of a continuity equation for the probability current,

$$\partial_t p + \nabla \cdot \mathbf{J} = 0. \tag{49}$$

For a stationary distribution, $\partial_t p = 0$, implying

$$\nabla \cdot \mathbf{J} = 0. \tag{50}$$

Thus, the probability current is divergence-free and admits a stream-function representation.

7.1 Probability Current in Polar Coordinates

In polar coordinates, the probability current [4] associated with the deterministic drift is given by,

$$\mathbf{J} = (J_r, J_\theta) = (A_r p, A_\theta p)\,, \tag{51}$$

where A_r and A_θ are the radial and angular drift components, respectively.

The condition of probability conservation in the steady state reads

$$\frac{1}{r}\frac{\partial}{\partial r}(rJ_r) + \frac{1}{r}\frac{\partial J_\theta}{\partial \theta} = 0. \tag{52}$$

7.2 Definition of the Stream Function

Since the probability current is divergence-free, there exists a scalar stream function $\Phi(r, \theta)$ such that

$$\boxed{J_r = \frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \qquad J_\theta = -\frac{\partial \Phi}{\partial r}.} \tag{53}$$

In terms of the probability density and drift fields, this implies

$$\frac{\partial \Phi}{\partial r} = -A_\theta p, \qquad \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = A_r p. \tag{54}$$

These relations define the stream function up to an additive constant and ensure that $\nabla \cdot \mathbf{J} = 0$ identically.

7.3 Physical Interpretation

The stream function $\Phi(r, \theta)$ provides a geometric description of probability flow in phase space. Contours of constant Φ correspond to streamlines of the probability current, along which probability circulates without accumulation or loss.

In systems with detailed balance, the probability current vanishes identically and the stream function is constant. In contrast, for the stochastic Hopf oscillator considered here, the angular drift generates non-vanishing probability currents, leading to closed streamlines in phase space. The existence of a nontrivial stream function therefore signals the breakdown of detailed balance and characterizes the non-equilibrium nature of the steady state.

7.4 Relation to the Fourier Representation

Using the truncated Fourier expansion

$$p(r, \theta) = a_0(r) + a_1(r)e^{i\theta} + a_1^*(r)e^{-i\theta}, \tag{55}$$

the stream function naturally inherits the same angular structure. The zeroth mode produces no angular current, while the first Fourier mode generates circulating probability flow. Consequently, the stream function provides a compact way of visualizing the angular modulation and rotational probability currents encoded in $a_1(r)$.

Although the stream function need not be evaluated explicitly for all purposes, its existence offers a useful conceptual framework for understanding the geometry of probability transport in the steady state.

8 Time-Dependent Analysis via the Method of Characteristics

To gain insight into the transient dynamics of the probability density, we analyze the full non-stationary Fokker–Planck equation using the method of characteristics [2]. This approach provides a geometric interpretation of probability transport in phase space and clarifies how the steady state emerges from the underlying deterministic flow.

8.1 Characteristic Equations

In polar coordinates, neglecting diffusion, the Fokker–Planck equation can be written as

$$\partial_t p + \frac{1}{r}\frac{\partial}{\partial r}(rA_r p) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta p) = 0, \tag{56}$$

where the drift components for the Hopf normal form are

$$A_r(r) = \lambda r + \alpha r^3, \qquad A_\theta(r) = \omega + \beta r^2. \tag{57}$$

The corresponding characteristic equations are

$$\frac{dr}{dt} = A_r(r), \tag{58}$$

$$\frac{d\theta}{dt} = A_\theta(r), \tag{59}$$

$$\frac{dp}{dt} = -p\,\nabla \cdot \mathbf{A}. \tag{60}$$

These equations describe, respectively, the radial motion, angular evolution, and the change in probability density along characteristic curves in phase space.

8.2 Solutions of the Characteristic Curves

The radial characteristic equation,

$$\frac{dr}{dt} = \lambda r + \alpha r^3, \tag{61}$$

can be integrated to give

$$\boxed{\frac{1}{2\lambda}\ln\left|\frac{r^2}{\lambda+\alpha r^2}\right| = t + C_1,} \tag{62}$$

where C_1 is a constant labeling different radial characteristics.

Eliminating time between Eqs. (58) and (59) yields the equation for the angular characteristic curves,

$$\frac{dr}{d\theta} = \frac{r^2(\lambda+\alpha r^2)}{\omega+\beta r^2}. \tag{63}$$

Integration leads to the implicit relation

$$\boxed{-\frac{\omega}{\lambda r} + \sqrt{\frac{\alpha}{\lambda}}\ln\left|\frac{r-\sqrt{-\lambda/\alpha}}{r+\sqrt{-\lambda/\alpha}}\right|\left(\frac{\beta}{2\alpha}-\frac{\omega}{2\lambda}\right) = \theta + C_2,} \tag{64}$$

where C_2 labels distinct angular characteristics.

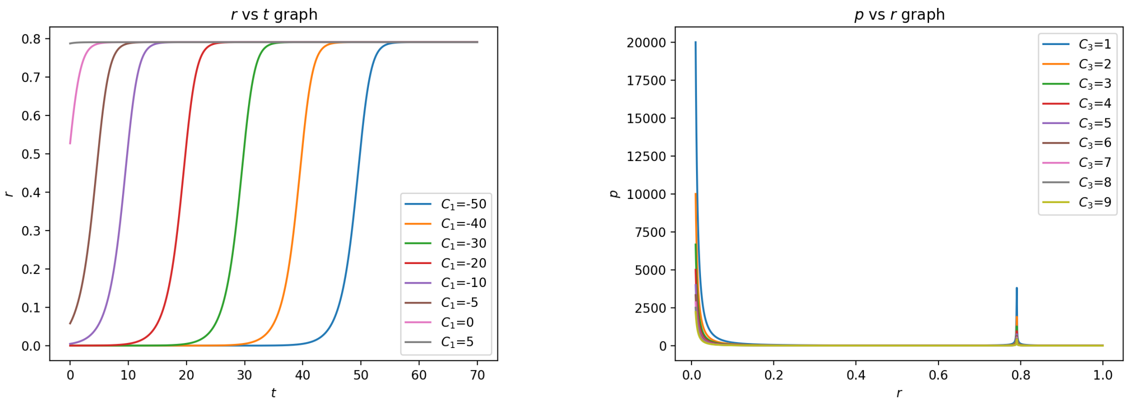
Finally, the evolution of the probability density along a characteristic is governed by

$$\frac{dp}{dr} = \frac{-2p(\lambda+2\alpha r^2)}{r(\lambda+\alpha r^2)} \tag{65}$$

which integrates to

$$\boxed{p = \frac{C_3}{r^2|\lambda+\alpha r^2|},} \tag{66}$$

where C_3 is a constant along each characteristic curve.



(a) $r(t)$ along characteristic curves.

(b) Radial probability density $p(r)$ for different C_3.

Figure 2: Characteristic behavior of the non-stationary Fokker–Planck equation.

Figure 2(a) Radial trajectories converge to the analytical limit-cycle radius $r_0 = \sqrt{-\lambda/\alpha} \approx 0.790569.$, *while Fig. 2(b) illustrates the corresponding probability accumulation.*

Before visualizing the angular characteristic curves, it is useful to rewrite Eq. (64) in a form that makes its connection to the Fourier-mode analysis explicit. Equation (64) can be expressed as

$$\boxed{\theta(r) = -\frac{\omega}{\lambda r} + 2\sqrt{\frac{-\alpha}{\lambda}}\,\tanh^{-1}\left(r\sqrt{\frac{-\alpha}{\lambda}}\right)\left(\frac{\omega}{2\lambda} - \frac{\beta}{2\alpha}\right) - C_2.} \tag{67}$$

This expression is identical, up to an additive constant, to the phase appearing in the first angular Fourier mode of the steady-state probability density. Thus, the angular characteristic curves encode the same phase structure that governs the angular modulation of the steady-state distribution.

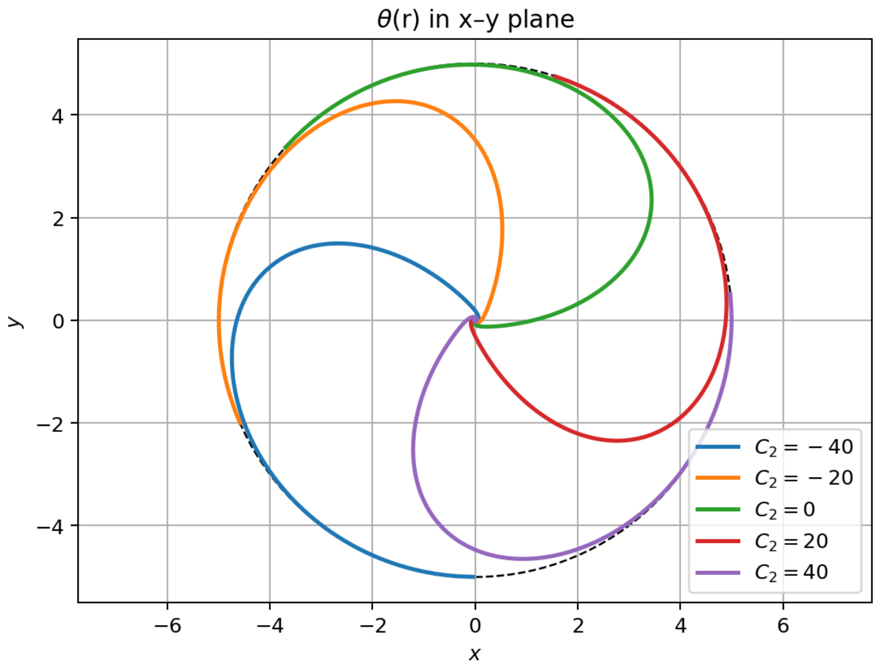


Figure 3: Angular characteristic curves $\theta(r)$ plotted in the x–y plane. Here $\alpha = -0.8, \beta = 2, \lambda = 20, \omega = 2.8$

8.3 Interpretation and Relation to the Steady State

The characteristic curves describe the flow of probability in phase space. Radially, trajectories are attracted toward the deterministic limit cycle, $r_0 = \sqrt{-\lambda/\alpha}$, while angular motion proceeds with a radius-dependent frequency. Along these curves, the probability density is amplified or depleted according to the local phase-space contraction rate.

The form of $p(r)$ obtained from the characteristic analysis coincides with the radial dependence of the steady-state solution derived earlier. Similarly the Angular characteristic curves is same as the phase of the First Fourier mode of the steady-state probability density. This demonstrates that the stationary distribution arises naturally from the accumulation of probability along characteristic curves and provides a geometric foundation for the steady-state and stream function analysis presented in previous sections.

9 Conclusion

In this work, the stochastic Hopf oscillator was analysed within the Fokker–Planck framework to understand the structure of its probability dynamics beyond individual trajectories. Starting from the time-dependent Fokker–Planck equation, the problem was reformulated in polar coordinates, allowing a clear separation of radial and angular dynamics. Exploiting the periodicity of the angular variable, a Fourier-mode reduction was employed to obtain closed-form expressions for the leading modes of the steady-state probability density.

The steady-state analysis revealed a divergence of the probability density near the deterministic limit cycle, reflecting the accumulation of probability induced by radial attraction. The first angular Fourier mode was shown to generate non-vanishing probability currents, indicating a breakdown of detailed balance and confirming the non-equilibrium nature of the stationary state. The introduction of a stream-function formulation provided a geometric interpretation of these currents, with closed streamlines circulating around the limit cycle.

Complementary insight was obtained from the method of characteristics applied to the full non-stationary equation. The characteristic curves demonstrated how probability is transported in phase space and how the steady-state structure emerges dynamically. In particular, the angular characteristic solution was shown to coincide with the phase of the first Fourier mode, establishing a direct connection between time-dependent probability flow and stationary angular modulation.

Several extensions of the present work are natural. The inclusion of diffusion would regularize the singularities observed in the deterministic steady state. Higher angular Fourier modes could be incorporated to capture finer angular structure, and the approach may be generalized to coupled oscillators or systems exhibiting non-smooth or noise-induced bifurcations. These directions offer promising routes for further exploration of non-equilibrium phenomena in stochastic nonlinear oscillators.

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