AS3020: Aerospace Structures

Module 5: Torsion of Beams

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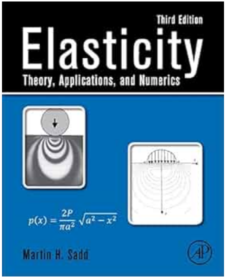
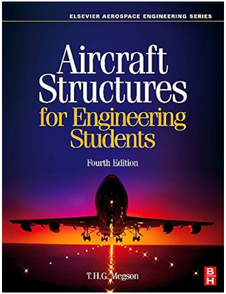


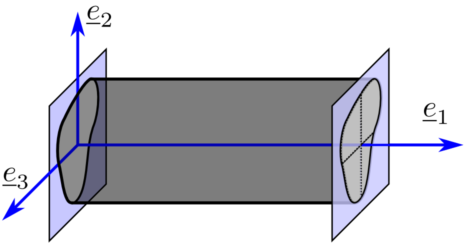
Chapter 9 in Sadd (2009)

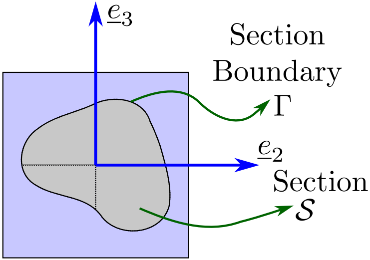


Chapters 3, 17-19 in Megson (2013)

1. Solid Section Torsion

Basic Setup





- We assume:
  1. No direct stresses applied:
  $$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$$
  2. Sections "rotate rigidly":
  $$\gamma_{23} = 0 \implies \sigma_{23} = 0.$$
  3. Body is at equilibrium under constant torque applied at right end.
- We will denote the section by $\mathcal{S}$ and the section-boundary by Γ.
- The words "torque" and "twisting moment" will be used interchangeably.

1.1. Stress Formulation

Solid Section Torsion

- Since we assume $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, the equilibrium equations read,

$$\sigma_{12,2} + \sigma_{13,3} = 0, \quad \sigma_{12,1} = 0, \quad \sigma_{13,1} = 0.$$

- We introduce the **Prandtl Stress Function** $\phi(X_2, X_3)$ (no dependence on X_1) such that

$$\sigma_{12} = \phi_{,3}, \quad \sigma_{13} = -\phi_{,2}.$$

  This satisfies equilibrium by definition.

- In terms of strains the above assumptions imply that we only have E_{12} and E_{13} active. **Recall** that Strain compatibility is $\epsilon_{mjk}\epsilon_{nil}E_{ij,mn} = 0$ (see Module 3).
- The non-trivial compatibility equations read,

$$\left.\begin{aligned} E_{12,23} - E_{13,22} &= 0 \\ E_{12,33} - E_{13,23} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} \phi_{,332} + \phi_{,222} &= 0 \\ \phi_{,333} + \phi_{,322} &= 0 \end{aligned}\right\} \implies \boxed{\nabla^2\phi = \text{constant}}.$$

- This is known as the **Poisson's problem**. What about Boundary Conditions?

1.1. Stress Formulation

Solid Section Torsion

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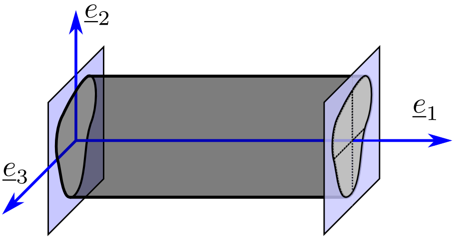
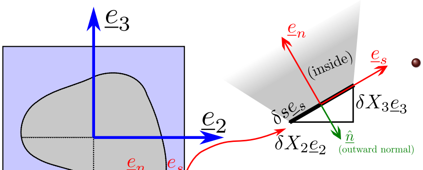
Kinematic considerations will give us this "constant".

- This is known as the **Poisson's problem**. What about Boundary Conditions?

1.1. Stress Formulation

Solid Section Torsion





$$\underline{e}_s = X_{2,s}\underline{e}_2 + X_{3,s}\underline{e}_3$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$$

Convention: $\underline{e}_2 \times \underline{e}_3 = \underline{e}_s \times \underline{e}_n = \underline{e}_1$

- We derive the coordinate transformation on the boundary as follows:

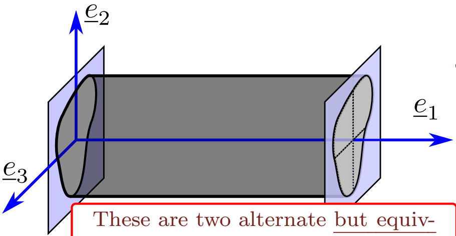
$$dX_2\underline{e}_2 + dX_3\underline{e}_3 = ds\underline{e}_s + dn\underline{e}_n$$
$$\implies \begin{bmatrix} dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_2, \underline{e}_n \rangle \\ \langle \underline{e}_3, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} ds \\ dn \end{bmatrix}$$
$$\text{and, } \begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_s \rangle \\ \langle \underline{e}_2, \underline{e}_n \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

- Considering 2D construction of normal vectors, we will also have

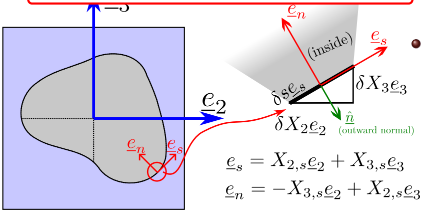
$$\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}.$$

1.1. Stress Formulation

Solid Section Torsion



These are two alternate but equivalent representations for $\underline{e}_s$ and $\underline{e}_n$ that we will invoke as convenient.



$$\underline{e}_s = X_{2,s}\underline{e}_2 + X_{3,s}\underline{e}_3$$
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- We derive the coordinate transformation on the boundary as follows:

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$$\implies \begin{bmatrix} dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_2, \underline{e}_n \rangle \\ \langle \underline{e}_3, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} ds \\ dn \end{bmatrix}$$

$$\text{and, } \begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_s \rangle \\ \langle \underline{e}_2, \underline{e}_n \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

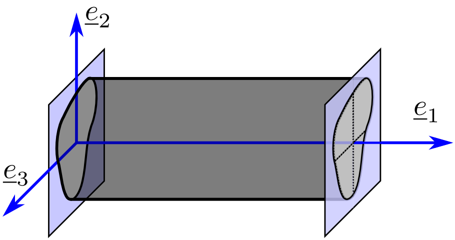
$$= \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

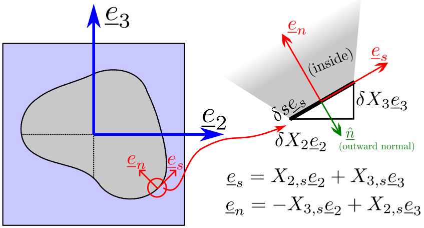
- Considering 2D construction of normal vectors, we will also have

$$\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}.$$

1.1. Stress Formulation

Solid Section Torsion





We invoke
$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$ here.

- Enforcing stress-free section boundary condtion leads to:

$$\begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} 0 \\ X_{3,s} \\ -X_{2,s} \end{bmatrix}}^{\hat{\underline{n}} = -\underline{e}_n} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \sigma_{12} X_{3,s} - \sigma_{13} X_{2,s} = 0$$

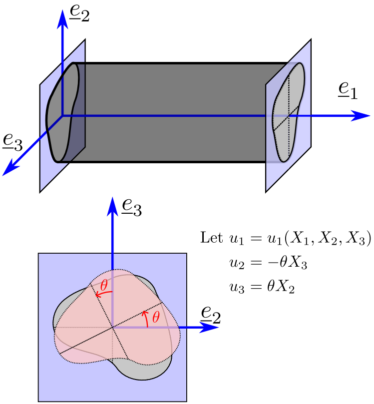
$$(\phi_{,3} X_{3,s} + \phi_2 X_{2,s}) = \phi_{,s} = 0$$

- That is, on the section-boundary, the stress function is constant, set to 0 w.l.o.g.:

$$\phi = \cancelto{0}{constant} \quad \text{on } \Gamma.$$

1.2. Displacement Formulation

Solid Section Torsion



- The strains are,

$$\begin{aligned} E_{11} &= u_{1,1} = 0 \\ E_{22} &= -\theta_{,2} X_3 = 0 \\ E_{33} &= \theta_{,3} X_2 = 0 \\ 2E_{23} &= \theta - \theta = 0 \\ 2E_{12} &= u_{1,2} - \theta_{,1} X_3 = \frac{\sigma_{12}}{G} = \frac{\phi_{,3}}{G} \\ 2E_{13} &= u_{1,3} + \theta_{,1} X_2 = \frac{\sigma_{13}}{G} = -\frac{\phi_{,2}}{G} \end{aligned}$$

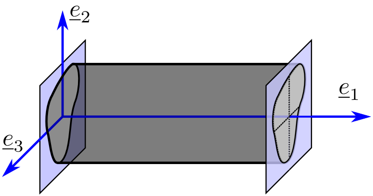
- Differentiating the strain expressions for σ_{12} and σ_{13} above allows us to write:

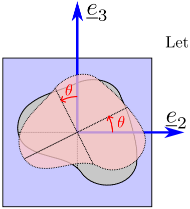
$$\boxed{\phi_{,kk} = -2G\theta_{,1}},$$

which gives us the "constant" required for the Poisson problem from before (along with the B.C. $\phi = 0 \text{ on } \Gamma$).

1.3. Section Moment

Solid Section Torsion





Let $u_1 = u_1(X_1, X_2, X_3)$
$u_2 = -\theta X_3$
$u_3 = \theta X_2$

- The non-trivial shear strains are:

$$\sigma_{12} = \phi_{,3} \quad = G(u_{1,2} - X_3\theta_{,1})$$
$$\sigma_{13} = -\phi_{,2} = G(u_{1,3} + X_2\theta_{,1})$$

- The moment about $\underline{e}_1$ is

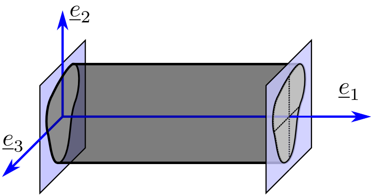
$$\boxed{M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA}.$$

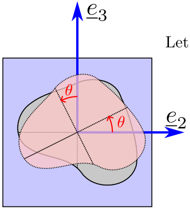
- Since σ_{12} and σ_{13} are expressed in terms of **kinematic quantities** as well as the **stress function** ϕ, we will write down relationships with both before proceeding.
- It is also obvious that $\phi_{,kk} = -2G\theta_{,1}$ implies

$$\boxed{u_{1,kk} = 0}.$$

1.3. Section Moment

Solid Section Torsion





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- It is also obvious that $\phi_{,kk} = -2G\theta_{,1}$ implies

$$\boxed{u_{1,kk} = 0}.$$

This is the governing equation in terms of the section-axial displacement field.

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA$$
$$= -\int_{\mathcal{S}} (\phi_{,2}X_2 + \phi_{,3}X_3) dA$$
$$\boxed{M_1 = 2\int_{\mathcal{S}} \phi dA}.$$

In terms of kinematic description

$$\begin{aligned} M_1 =& G\int_{\mathcal{S}} (X_2 u_{1,3} - X_3 u_{1,2}) dA \\ &+ G \underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2) dA}_{I_{11}} \theta_{,1} \\ =& GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{1jk} X_j u_{1,k} dA \\ =& GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{ijk} (X_j u_1)_{,k} dA \\ &- G\int_{\mathcal{S}} \cancel{\epsilon_{ijk}\delta_{jk}} u_1 dA \\ M_1 =& GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk} X_j n_k u_1 d|s| \end{aligned}$$
$$\boxed{M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|s|}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

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In terms of kinematic description

$$M_1 = G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2}) dA + G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2) dA}_{I_{11}}\,\theta_{,1}$$
$$= GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{1jk}X_ju_{1,k} dA$$
$$= GI_{11}\theta_{,1} + G\int \epsilon_{1jk}(X_ju_1)_{,k} dA$$

This term is clearly zero for a perfectly circular section. What about other types?

$$M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk}X_jn_ku_1 d|s|$$
$$\boxed{M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|s|}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA$$

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$$\boxed{M_1 = 2\int_{\mathcal{S}} \phi dA}.$$

In terms of kinematic description

$$M_1 = G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2}) dA + G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2) dA}_{I_{11}}\,\theta_{,1}$$

$$= GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{1jk}X_j u_{1,k} dA$$

$$= GI_{11}\theta_{,1} + G\int \epsilon_{1jk}(X_j u_1)_{,k} dA$$

This term is clearly zero for a perfectly circular section. What about other types?

$$M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk}X_j n_k u_1 d|s|$$

Not zero in the general case.

1.3. Section Moment: St. Venant's Warping Function

Solid Section Torsion

- For a "pure twist" condition, due to **translational symmetry**, u_1 can not depend on X_1. It also makes sense that u_1 has to be proportional to the twist θ somehow.
- Since θ depends on X_1, but $\theta_{,1}$ is a constant, St. Venant introduced a warping function $\psi(X_2, X_3)$ such that

$$\boxed{u_1 = \theta_{,1}\psi(X_2, X_3)}.$$

- Under this definition, the effective moment M_1 can be given as,

$$\boxed{M_1 = G\underbrace{\left(I_{11} + \int_\Gamma (\underline{X} \times \underline{n})_1 \psi d|s|\right)}_{J}\theta_{,1} = GJ\theta_{,1}}.$$

- Alternatively, J can also be written as,

$$\boxed{J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA}$$

1.3. Section Moment: St. Venant's Warping Function

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- Alternatively, J can also be written as,

The product GJ is also known as **Torsional Rigidity**

1.4. Membrane Analogy

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\text{on}\, \Gamma,$$

along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

1.4. Membrane Analogy

Solid Section Torsion

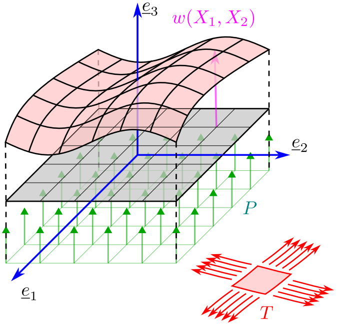
- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\text{on}\, \Gamma,$$

along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Uniform Planar Load Density P

- **The displacement field**
  $u_1 = 0, \quad u_2 = 0, \quad u_3 = w(X_1, X_2)$
- **The strain Field**
  $E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$
- **The Stress Field**
  $\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$



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Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Uniform Planar Load Density P

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  $u_1 = 0, \quad u_2 = 0, \quad u_3 = w(X_1, X_2)$
- **The strain Field**
  $E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$
- **The Stress Field**
  $\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$

- **Strain Energy Density (Integrated over thickness)**

$$\mathcal{U} = \frac{1}{2}\left(w_{,1}^2 + w_{,2}^2\right)T + Pw$$

- Equations of Motion [a]: $\frac{\partial}{\partial X_k}\frac{\partial \mathcal{U}}{\partial w_{,k}} - \frac{\partial \mathcal{U}}{\partial w} = 0$:

$$\boxed{T\left(w_{,11} + w_{,22}\right) - P = 0}$$

[a] Euler-Ostrogradsky

1.4. Membrane Analogy

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \, \text{on} \, \Gamma,$$

along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Uniform Planar Load Density P

The governing equations, therefore, are identical to that of a **membrane undergoing deformation under the action of a uniform area-load P.**

- **The displacement**

$u_1 = 0, \quad u_2 = 0$

- **The strain Field**

$$E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$$

- **The Stress Field**

$$\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$$

nergy Density (Integrated ckness)

$$= \frac{1}{2}\left(w_{,1}^2 + w_{,2}^2\right)T + Pw$$

- Equations of Motion [a]: $\frac{\partial}{\partial X_k}\frac{\partial \mathcal{U}}{\partial w_{,k}} - \frac{\partial \mathcal{U}}{\partial w} = 0$:

$$\boxed{T\left(w_{,11} + w_{,22}\right) - P = 0}$$

[a] Euler-Ostrogradsky

1.4. Membrane Analogy: Governing Equations of u_1 (Warping)

Solid Section Torsion

- The governing equations in terms of u_1 is the **Laplace equation**:

$$u_{1,kk} = 0,$$

and its boundary conditions **(Neumann B.C.s)** are written as (again based on zero traction at free end:

$$G \left\langle (u_{1,2} - X_3\theta_{,1})\underline{e}_2 + (u_{1,3} + X_2\theta_{,1})\underline{e}_3, \underline{e}_n \right\rangle = 0$$

$$\implies \langle u_{1,2}\underline{e}_2 + u_{1,3}\underline{e}_3, X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3 \rangle$$

$$- \theta_{,1} \langle X_3\underline{e}_2 - X_2\underline{e}_3, -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3 \rangle = 0$$

$$\implies \boxed{u_{1,n} = -\frac{\theta_{,1}}{2}\frac{d}{ds}\left(X_2^2 + X_3^2\right)} = -\theta_{,1}\left(X_3 \underbrace{X_{2,n}}_{-n_2} - X_2 \underbrace{X_{3,n}}_{-n_3}\right).$$

1.4. Membrane Analogy: Governing Equations of u_1 (Warping)

Solid Section Torsion

- The governing equatio **tion**:

Note: We have used two different representations of $\underline{e}_n$ here:

$$\underline{e}_n = X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3, \text{ and}$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3.$$

Also, we are representing the outward normal as

$$\hat{\underline{n}} = n_2\underline{e}_2 + n_3\underline{e}_3 = -\underline{e}_n.$$

and its boundary cond as (again based on zero traction at free end:

$$G\left\langle (u_{1,2}\right.$$

$$\implies \langle u_{1,2}\underline{e}_2$$
$$-\theta_{,1}\langle X_3\underline{e}_2 - X_2\underline{e}_3, -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3\rangle = 0$$

$$\implies \boxed{u_{1,n} = -\frac{\theta_{,1}}{2}\frac{d}{ds}\left(X_2^2 + X_3^2\right)} = -\theta_{,1}\left(X_3\underbrace{X_{2,n}}_{-n_2} - X_2\underbrace{X_{3,n}}_{-n_3}\right).$$

1.4. Membrane Analogy

Solid Section Torsion

Equations in the Stress Function

$$\nabla^2\phi = -2G\theta_{,1},$$
$$\phi = 0 \text{ on } \Gamma,$$
$$M_1 = 2\int_{\mathcal{S}} \phi dA.$$

Equations in Warping

$$\nabla^2 u_1 = 0,$$
$$\frac{\partial u_1}{\partial n} = \theta_{,1}\left(X_3 n_2 - X_2 n_3\right) \text{ on } \Gamma.$$
$$M_1 = GJ\theta_{,1}$$

Relating the two

- Once we find ϕ, we can integrate the following to get u_1:

$$\frac{1}{G}\phi_{,3} = u_{1,2} - X_3\theta_{,1}$$
$$-\frac{1}{G}\phi_{,2} = u_{1,3} + X_2\theta_{,1}$$

1.4. Membrane Analogy

Solid Section Torsion

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$$-\frac{1}{G}\phi_{,2} = u_{1,3} + X_2\theta_{,1}$$

If interested, you can see the FreeFem scripts in the website for numerical implementations of these. You need to know just a little bit about weak forms to understand the code, it is very straightforward.

(not for exam)

1.5. Tutorial: Elliptical Section

Solid Section Torsion

- Let us consider an elliptical section and choose the stress function as

$$\phi = C\left(\frac{X_2^2}{a^2} + \frac{X_3^2}{b^2} - 1\right).$$

- The Laplacian of ϕ evaluates as,

$$\nabla^2\phi = 2C\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\theta_{,1} \implies C = -G\theta_{,1}\frac{a^2b^2}{a^2+b^2}.$$

- Let us first compute the total resultant twisting moment M_1 that this represents:

$$M_1 = 2\int_{\mathcal{S}}\phi = 2C\left(\frac{1}{a^2}\overbrace{\int_{\mathcal{S}} X_2^2 dA}^{\frac{\pi a^3 b}{4}} + \frac{1}{b^2}\overbrace{\int_{\mathcal{S}} X_3^2 dA}^{\frac{\pi a b^3}{4}} - \overbrace{\int_{\mathcal{S}} dA}^{\pi ab}\right) = -C\pi ab$$

$$\boxed{M_1 = G\frac{\pi a^3 b^3}{a^2+b^2}\theta_{,1}}.$$

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$$\boxed{M_1 = G\frac{\pi a^3 b^3}{a^2+b^2}\theta_{,1}}.$$

The torsional rigidity reads,

$$GJ = G\frac{\pi a^3 b^3}{a^2+b^2}$$

1.5. Tutorial: Elliptical Section

Solid Section Torsion

- For the axial deflection we have two equations (by equating shear stress expressions),

$$u_{1,2} = \theta_{,1}\psi_{,2} = -\frac{2a^2}{a^2+b^2}\theta_{,1}X_3 + \theta_{,1}X_3 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_3$$
$$u_{1,3} = \theta_{,1}\psi_{,3} = \frac{2b^2}{a^2+b^2}\theta_{,1}X_2 - \theta_{,1}X_2 \quad = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2$$

- Integrating them separately we have,

$$u_1 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2 + f_1(X_3)$$
$$= -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2 + f_2(X_2)$$

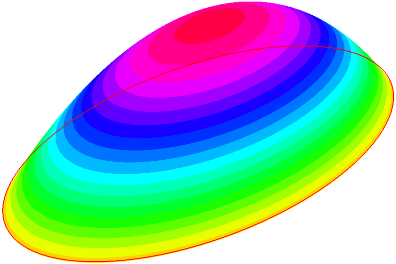
- f_1 and f_2 **have to be constant**. Setting it to zero we have,

$$\boxed{u_1 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2X_3 = -\frac{a^2-b^2}{G\pi a^3b^3}M_1X_2X_3}.$$

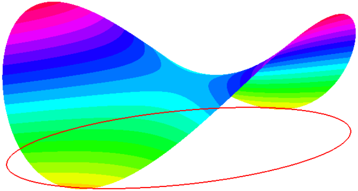
1.5. Tutorial: Elliptical Section

Solid Section Torsion

Stress Function



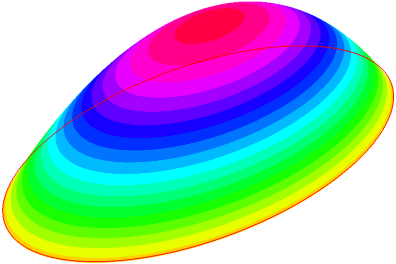
Section Warping



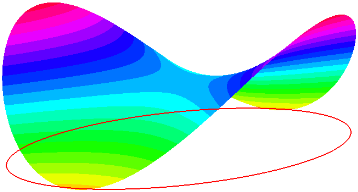
1.5. Tutorial: Elliptical Section

Solid Section Torsion

Stress Function



Section Warping



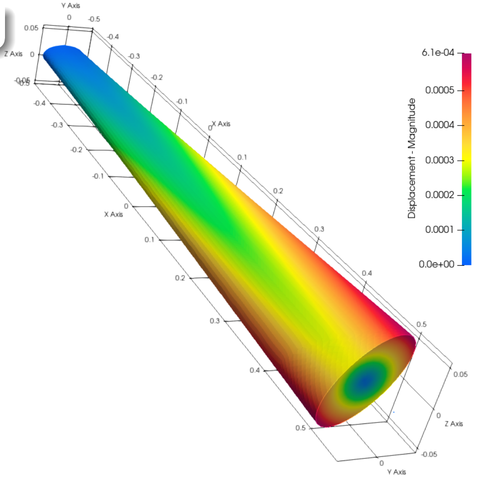
General Sections

- Torsion is amenable to analysis when the solid section boundary can be expressed in closed form **AND** its Laplacian evaluates to a constant. (See Chapter 9 in Sadd 2009)
- Every assumed form of ϕ will give us a warping field. For an application wherein the section warping is also constrained, **this solution is not exact**. (St. Venant's principle can be invoked, however).
- Several analytical techniques exist (check Sadd 2009 and references therein).
- Fully numerical approaches are also possible, see the FreeFem scripts in the website.

1.5. Tutorial: Elliptical Section: Results in 3D

Solid Section Torsion

Here is a 3D FE Result.
(Salome_Meca HDF Files in website)



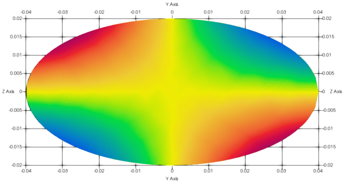
CODE aster
salome_meca 2023

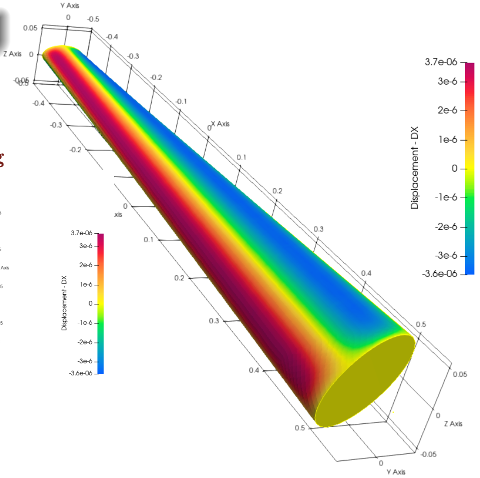
1.5. Tutorial: Elliptical Section: Results in 3D

Solid Section Torsion

Here is a 3D FE Result.
(Salome_Meca HDF Files in website)

Mid-Section Warping





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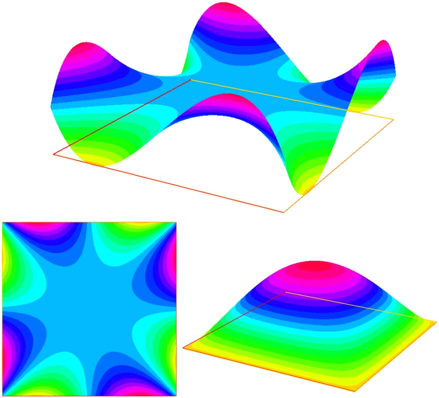
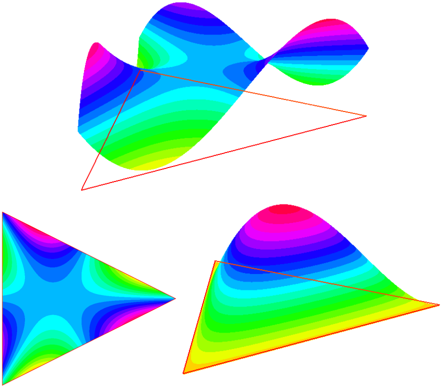
1.5. General Sections

Solid Section Torsion

Square Section



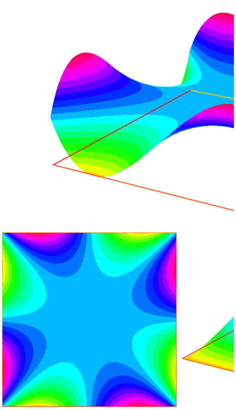
Triangular Section



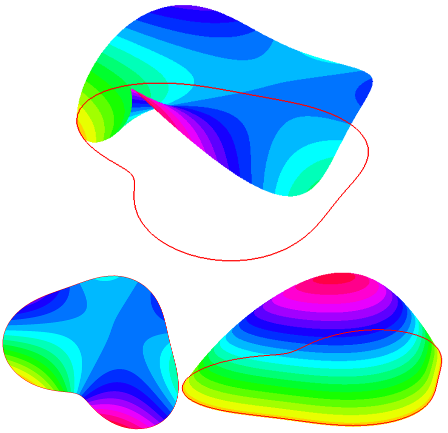
1.5. General Sections

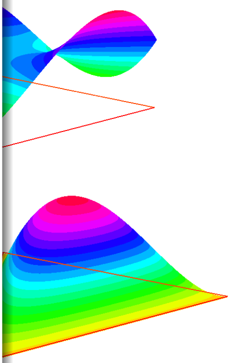
Solid Section Torsion

Square Section



An Arbitrary Hand-drawn Section

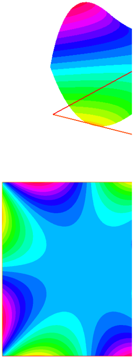




1.5. General Sections

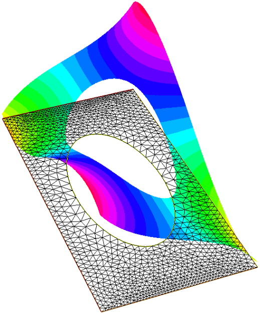
Solid Section Torsion

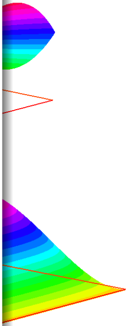
Square Secti



Sections with Holes

The validity of the governing equations extend beyond singly connected sections. Nothing stops us from applying it for multiply connected sections also for the warping formulation. (Some additional considerations necessary for the stress function, see sec. 9.3.3 in Sadd 2009).

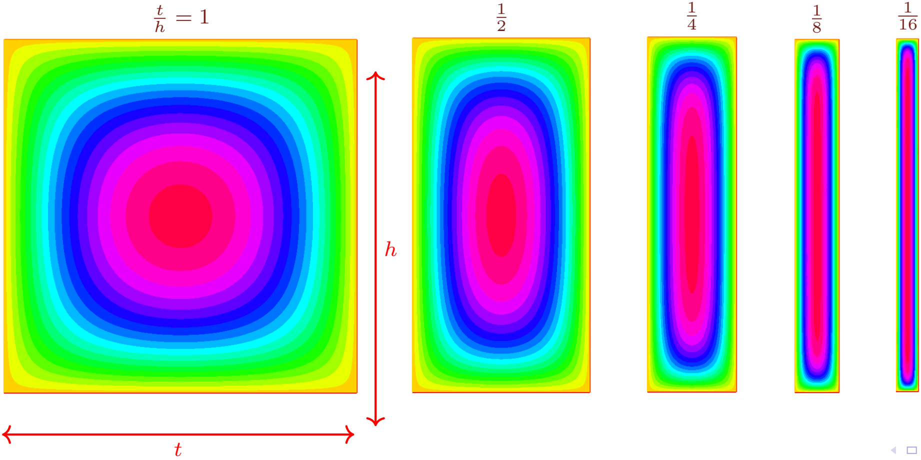




1.6. Rectangular Sections

Solid Section Torsion

- Rectangular sections are slightly more involved, in general. But an important simplifcation is achieved for thin sections.
- Let us look at some numerical results for motivation (FreeFem code **b_rectangle.edp**).



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Solid Section Torsion

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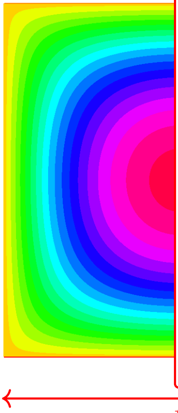
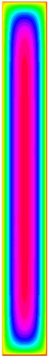
- Let us look a … rectangle.edp).

$\frac{t}{h}$

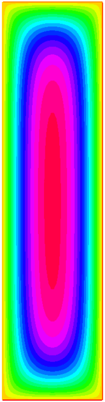


$\frac{1}{8}$

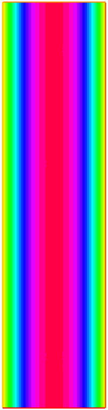


$\frac{1}{16}$

For the thin cases, we will approximate the shear function as



$\approx$



This will form the basis for the study of thin sections.

1.6. Rectangular Sections: Thin Strip Idealization

Solid Section Torsion

- Idealizing the rectangle as a "strip" (t/h is very small), we can write the stress function Poisson problem as,

$$\phi_{,22} = -2G\theta', \quad \text{with} \quad \phi = 0 \text{ at } X_2 \in \left\{-\frac{t}{2}, \frac{t}{2}\right\}, X_3 \in \left\{-\frac{h}{2}, \frac{h}{2}\right\},$$

solved by $\boxed{\phi(X_2, X_3) = -G\theta'\left(X_2^2 - \left(\frac{t}{2}\right)^2\right)}$.

- This implies the following shear stress and resultant moment:

$$\sigma_{12} = \overbrace{0}^{\phi_{,3}}, \qquad \sigma_{13} = \overbrace{2GX_2\theta'}^{-\phi_{,2}}, \qquad M_1 = 2\int_{\mathcal{S}} \phi dA = G\overbrace{\frac{ht^3}{3}}^{J}\theta'.$$

- The shear strain is $\gamma_{13} = u_{1,3} + u_{3,1} = u_{1,3} + X_2\theta_{,1}$, which implies $\boxed{u_1 = \theta' X_2 X_3}$ as the warping field (setting integration constant to zero).

1.6. Rectangular Sections: Thin Strip Idealization

Solid Section Torsion

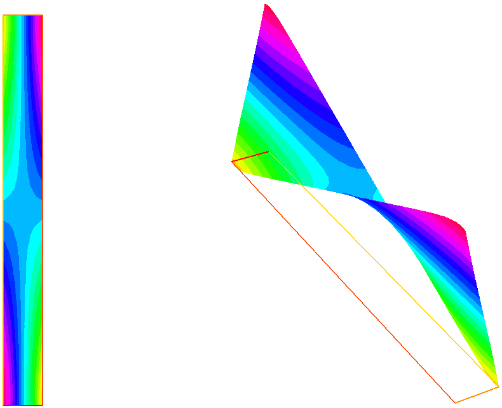
- Ideali ... s function
  Poisso

  solved

- This i

- The s ... $X_2 X_3$ as
  the w

Warping Profile



2. Torsion of Thin-Walled Sections

- Using the same notation as in Module 4, the equilibrium equations, for a thin-walled section undergoing pure torsion ($\sigma_{11} = \sigma_{ss} = \sigma_{nn} = \sigma_{sn} = 0$) can be written as

$$\cancelto{0}{\sigma_{11,1}} + \sigma_{1s,s} = 0, \quad \sigma_{1s,1} = 0, \quad (\sigma_{1n} \approx 0).$$

- This implies, when in "pure torsion", σ_{1s} is constant along the section arc.
- Since $q(s) = \int \sigma_{1s} dX_n$, this shows that **shear flow is constant across the section (along $\underline{e_s}$) under pure torsion.**
- The resultant moment of a shear flow distribution $q(s)$ can be given by

$$M_1 = \int_{\mathcal{S}} \underline{X} \times (q(s) ds \underline{e}_s) = q \int_{\mathcal{S}} p ds,$$

where p is the perpendicular distance to the point on the skin under consideration.

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An important simplification occurs when $\mathcal{S}$ is a *closed section*. This leads to the **Bredt-Batho Formula**:

$$M_1 = 2\mathcal{A}q.$$

- This implies, whe… …ction arc.
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where p is the perpendicular distance to the point on the skin under consideration.

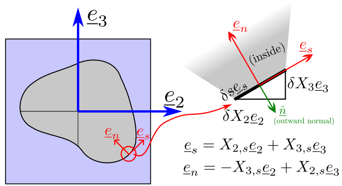
2.1. Transformation of Displacement Field to Skin-local Coordinates

Torsion of Thin-Walled Sections

We will consider the bending-torsion combined displacement field:

$$u_2 = v - X_3\theta$$
$$u_3 = w + X_3\theta,$$

and transform this to the **skin local coordinate system**.



- The section displacement field transforms as,

$$\begin{bmatrix} u_s \\ u_n \end{bmatrix} = \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}.$$

- The **tangential component** of displacement along the boundary Γ can be written as,

$$u_s = X_{2,s}(v - X_3\theta) + X_{3,s}(w + X_2\theta)$$
$$= X_{2,s}v + X_{3,s}w + \theta\underbrace{(X_{3,s}X_2 - X_{2,s}X_3)}_{-X_n = p}$$

$$\implies \boxed{u_s = p\theta + vX_{2,s} + wX_{3,s}}.$$

2.1. Transformation of Displacement Field to Skin-local Coordinates

Torsion of Thin-Walled Sections

- The transformed displacement field combining bending and torsion is:

$$\left.\begin{aligned} u_1 &= -X_3 v' - X_2 w' + \theta' \psi \\ u_2 &= v - X_3 \theta \\ u_3 &= w + X_2 \theta \end{aligned}\right\} \implies \begin{aligned} &u_1 && \text{(unchanged)} \\ &u_s && = p\theta + v X_{2,s} + w X_{3,s} \\ &u_n && = X_s \theta - v X_{3,s} + w X_{2,s} \end{aligned}$$

- The shear strain along a thin section between the $\underline{e}_1$, $\underline{e}_s$ directions is

$$\gamma_{1s} = u_{1,s} + u_{s,1} = u_{1,s} + p\theta' + X_{2,s} v' + X_{3,s} w' = \frac{\tau}{G} = \frac{q(s)}{Gt}.$$

- Integrating this over the skin, we get

$$\begin{aligned} \int_0^s \frac{q(x)}{Gt} dx =& (u_1(s) - u_1(0)) + \theta' \int_0^s p dx + v' \int_0^s X_{2,x} dx + w' \int_0^s X_{3,x} dx \\ =& (u_1(s) - u_1(0)) + \theta' 2\mathcal{A}_{Os}(s) + v'(X_2(s) - X_2(0)) + w'(X_3(s) - X_3(0)). \end{aligned}$$

- Over a **completely closed section** we have,

$$\boxed{\oint \frac{q(s)}{Gt} ds = 2\mathcal{A}\theta'}$$

2.2. Closed Sections: Bredt-Batho Theory

Torsion of Thin-Walled Sections

- For closed sections under *pure torsion*, we will set $v = w = 0$.
- So q is constant over the section and is written with the *Bredt-Batho Formula* based on the resultant twisting moment M_1 as

$$M_1 = 2\mathcal{A}q \implies q = \frac{M_1}{2\mathcal{A}}.$$

- The shear flow integral reads,

$$\underbrace{q\int_0^s \frac{1}{Gt}dx}_{\delta_{Os}(s)} = (u_1(s) - u_1(0)) + \theta' \underbrace{\int_0^s p dx}_{2\mathcal{A}_{Os}(s)} .$$

For the whole section, this becomes

$$q\oint \frac{1}{Gt}ds = \theta' 2\mathcal{A} \implies \theta' = \frac{q}{2\mathcal{A}}\oint \frac{1}{Gt}ds.$$

- So we can write the warping as

$$u_1(s) - u_1(0) = \overbrace{\frac{M_1\delta}{2\mathcal{A}}}^{q\delta}\left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}}\right)$$

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The integration constant $u_1(0)$ can be found by enforcing $\sigma_{11} = 0$ on the section after assuming $\sigma_{11} \propto u_1$. So $\oint u_1(s) ds = 0$ in the section, leading to:

$$u_1(0) = \frac{\oint u_{10}(s) t ds}{\oint t ds},$$

where $u_{10}(s)$ is the warping distriution assuming $u_1(0) = 0$.

For

$$q \oint \frac{\cdots}{Gt} ds = \theta\, 2\mathcal{A} \implies \theta = \frac{\cdots}{2\mathcal{A}} \oint \frac{\cdots}{Gt} ds.$$

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$$u_1(s) - u_1(0) = \overbrace{\frac{M_1 \delta}{2\mathcal{A}}}^{q\delta} \left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}} \right)$$

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- The shear flow integral reads,

$$q \underbrace{\int_0^s \frac{1}{Gt} dx}_{\delta_{Os}(s)} = (u_1(s) - u_1(0)) + \theta' \underbrace{\int_0^s p dx}_{2\mathcal{A}_{Os}(s)} .$$

For the whole section, this becomes

$$q \oint \frac{1}{Gt} ds = \theta' 2\mathcal{A} \implies \boxed{\theta' = \frac{q}{2\mathcal{A}} \oint \frac{1}{Gt} ds.}$$

- So we can write the warping as

$$u_1(s) - u_1(0) = \overbrace{\frac{M_1 \delta}{2\mathcal{A}}}^{q\delta} \left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}} \right)$$

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- T

$$u_1(0)) + \theta' \underbrace{\int_0^s p dx}_{2\mathcal{A}_{Os}(s)}.$$

$$\implies \boxed{\theta' = \frac{q}{2\mathcal{A}} \oint \frac{1}{Gt} ds}.$$

- S

$$2\mathcal{A}\left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}}\right)$$

Combining these two, we get the torsional rigidity:

$$M_1 = 2\mathcal{A}q = \frac{4A^2}{\delta}\theta'.$$

For constant G, t, we get,

$$M_1 = \frac{4A^2}{|\Gamma|}Gt\theta' = GJ\theta'$$

$$\implies \boxed{J = \frac{4tA^2}{|\Gamma|}}.$$

$|\Gamma|$ is the section circumference.

2.2. Closed Sections: The Neuber Beam

Torsion of Thin-Walled Sections

- A natural question arises: what should I do if I want to minimize/eliminate warping?
- We want to set $u_1(s) - u_1(0) = 0, \forall\, s \in \Gamma$. This implies:

$$\frac{\delta_{Os}(s)}{\delta} = \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}},$$

which is satisfied iff

$$\frac{1}{\delta} \overbrace{\frac{1}{Gt}}^{\frac{d\delta_{Os}(s)}{ds}} = \frac{p}{2\mathcal{A}}.$$

- This implies that the quantity pGt (modulus as well as thickness can vary along section) has to be a constant:

$$pGt = \frac{2\mathcal{A}}{\delta}.$$

- It is known as a Neuber Beam if this is satisfied. (eg., circular sections)

2.2. Closed Sections: The Shear Center

Torsion of Thin-Walled Sections

- Based on relating the kinematics to stress (through linear elastic constitutive relationships), we have written the shear flow integral as:

$$\oint \frac{q(s; \xi_2, \xi_3)}{Gt} ds = 2\mathcal{A}\theta'.$$

- Suppose, for a closed section, we evaluated the shear flow by the approach in Module 4. **Recall** that we required the resultant moment M_1 to be zero for this:

$$\oint p \overbrace{\left(q_b(s; \xi_2, \xi_3) + q_0(\xi_2, \xi_3)\right)}^{q(s; \xi_2, \xi_3)} ds = 0.$$

- We can not take it for granted that the section does not twist when no moment is applied. So we add this additional consideration in our definition of shear center. We posit that the resultant twist angle must also be zero when the shear resultants act along the shear center:

$$\theta' = 0 \implies \oint \frac{q_b(s; \xi_2, \xi_3) + q_0(\xi_2, \xi_3)}{Gt} ds = 0$$

- Considering V_2, V_3 separately, we can get 3 equations in the 3 unknowns and can solve it.

2.2. Closed Sections: The Shear Center

Torsion of Thin-Walled Sections

One possible sequence of analysis is this (for shear):

1. We choose some convenient point as origin, say $\mathcal{O}$.
2. We first obtain the "baseline" shear flow $q_b(s)$ using some arbitrary starting point for the shear flow integral.
3. We estimate q_0 by requiring zero twist:

$$\oint \frac{q_b(s) + q_0}{Gt} ds = 0 \implies \boxed{q_0 = -\frac{\oint \frac{q_b(s)}{Gt} ds}{\oint \frac{1}{Gt} ds}}.$$

4. We write down the resultant moment as

$$\oint p(q_b(s) + q_0(s)) ds = V_2(-\xi_3) + V_3(\xi_2).$$

The shear center coordinates (ξ_2, ξ_3) are estimated by comparing the coefficients of V_2 & V_3 in the above.

2.2. Closed Sections: The Shear Center

Torsion of Thin-Walled Sections

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1. We choose some convenient point as origin, say $\mathcal{O}$.
2. We first obtain the "baseline" shear flow $q_b(s)$ using some arbitrary starting point for the shear flow integral.
3. We estimate q_0 by

Question: We never required the zero twist condition for open sections. Does this mean open sections can undergo twisting even when $M_1 = 0$?

$$\frac{\frac{q_b(s)}{Gt}ds}{\frac{1}{Gt}ds}.$$

4. We write down the resultant moment as

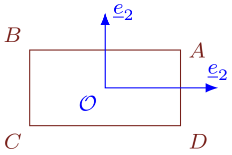
$$\oint p(q_b(s) + q_0(s))ds = V_2(-\xi_3) + V_3(\xi_2).$$

The shear center coordinates (ξ_2, ξ_3) are estimated by comparing the coefficients of V_2 & V_3 in the above.

2.2. Closed Sections: Tutorial on Rectangular Closed Sections

Torsion of Thin-Walled Sections

- Consider this rectangular Section:



- We will write out the warping quantity $\frac{1}{2\mathcal{A}\theta'}(u(s) - u(0)) = \frac{\delta_{OS}(s)}{\delta} - \frac{\mathcal{A}_{OS}(s)}{\mathcal{A}}$ as a table in the following fashion:

Section	$\delta_{OS}(s)$	$\mathcal{A}_{OS}(s)$	$\frac{\delta_{OS}(s)}{\delta} - \frac{\mathcal{A}_{OS}(s)}{\mathcal{A}}$	$\frac{1}{2\mathcal{A}\theta'}(u_{end} - u_{start})$
A→B	$\frac{\frac{a}{2}-X_2}{Gt}$	$\frac{b}{4}\left(\frac{a}{2} - X_2\right)$	$\frac{a-b}{4a(a+b)}\left(\frac{a}{2} - X_2\right)$	$\frac{a-b}{4(a+b)}$
B→C	$\frac{\frac{b}{2}-X_3}{Gt}$	$\frac{a}{4}\left(\frac{b}{2} - X_3\right)$	$-\frac{a-b}{4b(a+b)}\left(\frac{b}{2} - X_3\right)$	$-\frac{a-b}{4a(a+b)}$
C→D	$\frac{\frac{a}{2}+X_2}{Gt}$	$\frac{b}{4}\left(\frac{a}{2} + X_2\right)$	$\frac{a-b}{4a(a+b)}\left(\frac{a}{2} + X_2\right)$	$\frac{a-b}{4a(a+b)}$
D→A	$\frac{\frac{b}{2}+X_3}{Gt}$	$\frac{a}{4}\left(\frac{b}{2} + X_3\right)$	$-\frac{a-b}{4a(a+b)}\left(\frac{b}{2} + X_3\right)$	$-\frac{a-b}{4a(a+b)}$

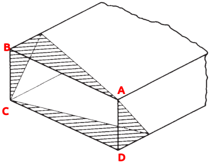
2.2. Closed Sections: Tutorial on Rectangular Closed Section

Torsion of Thin-Walled Sections

- Letting u_A be some constant, we have the following:

$$u_B = u_A + 2\mathcal{A}\theta' \frac{a-b}{4(a+b)}, \quad uC = u_A, \quad u_D = u_A + 2\mathcal{A}\theta' \frac{a-b}{4(a+b)}.$$

- In each member, the warping function is distributed linearly in each member such that the warped shape looks like:



Figures from Megson 2013

- Imposing zero net translation of section we get,

$$\oint u(s) ds = u_A 2(a+b) + \frac{a-b}{4} := 0 \implies u_A = -\frac{a-b}{8(a+b)}.$$

2.2. Closed Sections: Tutorial on Rectangular Closed Section

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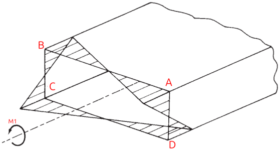

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2.3. Open Sections

Torsion of Thin-Walled Sections

- We will invoke the thin-strip idealization for this. The main results from the idealization are:

$$\phi = -G\theta' \left(X_2^2 - \frac{t^2}{4}\right); \quad M_1 = G\frac{ht^3}{3}\theta';$$
$$\sigma_{12} = 0, \quad \sigma_{13} = 2GX_2\theta', \quad u_1 = \theta' X_2 X_3.$$

- For general thin-walled sections, the torsion constant J is generalized as,

$$J = \frac{1}{3}\int_{\mathcal{S}} t^3 ds, \quad \text{s.t.} \quad M_1 = GJ\theta'.$$

Thin Section Kinematics

The kinematics of thin sections can be given as

$$u_s = -X_n\theta + vX_{2,s} + wX_{3,s} \xrightarrow{X_n=-p} p\theta + vX_{2,s} + wX_{3,s}$$
$$u_n = X_s\theta - vX_{3,s} + wX_{2,s} \xrightarrow{X_s=s} s\theta - vX_{3,s} + wX_{2,s}.$$

2.3. Open Sections: Warping

Torsion of Thin-Walled Sections

- Along the centerline $\sigma_{1n} = \sigma_{1s} = 0$ (**Note**: shear flow is zero under the idealization!). So we have (on the centerline),

$$\gamma_{1s} = 0 = u_{1,s} + u_{s,1} = u_{1,s} + p\theta',$$

where p is the perpendicular distance to the point on the skin. This can be integrated to

$$u_1(s) - u_1(0) = -\theta' \int_0^s p ds = -2\theta' \mathcal{A}_{Os}(s).$$

- $u_1(0)$ can be fixed based on enforcing the zero straight-stress ($\sigma_{11} = 0$, $\sigma_{11} \propto u_1$) assumption which leads to

$$\int_\Gamma u_1(s) ds = 0 \implies u_1(0) = \frac{1}{|\Gamma|} 2\theta' \int_\Gamma \mathcal{A}_{Os}(s) ds.$$

$|\Gamma|$ is the total *circumference*.

2.3. Open Sections

Torsion of Thin-Walled Sections

- For points off of the centerline, we consider $\sigma_{1n} = 0$, which implies,

$$\gamma_{1n} = u_{1,n} + u_{n,1} = u_{1,n} + s\theta' = 0 \implies u_{1,n} = -s\theta',$$

  where s is the position of the point along the skin (measured relative to the central line).

- This can be integrated to

$$u_1 = -\theta' ns + u_1(n = 0),$$

  where n is the position with respect to the centerline along $\underline{e}_n$.
  - Note that while $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$, we have $\underline{e}_s \times \underline{e}_n = \underline{e}_1$. Hence the negative sign in comparison to the thin-strip expression.

- $u_1(n = 0) = u_0 - 2\theta' \mathcal{A}_{Os}(s)$ from the centerline considerations above.

2.3. Open Sections

Torsion of Thin-Walled Sections

- In summary, the warping can be written in terms of section-local coordinates as,

$$\boxed{u_1 = \underbrace{u_0 - 2\mathcal{A}_{Os}(s)\theta'}_{u_1(n=0)} - \theta' n s}.$$

- The first term in the above, representing center-line warping, is known as **primary warping**, and the second term, representing section warping, is known as **secondary warping**.
- For sufficiently thin sections, the latter is usually neglected.

2.3. Open Sections Tutorial: C-Section

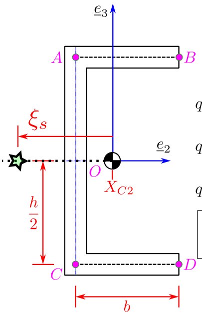
Torsion of Thin-Walled Sections

- Let us consider the C-Section from Module 4.
- We will shift the origin to the shear center and consider the integrals.
- The torsional rigidity is given by:

$$GJ = \frac{Gt^3}{3}\int_\Gamma ds = G\frac{t^3(h+2b)}{3}$$

- Warping is worked out as,

	$\mathcal{A}_{Os}(s)$	end
$B\to A$	$\frac{h}{2}(b+\xi_s-X_2)$	$\frac{bh}{2}$
$A\to C$	$-\xi_s(\frac{h}{2}-X_3)$	$-\xi_s h$
$C\to D$	$\frac{h}{2}(X_2-\xi_s)$	$\frac{bh}{2}$



$$I_{22} \approx \frac{(h^3+6bh^2)t}{12} + \mathcal{O}(t^2)$$

$$q_{BA}(X_2) = -\frac{htV_3}{2I_{22}}(b-X_2)$$

$$q_{AC}(X_3) = -\frac{htV_3}{2I_{22}}\left(b+\frac{h}{4}\right) + \frac{tV_3}{2I_{22}}X_3^2$$

$$q_{CD}(X_2) = -\frac{htV_3}{2I_{22}}(b-X_2)$$

$$M_1 = \oint pq ds = -\frac{b^2h^2tV_3}{4I_{22}} := -V_3\xi_s$$

$$\xi_s \approx \frac{3b^2}{h+6b} + \mathcal{O}(t)$$

- Using the table we can write:

$$u_b(s) = -\theta'\begin{cases} \frac{h}{2}(b+\xi_s-X_2) & B\to A\\ \frac{bh}{2}-\xi_s(\frac{h}{2}-X_3) & A\to C\\ \frac{bh}{2}-\frac{h}{2}(X_2-2\xi_s) & C\to D\end{cases}.$$

2.3. Open Sections Tutorial: C-Section I

Torsion of Thin-Walled Sections

- Since warping is linear in each segment, it is sufficient to look at points A, B, C, D to visualize it completely.
- Here we have:

$$u_B = 0, \quad u_A = -\theta' \frac{bh}{2}, \quad u_C = -\theta' \frac{bh}{2}\left(1 - 2\frac{\xi_s}{b}\right), \quad u_D = -\theta' \frac{bh}{2}\left(2 - 2\frac{\xi_s}{b}\right).$$

- The integral of warping over the complete section comes out to be

$$\int_\Gamma u_b ds = -\theta' \left(\frac{b^2 h}{4} + \frac{bh^2}{2}(1 - \frac{\xi_s}{b}) + \frac{b^2 h}{4}(3 - 4\frac{\xi_s}{b})\right)$$
$$= -\theta' \frac{bh(h+2b)}{2}\left(1 - \frac{\xi_s}{b}\right)$$

- Requiring $\int_\Gamma u ds = 0$ implies, since $u = u_b + u_0$,

$$u_0 = -\frac{1}{|\Gamma|}\int_\Gamma u_b ds = \theta' \frac{bh}{2}\left(1 - \frac{\xi_s}{b}\right).$$

- Notice that u_o is exactly the negative of the warping at the mid-point between points A and C (marked $\mathcal{O}$ in figure). The warping at this point is given by:

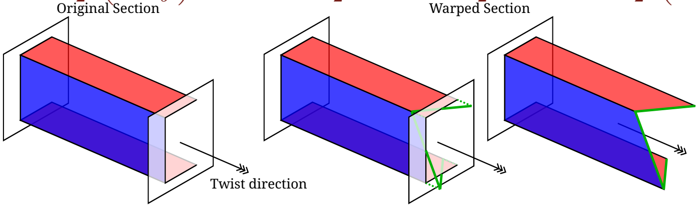
$$u_{\mathcal{O}} = \frac{u_A + u_C}{2} = -\theta' \frac{bh}{2}\left(1 - \frac{\xi_s}{b}\right).$$

2.3. Open Sections Tutorial: C-Section II

Torsion of Thin-Walled Sections

- This implies that the section warps in such a manner as to ensure that point $\mathcal{O}$ does not move at all ($u_o + u_{\mathcal{O}} = 0$).
- Finally the warping function at the corner points come out to be,

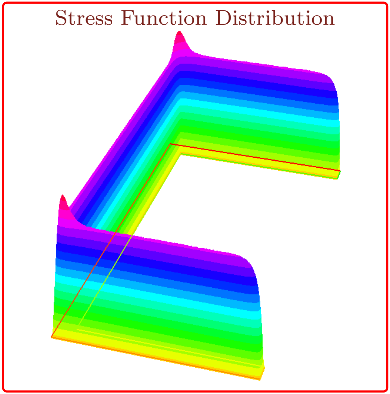
$$u_B = \theta' \frac{bh}{2}\left(1 - \frac{\xi_s}{b}\right), \quad u_A = -\theta' \frac{\xi_s h}{2}, \quad u_C = \theta' \frac{\xi_s h}{2}, \quad u_D = -\theta' \frac{bh}{2}\left(1 - \frac{\xi_s}{b}\right)$$

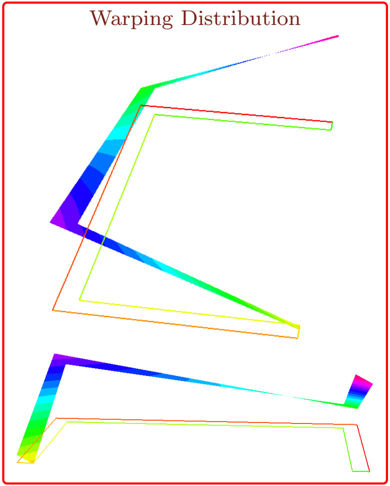


2.3. Open Sections Tutorial: C-Section

Torsion of Thin-Walled Sections

- Let us also illustrate the above with exact (numerical) results.





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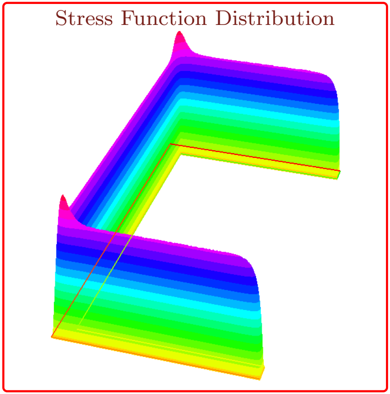

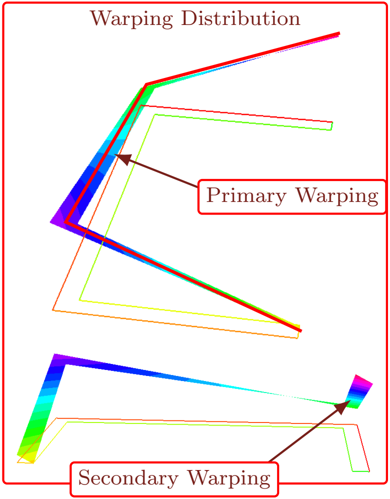

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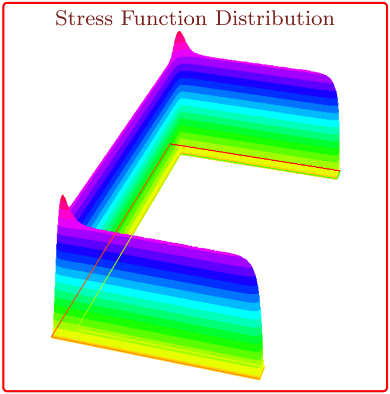

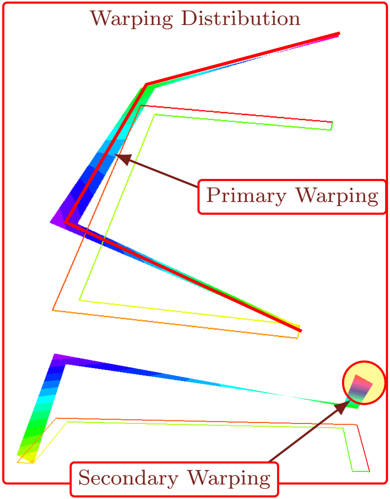

2.4. Combined Cells

Torsion of Thin-Walled Sections

- It is instructive to now take stock of what we have obtained so far. The moment-twist relationship is generically written by

$$M_1 = GJ\theta',$$

with J being the torsion constant.

Solid Sections

$$J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA$$

Closed Sections

$$J = \frac{4t\mathcal{A}^2}{|\Gamma|}$$

Open Sections

$$J = \frac{t^3|\Gamma|}{3}$$

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Solid Sections	Closed Sections	Open Sections
$J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA$	$J = \frac{4t\mathcal{A}^2}{\lvert\Gamma\rvert}$	$J = \frac{t^3\lvert\Gamma\rvert}{3}$

Let us consider the implications on a **Circular Section** of radius R.

Solid Section $J_s = I_{11} = \frac{\pi R^4}{2}$.

Closed Section $J_c = \frac{4t\times(\pi R^2)^2}{2\pi R} = 2\pi R^3 t$

Open Section $J_o = \frac{t^3}{3} 2\pi R = \frac{2\pi}{3} R t^3$

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Solid Sections	Closed Sections	Open Sections
$J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA$	$J = \dfrac{4t\mathcal{A}^2}{\lvert\Gamma\rvert}$	$J = \dfrac{t^3\lvert\Gamma\rvert}{3}$

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Open Section $J_o = \frac{t^3}{3}2\pi R = \frac{2\pi}{3}Rt^3$

For $J_c = J_s$, we need $t = \frac{1}{4}R = 0.25R$.

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Solid Sections	Closed Sections	Open Sections
$J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA$	$J = \frac{4t\mathcal{A}^2}{\lvert\Gamma\rvert}$	$J = \frac{t^3\lvert\Gamma\rvert}{3}$

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Open Section $J_o = \frac{t^3}{3}2\pi R = \frac{2\pi}{3}Rt^3$

For $J_o = J_s$, we need $t = \sqrt[3]{\frac{3}{4}}R \approx 0.91R.$

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Torsion of Thin-Walled Sections

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Solid Sections	Closed Sections	Open Sections
$J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA$	$J = \frac{4t\mathcal{A}^2}{\lvert\Gamma\rvert}$	$J = \frac{t^3\lvert\Gamma\rvert}{3}$

Let us consider the implications on a **Circular Section** of radius R.

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Open Section $J_o = \frac{t^3}{3} 2\pi R = \frac{2\pi}{3} R t^3$

For a given thickness,

$$\frac{J_o}{J_c} = \frac{1}{3}\left(\frac{t}{R}\right)^2 = \mathcal{O}(t^2).$$

2.4. Combined Cells

Torsion of Thin-Walled Sections

- It is instructive to now take stock of what we have obtained so far. The moment-twist relationship is generically written by

$$M_1 = GJ\theta',$$

with J being the torsion constant.

Solid Sections

$$J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA$$

$$|\Gamma|$$

Open Sections

$$J = \frac{t^3|\Gamma|}{3}$$

So open sections can safely be ignored for torsion calculations in the combined context!

Let us consider the implications on a **Circular Section** of radius R.

Solid Section $J_s = I_{11} = \frac{\pi R^4}{2}$.

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Solid Sections		Open Sections
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So open sections can safely be ignored for torsion calculations in the combined context!

For shear, we can follow exactly the same procedure as in module 4 for combined sections.

Let us consider the im

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For a given thickness,

$$\frac{J_o}{J_c} = \frac{1}{3}\left(\frac{t}{R}\right)^2 = \mathcal{O}(t^2).$$

3. Summary of Final Expressions

Solid Sections

$$J = I_{11} + \int_{\mathcal{S}} X_2 \psi_{,3} - X_3 \psi_{,2} dA$$

$$u_1 = \theta' \psi(X_2, X_3)$$

Thin Strip Idealization

$$J = \frac{ht^3}{3}$$

$$u_1 = X_2 X_3 \theta'$$

Closed Sections

$$GJ = \frac{4\mathcal{A}^2}{\delta}$$

$$u_1(s) = u_0 + 2\mathcal{A}\theta' \left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}} \right)$$

Open Sections

$$GJ = \frac{1}{3} \int_{\mathcal{S}} Gt^3 ds$$

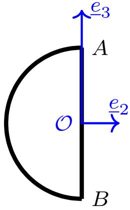
$$u_1(s) = u_0 - 2\theta' \mathcal{A}_{Os}(s) - \theta' ns$$

$$\delta_{Os}(s) = \int_0^s \frac{1}{Gt} dx; \quad \mathcal{A}_{Os}(s) = \frac{1}{2} \int_0^s p dx$$

4. Example: Shear Center of Closed Section

- Let us consider the "inverted D" section with radius R as shown.
- The shear center lies on the $\underline{e}_2$ axis due to symmetry so we only consider the shear flow distribution due to resultant $V_3\underline{e}_3$.
- So we have, $\boxed{q(s) = q_0 - \frac{tV_3}{I_{22}} \int\limits_0^s X_3 ds}$.
- Starting integration at A we have,

$$q(s) = q_0 + \underbrace{\frac{tV_3}{2I_{22}} \begin{cases} 2R^2\cos\theta & A \to B \\ R^2 - X_3^2 & B \to A \end{cases}}_{q_b(s)}$$



- Enforcing **zero twist** we get,

$$\oint q(s)ds = q_0|\Gamma| + \oint q_b(s)ds = q_0(\pi+2)R - \frac{4R^3tV_3}{3I_{22}} = 0.$$

$$\implies \boxed{q_0 = \frac{4R^2tV_3}{3(\pi+2)I_{22}}}.$$

4. Example: Shear Center of Closed Section

- Now we have the complete shear flow distribution:

$$q(s) = \frac{4R^2t}{3(\pi+2)I_{22}}V_3 + \frac{tV_3}{2I_{22}}\begin{cases}2R^2\cos\theta & A\to B\\ R^2 - X_3^2 & B\to A\end{cases}.$$

- We now take the moment about the point $\mathcal{O}$ and write it as follows. Note that the shear flow on the vertical member $B \to A$ does not contribute to moment about $\mathcal{O}$.

$$M_{\mathcal{O}} = q_0 \underbrace{\oint p ds}_{2\mathcal{A}} + \oint p q_b ds = \pi R^2 q_0 + \frac{R^2 t V_3}{I_{22}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} R \times \cos\theta \times R d\theta$$

$$= \frac{4\pi R^4 t}{3(\pi+2)I_{22}}V_3 - \frac{2R^4t}{I_{22}}V_3 = -\frac{2R^4t}{3I_{22}}\frac{(\pi+6)}{(\pi+2)}V_3 \equiv \xi_2 V_3.$$

- The second moment of area of the section I_{22} is written as $I_{22} = \frac{3\pi+4}{6}R^3t$.

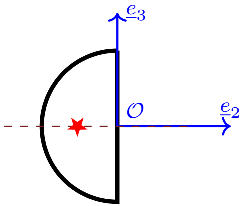
4. Example: Shear Center of Closed Section

- The shear center coordinate ξ_2 simplifies as,

$$\xi_2 = -\frac{2R^4t}{3I_{22}}\frac{(\pi+6)}{(\pi+2)} = -\frac{2R^4t}{3}\frac{6}{3\pi+4}\frac{1}{R^3t}\frac{(\pi+6)}{(\pi+2)}$$
$$= -\frac{4(\pi+6)}{(3\pi+4)(\pi+2)}R \approx -0.53R.$$

which shows that the shear center is approximately at the mid-point of the horizontal, **inside the section**.

- The shear center is marked with a red star in this figure:



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[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on pp. 2, 52, 53).