AS3020: Aerospace Structures

Module 4: Bending of Beam-Like Structures

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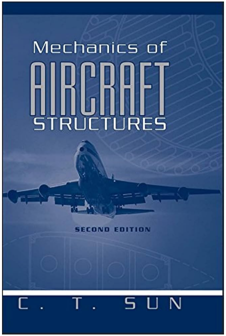
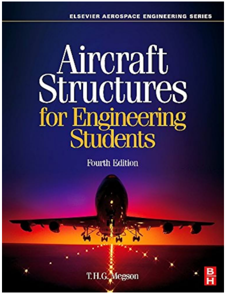


Chapters 4-5 in Sun (2006)



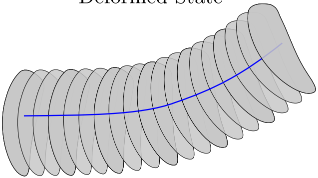
Chapters 16-20 in Megson (2013)

1. Unsymmetrical Bending

Assumptions

1. Plane sections remain planar.
2. Sections remain perpendicular to neutral axis: $\gamma_{12} = \gamma_{13} = 0$.
3. Plane Stress: $\sigma_{22} = \sigma_{33} = 0$.

Deformed State



1. Rigid Section Displacement Field

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ v(X_1) \\ w(X_1) \end{bmatrix} + \underbrace{\begin{bmatrix} X_3\theta_2 - X_2\theta_3 \\ 0 \\ 0 \end{bmatrix}}_{\underset{\sim}{\theta} \times \underset{\sim}{X}}$$

2. Zero Shear Strain Simplification

$$\gamma_{12} = \gamma_{13} = 0 \implies \boxed{\theta_2 = -w', \quad \theta_3 = v'}$$

3. Plane Stress Constitution

$$\boxed{\sigma_{11} = E_Y \underline{E}_{11}}$$

$$\implies \mathcal{E}_{11} = u_{1,1} = \begin{bmatrix} X_3 & -X_2 \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

We shall develop the theory without the zero strain simplification first.

1.1. Axial Stress and its Moments

Unsymmetrical Bending

- The axial stress distribution is $\boxed{\sigma_{11} = E_Y \begin{bmatrix} X_3 & -X_2 \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}}$. The traction vector in the section is $\underline{t} = \sigma_{11}\underline{e_1} + \sigma_{12}\underline{e_2} + \sigma_{13}\underline{e_3}$.
- Considering just the axial component ($\sigma_{11}\underline{e_1}$), we write the overall axial force as the area integral (**zeroth moment**):

$$N_1 = \int_{\mathcal{A}} \sigma_{11} = E_Y \begin{bmatrix} \int_{\mathcal{A}} X_3 dA & -\int_{\mathcal{A}} X_2 dA \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

- Recall that we have already chosen then origin as the section centroid for expressing the rigid rotation displacement field, s.t. $\int_{\mathcal{A}} \underline{X}\, dA = \underline{0}$. Therefore $N_1 = 0$ for pure bending.
- Considering the moment due to the axial component ($d\underline{m} = (X_k\underline{e_k}) \times (\sigma_{11}\underline{e_1}\, dA)$ we have (**first moment**):

$$\begin{aligned} \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} &= \int_{\mathcal{A}} \begin{bmatrix} X_3 \\ -X_2 \end{bmatrix} \sigma_{11} dA = \int_{\mathcal{A}} E_Y \begin{bmatrix} X_3 \\ -X_2 \end{bmatrix} \begin{bmatrix} X_3 & -X_2 \end{bmatrix} dA \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix} \\ &= \int_{\mathcal{A}} E_Y \begin{bmatrix} X_3^2 & -X_2X_3 \\ -X_2X_3 & X_2^2 \end{bmatrix} dA \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}. \end{aligned}$$

For constant E_Y through section,

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = E_Y \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

Second Moments of Area

$$I_{22} = \int_{\mathcal{A}} X_3^2 dA$$

$$I_{33} = \int_{\mathcal{A}} X_2^2 dA$$

$$I_{23} = \int_{\mathcal{A}} X_2 X_3 dA$$

1.2. Axial Stress In Terms of Moments and Forces

Unsymmetrical Bending

- It is sometimes convenient to have the stress σ_{11} expressed in terms of its resultant moments instead of kinematic quantities like θ_2 and θ_3. So we will invert the relationship that we have to first get:

$$\begin{bmatrix}\theta_2'\\ \theta_3'\end{bmatrix} = \frac{1}{E_Y}\frac{1}{I_{22}I_{33}-I_{23}^2}\begin{bmatrix}I_{33} & I_{23}\\ I_{23} & I_{22}\end{bmatrix}\begin{bmatrix}M_2\\ M_3\end{bmatrix}.$$

- Stress simplifies as

$$\begin{aligned}\sigma_{11} &= E_Y\begin{bmatrix}X_3 & -X_2\end{bmatrix}\begin{bmatrix}\theta_2'\\ \theta_3'\end{bmatrix}\\ &= \frac{\begin{bmatrix}X_3 & -X_2\end{bmatrix}}{I_{22}I_{33}-I_{23}^2}\begin{bmatrix}I_{33} & I_{23}\\ I_{23} & I_{22}\end{bmatrix}\begin{bmatrix}M_2\\ M_3\end{bmatrix}.\end{aligned}$$

- **Observe that we have gotten to the above without requiring shear strains to be zero.**

1.3. Equilibrium Equations

Unsymmetrical Bending

- We shall invoke and simplify the equilibrium equations in an integral sense in the presence of transverse forces only (**stress assumptions**: $\sigma_{22} = \sigma_{33} = \sigma_{23} = 0$).

$$\sigma_{1j,j} = 0 \implies \int_{\mathcal{A}} \sigma_{1j,j} dA = 0$$

$$\sigma_{12,1} + f_2 = 0 \implies \int_{\mathcal{A}} \sigma_{12,1} dA + \int_{\mathcal{A}} f_2 dA = 0$$

$$\sigma_{13,1} + f_3 = 0 \implies \int_{\mathcal{A}} \sigma_{13,1} dA + \int_{\mathcal{A}} f_3 dA = 0$$

- $\int_{\mathcal{A}} \sigma_{1j,j} dA$ is simplified as

$$\int_{\mathcal{A}} \sigma_{1j,j} dA = \int_{\mathcal{A}} \sigma_{11,1} dA + \overbrace{\int_{\mathcal{A}} \sigma_{12,2} + \sigma_{13,3} dA}^{\text{Gauss divergence in 2D: } \cancel{\int_{\mathcal{A}} \sigma_{1j} n_j dA}} = N_{1,1}$$

where $\underline{n} = n_j \underline{e_j}$ is the **outward pointing normal** on the boundary of the section ($n_1 = 0$).

- $\sigma_{1j} n_j$ is the $\underline{e_1}$ component of the traction vector on the free surface. By definition this has to be zero, so we have $N_{1,1} = 0$.
- Defining the shearing forces as $V_2 = \int_{\mathcal{A}} \sigma_{12} dA$ and $V_3 = \int_{\mathcal{A}} \sigma_{13} dA$, the second two equations can be read as:

$$\boxed{V_{2,1} + F_2 = 0}, \qquad \boxed{V_{3,1} + F_3 = 0}.$$

1.3. Equilibrium Equations

Unsymmetrical Bending

- In order to relate the different stresses, we invoke $M_2 = X_3\sigma_{11}$ and $M_3 = -X_2\sigma_{11}$ now.
- We first pre-multiply $\sigma_{1j,j}$ by X_3 **and then integrate** over the section:

$$\int_{\mathcal{A}} X_3\sigma_{11,1} dA + \int_{\mathcal{A}} X_3\sigma_{12,2} + X_3\sigma_{13,3} dA = M_{2,1} + \int_{\mathcal{A}} (X_3\sigma_{12})_{,2} + (X_3\sigma_{13})_{,3} - \sigma_{13} dA$$

$$\int_{\mathcal{A}} (X_3\sigma_{12})_{,2} + (X_3\sigma_{13})_{,3} dA = \cancel{\int_{\partial\mathcal{A}} X_3\sigma_{1k}n_k d\ell} \implies M_{2,1} - \int_{\mathcal{A}} \sigma_{13} dA = \boxed{M_{2,1} - V_3 = 0}.$$

- Next we pre-multiply $\sigma_{1j,j}$ by X_2 and repeat the same:

$$\int_{\mathcal{A}} X_2\sigma_{11,1} dA + \int_{\mathcal{A}} X_2\sigma_{12,2} + X_3\sigma_{13,3} dA = -M_{3,1} + \int_{\mathcal{A}} (X_2\sigma_{12})_{,2} - \sigma_{12} + (X_2\sigma_{13})_{,3} dA$$

$$\int_{\mathcal{A}} (X_2\sigma_{12})_{,2} + (X_2\sigma_{13})_{,3} dA = \cancel{\int_{\partial\mathcal{A}} X_2\sigma_{1k}n_k d\ell} \implies M_{3,1} + \int_{\mathcal{A}} \sigma_{12} dA = \boxed{M_{3,1} + V_2 = 0}.$$

- We are finally left with 4 equilibrium equations applicable for beam theory:

$$\boxed{V_{2,1} + F_2 = 0}, \quad \boxed{V_{3,1} + F_3 = 0}, \quad \boxed{M_{2,1} - V_3 = 0}, \quad \boxed{M_{3,1} + V_2 = 0}.$$

These are independent of any kinematic assumptions that we may make.

1.3. Equilibrium Equations

Unsymmetrical Bending

- In order to relate the different stresses, we invoke $M_2 = X_3\sigma_{11}$ and $M_3 = -X_2\sigma_{11}$ now.
- We first pre-multiply $\sigma_{1j,j}$ by X_3 **and then integrate** over the section:

$$\int X_3\sigma_{11,1} dA + \int X_3\sigma_{12,2} + X_3\sigma_{13,3} dA = M_{2,1} + \int (X_3\sigma_{12})_{,2} + (X_3\sigma_{13})_{,3} - \sigma_{13} dA$$

$$\int_{\mathcal{A}} (X \ldots \quad \ldots - V_3 = 0 .$$

Transverse Force-Bending Moment Relationship

$$V_{2,1} + F_2 = 0, \quad V_3,1 + F_3 = 0, \quad M_{2,1} - V_3 = 0, \quad M_{3,1} + V_2 = 0$$

$$\implies \begin{bmatrix} M_{3,11} \\ -M_{2,11} \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}.$$

- N…

$$\int_{\mathcal{A}} X_2\sigma_{11,1} dA + \int_{\mathcal{A}} X_2\sigma_{12,2} + X_3\sigma_{13,3} dA = -M_{3,1} + \int_{\mathcal{A}} (X_2\sigma_{12})_{,2} - \sigma_{12} + (X_2\sigma_{13})_{,3} dA$$

$$\int_{\mathcal{A}} (X_2\sigma_{12})_{,2} + (X_2\sigma_{13})_{,3} dA = \cancel{\int_{\partial\mathcal{A}} X_2\sigma_{1k} n_k d\ell} \implies M_{3,1} + \int_{\mathcal{A}} \sigma_{12} dA = \boxed{M_{3,1} + V_2 = 0}.$$

- We are finally left with 4 equilibrium equations applicable for beam theory:

$$\boxed{V_{2,1} + F_2 = 0}, \quad \boxed{V_{3,1} + F_3 = 0}, \quad \boxed{M_{2,1} - V_3 = 0}, \quad \boxed{M_{3,1} + V_2 = 0}.$$

These are independent of any kinematic assumptions that we may make.

1.4. Equations of Motion in Terms of Displacement

Unsymmetrical Bending

- The moments are related to the kinematics through

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = E_Y \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

- For the zero shear strain case ($\theta_2 = -w'$, $\theta_3 = v'$) the equilibrium equations simplify in the following manner:

$$\begin{bmatrix} M_{3,11} \\ -M_{2,11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} M_{2,11} \\ M_{3,11} \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix},$$

$$E_Y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v'''' \\ w'''' \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix},$$

$$\implies E_Y \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix} \begin{bmatrix} v'''' \\ w'''' \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix},$$

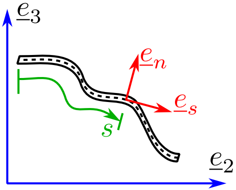
or in more compact notation,

$$\boxed{E_Y \underset{\approx}{I} \underset{\sim}{V}'''' = \underset{\sim}{F}}, \quad \underset{\approx}{I} = \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix}, \quad \underset{\sim}{V} = \begin{bmatrix} v \\ w \end{bmatrix}.$$

(Recall that the planar symmetric bending equation is $EIv'''' = F$)

2. Shear Stress and Flow in Sections

Thin Section: Plane Stress Assumption



- Now we shall pursue the equilibrium equations for thin-walled sections.
- We define the above section-local coordinate system and transform the elasticity equations to $\sigma_{11,1} + \sigma_{1s,s} + \sigma_{1n,n} = 0$. We integrate this along the thickness:

$$\int_{X_n - \frac{t}{2}}^{X_n + \frac{t}{2}} \sigma_{11,1} dX_n + \int \sigma_{1s,s} dX_n + \cancelto{0}{\int \sigma_{1n,n} dX_n} = 0$$

- σ_{1n} has to be zero on the surfaces with normal $\underline{e}_n$ since these are "free" surfaces; so the last integral goes to zero. The integral above simplifies (for constant thickness along s) to:

$$t\sigma_{11,1} + \int \sigma_{1s,s} dX_n = 0 \implies \boxed{t\sigma_{11,1} + q_{,s} = 0},$$

  where we define **shear flow** q, a new quantity that is basically the integral of the transverse shear stress along the thickness: $\boxed{q(s) = \int \sigma_{1s} dX_n}$.

2.1. Shear Flow Distribution

Shear Stress and Flow in Sections

- The stress distribution is written as:

$$\sigma_{11} = \frac{\begin{bmatrix} X_3 & -X_2 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \end{bmatrix}.$$

- Differentiating this we get:

$$\sigma_{11,1} = \frac{\begin{bmatrix} X_3 & -X_2 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix} \begin{bmatrix} M_{2,1} \to V_3 \\ M_{3,1} \to -V_2 \end{bmatrix}$$

$$\implies \sigma_{11,1} = \frac{\begin{bmatrix} X_2 & X_3 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}.$$

- Substituting this in $t\sigma_{11,1} + q_{,s} = 0$ we have,

You should be able to remember this formula!

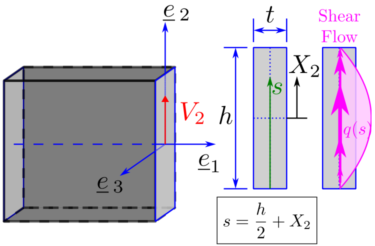
$$\frac{dq}{ds} = -\frac{\begin{bmatrix} tX_2 & tX_3 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}.$$

- Integrating this from some point we designate as $s = 0$, we have

$$\boxed{q(s) - q_0 = -\frac{\begin{bmatrix} \int_0^s tX_2 ds & \int_0^s tX_3 ds \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}}.$$

2.1.1. Shear Flow Distribution: The Simple Rectangular Section

- Consider the rectangular section with height h and thickness t:



$$q(s) = -\frac{V_2}{I_{33}} \int_0^s t X_2 ds = -\frac{tV_2}{I_{33}} \int_{-\frac{h}{2}}^{X_2} X_2 dX_2$$

$$= -\frac{tV_2}{2I_{33}} (X_2^2 - \frac{h^2}{4})$$

- Remember that V_2 is NOT any externally **applied force**. It is merely **the resultant of all the shear stresses in the section**.
- We are asking the question: **what SHOULD be the distribution of shear stresses (flow) so that their resultant is V_2?** It is incorrect to think that $q(s)$ is balancing out V_2.

2.1.2. Shear Flow Distribution: The "C" Section

Shear Stress and Flow in Sections

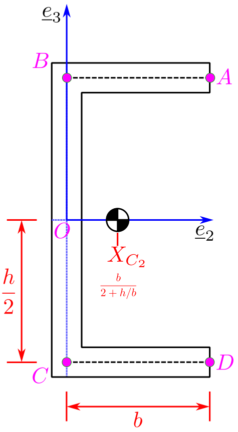
- Let us consider a "C" section with uniform thickness t. **Out of convenience we shall use a non-centroidal origin, so the shear flow expression is written as**

$$q(s) - q_0 = -\frac{tV_2}{I_{33}} \overbrace{\int_0^s (X_2 - X_{C_2})ds}^{z_2(s)} - \frac{tV_3}{I_{22}} \overbrace{\int_0^s X_3 ds}^{z_3(s)}.$$

- Doing shear flow calculations can get confusing because of the running integral. A nice way to keep things organized is to chart up a table and start filling it up:

	$s(X_2, X_3)$	X_2	X_3	$z_2 - z_{20}$	$z_3 - z_{30}$
A→B	$b - X_2$	$b - s$	$\frac{h}{2}$		
B→C	$b + \frac{h}{2} - X_3$	0	0		
C→D	$b + h + X_2$	$s - (b + h)$	$\frac{h}{2}$		

- Our task now boils down to filling this table carefully and then substituting in the equation above.

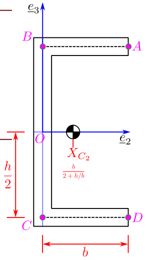


2.1.2. Shear Flow Distribution: The "C" Section

Shear Stress and Flow in Sections

- Let us first consider the case of having only V_2 and setting $V_3 = 0$, where the shear flow gets written as $q(s) = -\frac{tV_2}{I_{33}} z_2(s)$.

	$s(X_2, X_3)$	X_2	X_3	$z_2 - z_{20}$	$z_{21} - z_{20}$
A→B	$b - X_2$	$b - s$	$\frac{h}{2}$	$-\frac{(X_2-b)}{2}(X_2 + \frac{h}{2+\frac{h}{b}})$	$\frac{h}{2} X_{C_2}$
B→C	$b + \frac{h}{2} - X_3$	0	0	$X_{C_2}(X_3 - \frac{h}{2})$	$-hX_{C_2}$
C→D	$b + h + X_2$	$s - (b + h)$	$\frac{h}{2}$	$\frac{X_2}{2}(X_2 - 2X_{C_2})$	$\frac{h}{2} X_{C_2}$



2.1.2. Shear Flow Distribution: The "C" Section

Shear Stress and Flow in Sections

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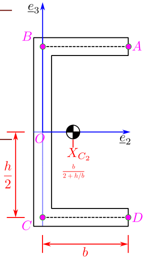
	$s(X_2, X_3)$	X_2	X_3	$z_2 - z_{20}$	$z_{21} - z_{20}$
A→B	$b - X_2$	$b - s$	$\frac{h}{2}$	$-\frac{(X_2-b)}{2}\left(X_2 + \frac{h}{2+\frac{h}{b}}\right)$	$\frac{h}{2} X_{C_2}$
B→C	$b + \frac{h}{2} - X_3$	0	0	$X_{C_2}(X_3 - \frac{h}{2})$	$-hX_{C_2}$
C→D	$b + h + X_2$	$s - (b+h)$	$\frac{h}{2}$	$\frac{X_2}{2}(X_2 - 2X_{C_2})$	$\frac{h}{2} X_{C_2}$



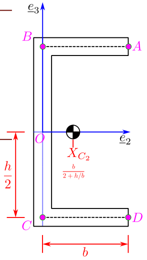
- Since $z_A = 0$, adding the last column cumulatively will give the value of $z(s)$. So we have $z_B = \frac{h}{2} X_{C_2}$, $z_C = -\frac{h}{2} X_{C_2}$, and $z_D = 0$. ($z_D = 0$ should also be a verification check for you since this will go to zero only if everything else is correct)
- The shear flow distribution is quadratic in the $A \to B$ and $C \to D$ segments and linear in the $B \to C$ segment.

2.1.2. Shear Flow Distribution: The “C” Section

Shear Stress and Flow in Sections

- Let us first consider the case of having only V_2 and setting $V_3 = 0$, where the shear flow gets written as $q(s) = -\frac{tV_2}{I_{33}} z_2(s)$.

	$s(X_2, X_3)$	X_2	X_3	$z_2 - z_{20}$	$z_{21} - z_{20}$
A→B	$b - X_2$	$b - s$	$\frac{h}{2}$	$-\frac{(X_2-b)}{2}(X_2 + \frac{h}{2+\frac{h}{b}})$	$\frac{h}{2} X_{C_2}$
B→C	$b + \frac{h}{2} - X_3$	0	0	$X_{C_2}(X_3 - \frac{h}{2})$	$-hX_{C_2}$
C→D	$b + h + X_2$	$s - (b+h)$	$\frac{h}{2}$	$\frac{X_2}{2}(X_2 - 2X_{C_2})$	$\frac{h}{2} X_{C_2}$



- Since $z_A = 0$, adding the last column cumulatively will give the value of $z(s)$. So we have $z_B = \frac{h}{2} X_{C_2}$, $z_C = -\frac{h}{2} X_{C_2}$, and $z_D = 0$. ($z_D = 0$ should also be a verification check for you since this will go to zero only if everything else is correct)
- The shear flow distribution is quadratic in the $A \to B$ and $C \to D$ segments and linear in the $B \to C$ segment.

Intuition Note

On segments along the direction of the resultant, the shear flow varies quadratically in space. Perpendicular to the direction of the resultant, the shear flow varies linearly in space.

2.1.2. Shear Flow Distribution: The "C" Section

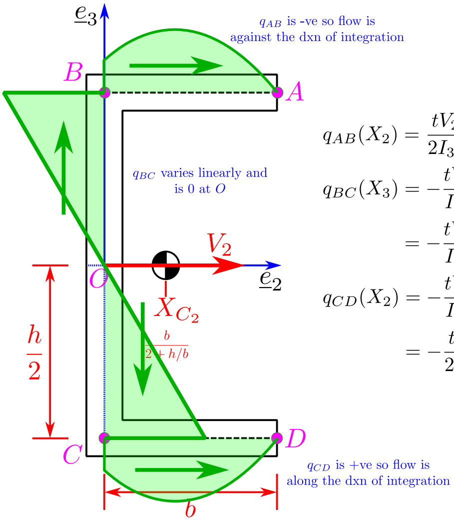
Shear Stress and Flow in Sections

- Le… th…

A→B…

B→C…

C→D…



Shear Flow

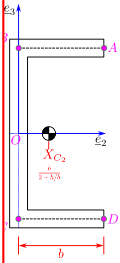
$$q_{AB}(X_2) = \frac{tV_2}{2I_{33}}(X_2 - b)(X_2 - \frac{h}{b}X_{C_2})$$

$$q_{BC}(X_3) = -\frac{tV_2}{I_{33}}\left(\frac{h}{2}X_{C_2} - X_{C_2}(X_3 - \frac{h}{2})\right)$$

$$= -\frac{tV_2}{I_{33}}X_{C_2}(h - X_3)$$

$$q_{CD}(X_2) = -\frac{tV_2}{I_{33}}\left(-\frac{h}{2}X_{C_2} + \frac{X_2}{2}(X_2 - 2X_{C_2})\right)$$

$$= -\frac{tV_2}{2I_{33}}\left(X_2^2 - X_{C_2}X_2 - hX_{C_2}\right)$$



- Si… $z($… als… ev…
- Th… seg…

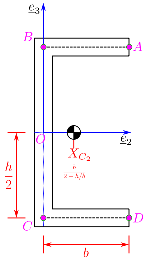
On seg… y in space. Perpendicular to the direction of the resultant, the shear flow varies linearly in space.

2.1.2. Shear Flow Distribution: The "C" Section

Shear Stress and Flow in Sections

- Now let us repeat the procedure for $V_2 = 0$ but $V_3 \neq 0$, so the shear flow gets written as $q(s) = -\frac{tV_3}{I_{22}} z_3(s)$.

	$s(X_2, X_3)$	X_2	X_3	$z_3 - z_{30}$	$z_{31} - z_{30}$
A→B	$b - X_2$	$b - s$	$\frac{h}{2}$	$-\frac{h}{2}(X_2 - b)$	$\frac{hb}{2}$
B→C	$b + \frac{h}{2} - X_3$	0	0	$-\frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$	0
C→D	$b + h + X_2$	$s - (b + h)$	$\frac{h}{2}$	$-\frac{h}{2} X_2$	$-\frac{hb}{2}$



- Once again we see that the flow is **quadratically varying along the resultant and linearly perpendicular to it**.
- We also ensure that the last column sums up to zero.

2.1.2. Shear Flow Distribution: The “C” Section

Shear Stress and Flow in Sections

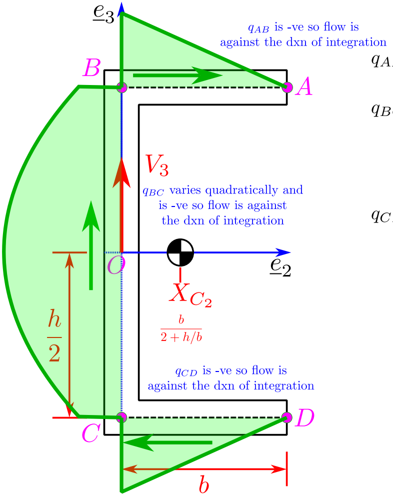
- No… ge…

A→

B→

C→

- Or… re…
- W…



Shear Flow

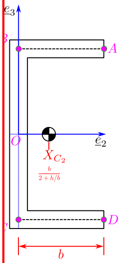
$$q_{AB}(X_2) = \frac{tV_3}{2I_{22}} h(X_2 - b)$$

$$q_{BC}(X_3) = -\frac{tV_3}{I_{22}} \left(\frac{hb}{2} - \frac{(X_3^2 - (\frac{h}{2})^2)}{2} \right)$$

$$= -\frac{tV_3}{2I_{22}} \left(hb + \frac{h^2}{4} - X_3^2 \right).$$

$$q_{CD}(X_2) = -\frac{tV_3}{I_{22}} \left(\frac{hb}{2} - \frac{h}{2} X_2 \right)$$

$$= \frac{tV_3}{2I_{22}} h(X_2 - b)$$



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Shear Stress and Flow in Sections

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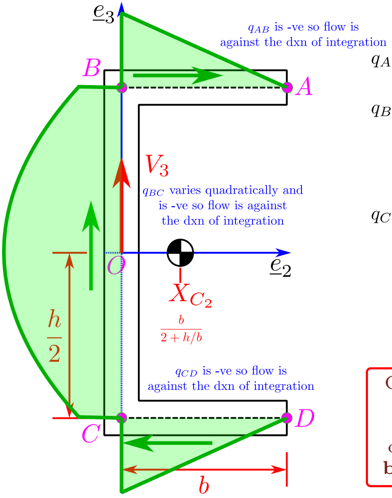

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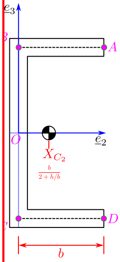
$$= \frac{tV_3}{2I_{22}} h(X_2 - b)$$

Given this flow, let us now work out the twisting moment M_1 that this results in about the origin O. (**Note that we have been assuming** $M_1 = 0$ **so far**)



2.1.2. Shear Flow Distribution: The "C" Section

Shear Stress and Flow in Sections

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A→
B→
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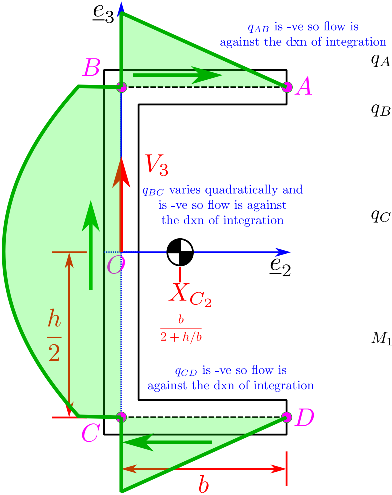

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$$= \frac{tV_3}{2I_{22}} h(X_2 - b)$$

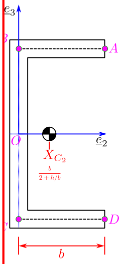
Twisting Moment M_1

$$M_1 = \int p\, q(s) ds$$

$$= \int_b^0 \frac{h}{2} q_{AB}(X_2)(-dX_2) + \int_0^b \frac{h}{2} q_{CD}(X_2) dX_2$$

$$= \frac{h^2 tV_3}{2I_{22}} \int_0^b (X_2 - b) dX_2 = -\frac{h^2 b^2 t}{4I_{22}} V_3$$

$$\boxed{M_1 = \xi_s V_3 => \xi_s = -\frac{h^2 b^2 t}{4I_{22}}}$$



2.1.2. Shear Flow Distribution: The “C” Section

Shear Stress and Flow in Sections

- Now let us repeat the procedure for $V_2 = 0$ but V_3
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	$s(X_2, X_3)$	X_2	X_3	z_3 −
A→B	$b - X_2$	$b - s$	$\frac{h}{2}$	$-\frac{h}{2}(X$
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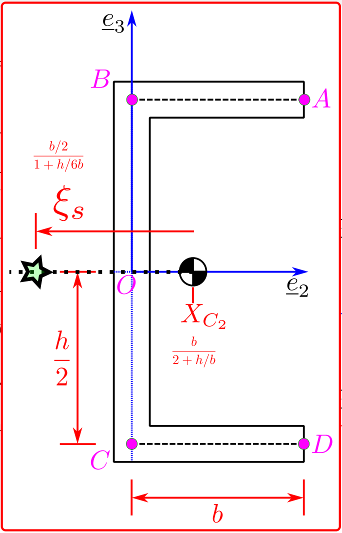
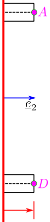
C→D	$b + h + X_2$	$s - (b + h)$	$\frac{h}{2}$	$-\frac{h}{2}$





- Once again we see that the flow is **quadratically v**
  resultant and linearly perpendicular to it.
- We also ensure that the last column sums up to zer
- After substituting for $I_{22} = \frac{h^2bt}{2}\left(1 + \frac{h}{6b}\right)$, we obta

$$\xi_s = -\frac{b/2}{1 + \frac{h}{6b}}$$

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Shear Stress and Flow in Sections

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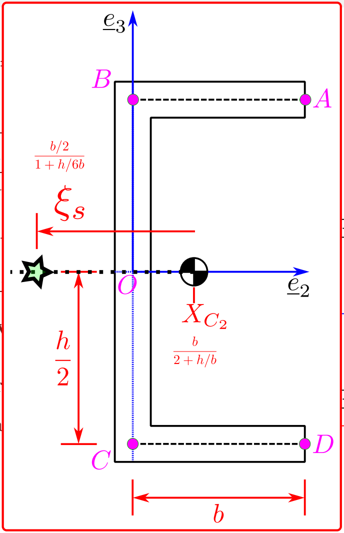
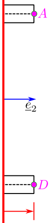
- We also ensure that the last column sums up to zer
- After substituting for $I_{22} = \frac{h^2bt}{2}\left(1 + \frac{h}{6b}\right)$, we obta

This point is known as the **Shear Centre** of the section.

$$\xi_s = -\frac{b/2}{1 + \frac{h}{6b}}$$





2.1.3. Shear Flow Distribution: The "I" Section

Shear Stress and Flow in Sections

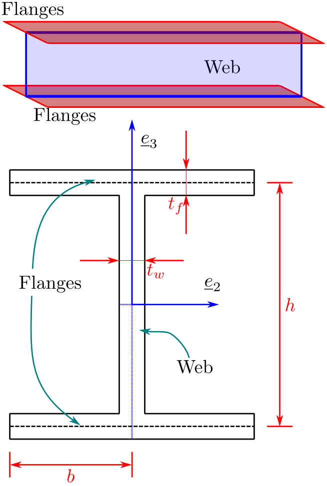
- Consider the shear distribution through an I-section as shown here
- The shear distribution looks like it is "flowing", with more "flow" occurring in the thin vertical web and less in the flanges.
- The second moment of area I_{22} sums up as,

$$I_{22} = \overbrace{\frac{h^3 t_w}{12}}^{web} + 2 \times \overbrace{\left(\frac{2bt_f^3}{12} + 2bt_f \times \frac{h^2}{4}\right)}^{flange} \approx \frac{h^3 t_w}{12} + h^2 b t_f.$$

- I_{33} sums up as,

$$I_{33} = \underbrace{\frac{h t_w^3}{12}}_{web} + 2 \times \overbrace{\left(\frac{2b^3 t_f}{3}\right)}^{flange} \approx \frac{4b^3 t_f}{3}.$$

- Recall that the stress distribution in this case ($I_{23} = 0$) is $\sigma_{11} = \frac{M_2}{I_{22}} X_3 - \frac{M_3}{I_{33}} X_2$. So I_{22} governs bending in the $\underline{e_2}$ direction and I_{33} governs bending in the $\underline{e_3}$ direction.



2.1.3. Shear Flow Distribution: The "I" Section

Shear Stress and Flow in Sections

- Consider the shear distribution through an I-section as shown here

Both I_{22} and I_{33} have a term that's proportional to t_f. For bending purposes, in fact, it is more efficient use of material to move the flanges far apart ($h \uparrow$) and make the web very thin ($t_w \downarrow$).

$$I_{22} = \overbrace{\frac{h^3 t_w}{12}}^{web} + 2 \times \overbrace{\left(\frac{2bt_f^3}{12} + 2bt_f \times \frac{h^2}{4}\right)}^{flange} \approx \frac{h^3 t_w}{12} + h^2 b t_f.$$

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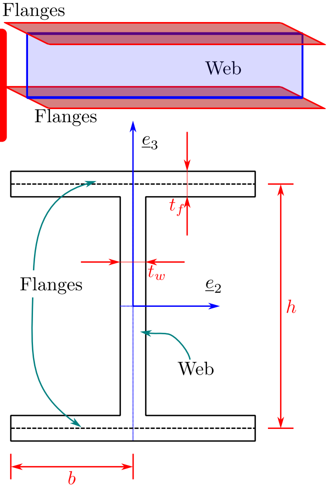

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- T… r… i…

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- T…

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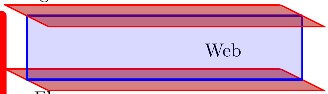
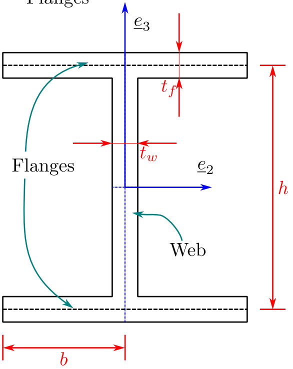
Design Principle

- Design the flanges to bear all the bending stresses.
- Design the web to "survive" the shear.

$$I_{33} = \underbrace{\frac{\cdots}{12}}_{web} + 2 \times \left(\frac{\cdots}{3}\right) \approx \frac{\cdots}{3}.$$

- Recall that the stress distribution in this case ($I_{23} = 0$) is $\sigma_{11} = \frac{M_2}{I_{22}} X_3 - \frac{M_3}{I_{33}} X_2$. So I_{22} governs bending in the $\underline{e_2}$ direction and I_{33} governs bending in the $\underline{e_3}$ direction.





2.1.3. Shear Flow Distribution: The "I" Section

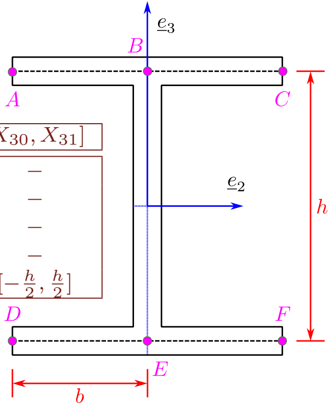
Shear Stress and Flow in Sections

- Let us now calculate the actual shear stress distribution for the I section. We label the section as shown and write up a table as follows:

***Shear flow table** for the I-section*

	t	$s(X_2, X_3)$	X_2	X_3	$[X_{20}, X_{21}]$	$[X_{30}, X_{31}]$
A-B	t_y	$b + X_2$	$s - b$	$\frac{h}{2}$	$[-b, 0]$	–
C-B	t_y	$b - X_2$	$b - s$	$\frac{h}{2}$	$[b, 0]$	–
D-E	t_y	$b + X_2$	$s - b$	$-\frac{h}{2}$	$[-b, 0]$	–
F-E	t_y	$b - X_2$	$b - s$	$-\frac{h}{2}$	$[b, 0]$	–
E-B	t_w	$\frac{h}{2} + X_3$	0	$s - \frac{h}{2}$	–	$[-\frac{h}{2}, \frac{h}{2}]$

- $[X_{20}, X_{21}]$ and $[X_{30}, X_{31}]$ are the domains of each of the segments.



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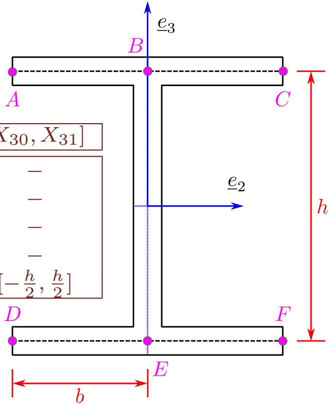
C-B	t_y	$b - X_2$	$b - s$	$\frac{h}{2}$	$[b, 0]$	–
D-E	t_y	$b + X_2$	$s - b$	$-\frac{h}{2}$	$[-b, 0]$	–
F-E	t_y	$b - X_2$	$b - s$	$-\frac{h}{2}$	$[b, 0]$	–
E-B	t_w	$\frac{h}{2} + X_3$	0	$s - \frac{h}{2}$	–	$[-\frac{h}{2}, \frac{h}{2}]$



- $[X_{20}, X_{21}]$ and $[X_{30}, X_{31}]$ are the domains of each of the segments.
- The shear flow integral is written as

$$q(s) - q_0 = -\frac{V_2}{I_{33}} \overbrace{\int_0^s t X_2 ds}^{z_2(s)} - \frac{V_3}{I_{22}} \overbrace{\int_0^s t X_3 ds}^{z_3(s)}$$
$$= -\frac{V_2}{I_{33}} z_2(s) - \frac{V_3}{I_{22}} z_3(s).$$

We will compute the z_2 and z_3 functions (in terms of X_2, X_3) and add them to the table above

2.1.3. Shear Flow Distribution: The "I" Section

Shear Stress and Flow in Sections

$$q(s) - q_0 = -\frac{V_2}{I_{33}}\int_0^s tX_2 ds - \frac{V_3}{I_{22}}\int_0^s tX_3 ds = -\frac{V_2}{I_{33}} z_2(s) - \frac{V_3}{I_{22}} z_3(s).$$

Shear flow table *for the I-section*

	t	$s(X_2, X_3)$	X_2	X_3	$[X_{20}, X_{21}]$	$[X_{30}, X_{31}]$	$z_2 - z_{20}$	$z_3 - z_{30}$
A-B	t_f	$b + X_2$	$s - b$	$\frac{h}{2}$	$[-b, 0]$	–	$t_f \frac{X_2^2 - b^2}{2}$	$t_f \frac{h(X_2+b)}{2}$
C-B	t_f	$b - X_2$	$b - s$	$\frac{h}{2}$	$[b, 0]$	–	$-t_f \frac{X_2^2 - b^2}{2}$	$-t_f \frac{h(X_2-b)}{2}$
D-E	t_f	$b + X_2$	$s - b$	$-\frac{h}{2}$	$[-b, 0]$	–	$t_f \frac{X_2^2 - b^2}{2}$	$-t_f \frac{h(X_2+b)}{2}$
F-E	t_f	$b - X_2$	$b - s$	$-\frac{h}{2}$	$[b, 0]$	–	$-t_f \frac{X_2^2 - b^2}{2}$	$t_f \frac{h(X_2-b)}{2}$
E-B	t_w	$\frac{h}{2} + X_3$	0	$s - \frac{h}{2}$	–	$[-\frac{h}{2}, \frac{h}{2}]$	0	$t_w \frac{X_3^2 - (\frac{h}{2})^2}{2}$

- We now have all the terms necessary for furnishing the shear flow formula above.
- Noting that $q(s) = 0$ at all the 4 free tips (A,B,C,D here), q_0 for these integrals can safely be taken as zero. **But what about the section $E - B$?**

2.1.3. Shear Flow Distribution: The "I" Section

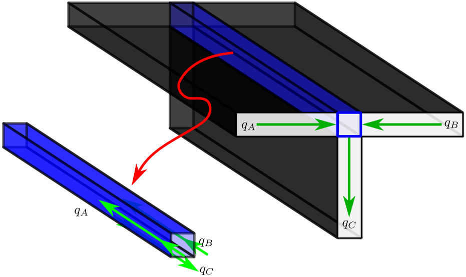
Shear Stress and Flow in Sections

$$q(s) - q_0 = -\frac{V_2}{I_{33}}\int_0^s tX_2 ds - \frac{V_3}{I_{22}}\int_0^s tX_3 ds = -\frac{V_2}{I_{33}} z_2(s) - \frac{V_3}{I_{22}} z_3(s).$$

Shear flow table *for the I-section*

	t	$z_2 - z_{20}$	$z_3 - z_{30}$
A-B	t_f	$t_f \frac{X_2^2 - b^2}{2}$	$t_f \frac{h(X_2+b)}{2}$
C-B	t_f	$-t_f \frac{X_2^2 - b^2}{2}$	$-t_f \frac{h(X_2-b)}{2}$
D-E	t_f	$t_f \frac{X_2^2 - b^2}{2}$	$-t_f \frac{h(X_2+b)}{2}$
F-E	t_f	$-t_f \frac{X_2^2 - b^2}{2}$	$t_f \frac{h(X_2-b)}{2}$
E-B	t_w	0	$t_w \frac{X_3^2 - (\frac{h}{2})^2}{2}$

Balance at the T-junction



$\sum F = q_A + q_B - q_C = 0 \Rightarrow q_C = q_A + q_B$

- We now h ... ar flow formula above.
- Noting that $q(s) = 0$ at all the 4 free tips (A,B,C,D here), q_0 for these integrals can safely be taken as zero. **But what about the section $E - B$?**

2.1.3. Shear Flow Distribution: The "I" Section

Shear Stress and Flow in Sections

- Since z_2 and z_3 are running integrals, we will also tabulate their values at the "end" of each segment.

Shear flow table *for the I-section*

	$z_2 - z_{20}$	$z_{21} - z_{20}$	$z_3 - z_{30}$	$z_{31} - z_{30}$
A-B	$t_f \frac{X_2^2 - b^2}{2}$	$-\frac{t_f b^2}{2}$	$t_f \frac{h(X_2 + b)}{2}$	$\frac{t_f hb}{2}$
C-B	$-t_f \frac{X_2^2 - b^2}{2}$	$\frac{t_f b^2}{2}$	$-t_f \frac{h(X_2 - b)}{2}$	$\frac{t_f hb}{2}$
D-E	$t_f \frac{X_2^2 - b^2}{2}$	$-\frac{t_f b^2}{2}$	$-t_f \frac{h(X_2 + b)}{2}$	$-\frac{t_f hb}{2}$
F-E	$-t_f \frac{X_2^2 - b^2}{2}$	$\frac{t_f b^2}{2}$	$t_f \frac{h(X_2 - b)}{2}$	$-\frac{t_f hb}{2}$
E-B	0	0	$t_w \frac{X_3^2 - (\frac{h}{2})^2}{2}$	0

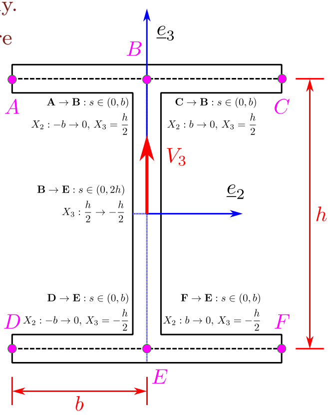
- At the junction E, the shear flow will be a sum of the contributions from the segments D-E and F-E. In terms of the z_2, z_3 functions this turns out as,

$$z_2\Big|_E = -\frac{t_f b^2}{2} + \frac{t_f b^2}{2} = 0, \quad z_3\Big|_E = -\frac{t_f hb}{2} - \frac{t_f hb}{2} = -t_f hb.$$

2.1.3. Shear Flow Distribution: The "I" Section

Shear Stress and Flow in Sections

- Let us consider the case with $V_2 = 0$, $V_3 \neq 0$ graphically.
- The segments $A \to B$, $C \to B$, $D \to E$, and $F \to E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).



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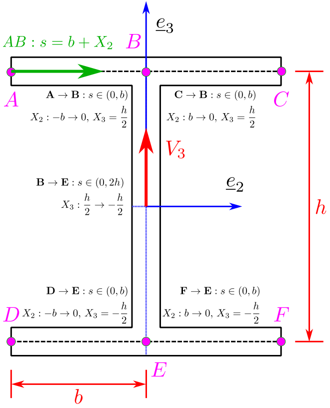
$$\mathbf{A} \to \mathbf{B} \; : q_{AB}(s) \equiv q_{AB}(X_2) = -\frac{V_3}{I_{22}} z_3(s) = -\frac{V_3}{I_{22}} t_f \frac{h(X_2+b)}{2}$$



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$$\mathbf{A} \to \mathbf{B} \;:\; q_{AB}(s) \equiv q_{AB}(X_2) = -\frac{V_3}{I_{22}} z_3(s) = -\frac{V_3}{I_{22}} t_f \frac{h(X_2+b)}{2}$$

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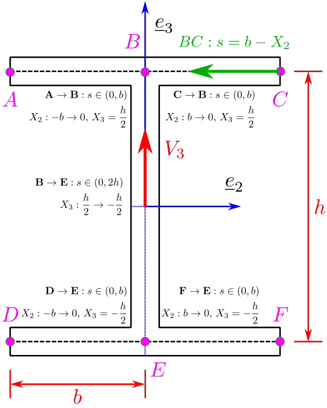

2.1.3. Shear Flow Distribution: The "I" Section

Shear Stress and Flow in Sections

- Let us consider the case with $V_2 = 0$, $V_3 \neq 0$ graphically.
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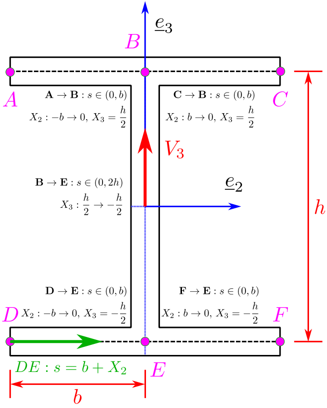

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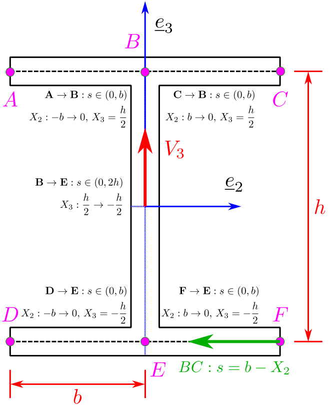

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Shear Stress and Flow in Sections

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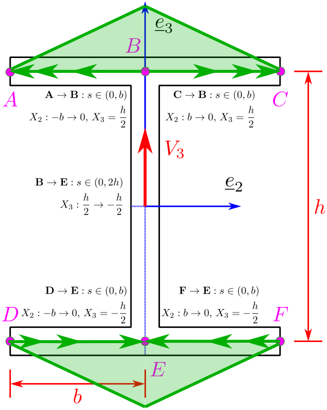
- In summary we have linear shear flow trends at the flanges.



2.1.3. Shear Flow Distribution: The "I" Section

Shear Stress and Flow in Sections

- Let us consider the case with $V_2 = 0$, $V_3 \neq 0$ graphically.
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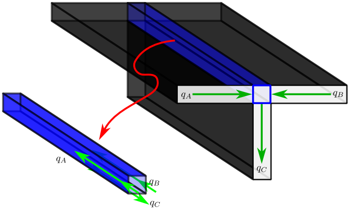
- In summary we have linear shear flow trends at the flanges.
- For the web ($E \rightarrow B$), we recall the balance at the "T" junction.

Balance at the T-junction



$$\sum F = q_A + q_B - q_C = 0 => q_C = q_A + q_B$$

2.1.3. Shear Flow Distribution: The “I” Section

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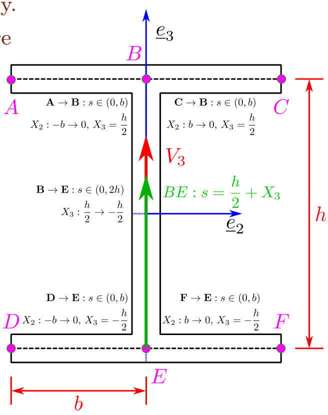
- The segments $A \to B$, $C \to B$, $D \to E$, and $F \to E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).
- On $\mathbf{E} \to \mathbf{B}$, we have $q_{EB}(0) = q_{DE}(0) + q_{FE}(0) = \frac{V_3}{I_{22}} t_f hb$.
- The integration evaluates as,

$$q_{EB}(X_3) = -\frac{V_3}{I_{22}}\left(-t_f hb + \frac{t_w}{2}(X_3^2 - (\frac{h}{2})^2)\right)$$
$$= \frac{hV_3}{I_{22}}(t_f b + \frac{t_w h}{8}) - \frac{V_3}{I_{22}}\frac{t_w}{2}X_3^2.$$



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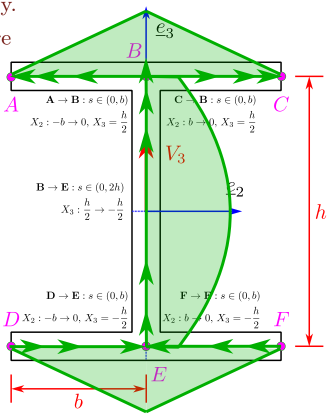
$$
\begin{aligned}
q_{EB}(X_3) &= -\frac{V_3}{I_{22}} \left(-t_f hb + \frac{t_w}{2}(X_3^2 - (\frac{h}{2})^2) \right) \\
&= \frac{hV_3}{I_{22}}(t_f b + \frac{t_w h}{8}) - \frac{V_3}{I_{22}} \frac{t_w}{2} X_3^2.
\end{aligned}
$$

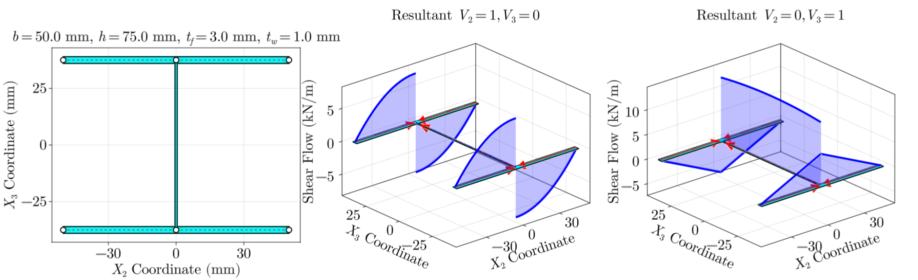
- We now have the complete shear flow in the section.



2.1.3. Shear Flow Distribution: The "I" Section

Shear Stress and Flow in Sections

Numerical Example



- We now have the complete shear flow in the section.

2. Shear Stress and Flow in Sections

The "I" section: Order of Magnitude Analysis

- Let us consider the "total" shear forces experienced by each member.
- **Flange AB**

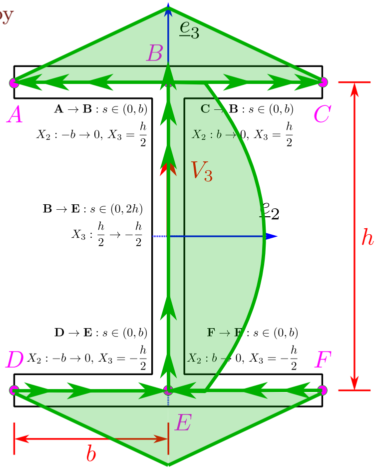
$$V_{AB} = -\frac{V_3}{I_{22}}\frac{ht_f}{2}\int_{-b}^{0}(X_2+b)dX_2 = -\frac{b^2ht_fV_3}{4I_{22}}$$

- **Web BE**

$$V_{EB} = \frac{h^2(h+12b)tV_3}{12I_{22}} = V_3$$

- For $b = \frac{h}{2}$, we have,

$$V_{AB} = -\frac{h^3tV_3}{16I_{22}} \approx -\frac{V_3}{8}$$
$$V_{EB} = V_3$$



2. Shear Stress and Flow in Sections

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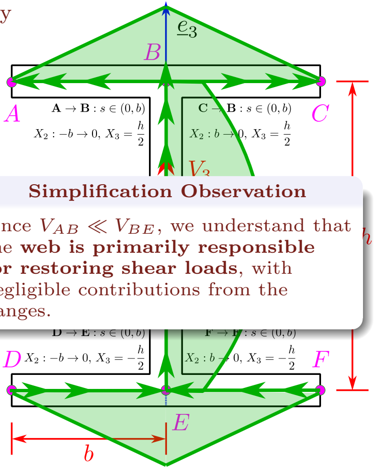
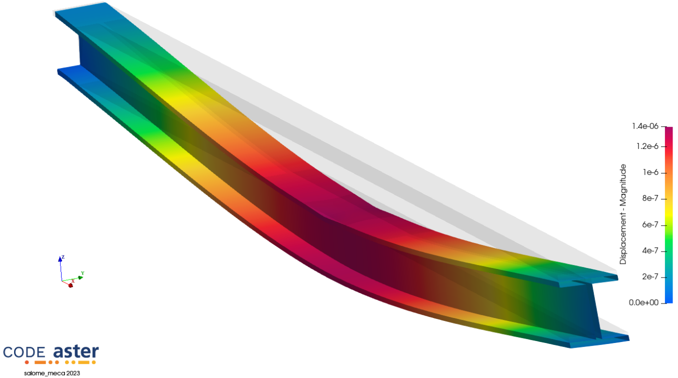


Simplification Observation

Since $V_{AB} \ll V_{BE}$, we understand that the **web is primarily responsible for restoring shear loads**, with negligible contributions from the flanges.

2. Shear Stress and Flow in Sections

An "I" section beam subjected to 3-point bending: Finite Element Results

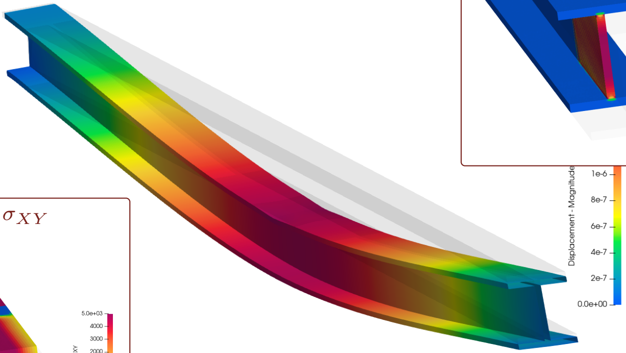


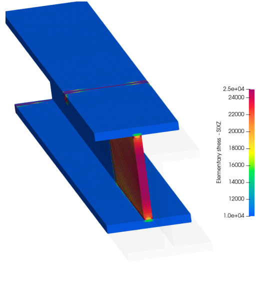
Code_Aster on the Salome Platform

Free and Open Source (FOSS) FE solver that comes with a fully functional frontend (Salome)! Please Do Explore!

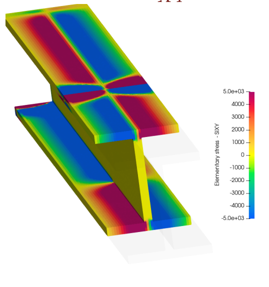
2. Shear Stress and Flow in Sections

An “I” section beam subjected to 3-point bending: Finite Element Results





Stress σ_{XZ}



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2. Shear Stress and Flow in Sections

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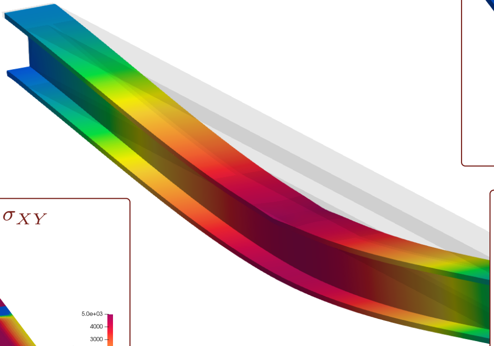

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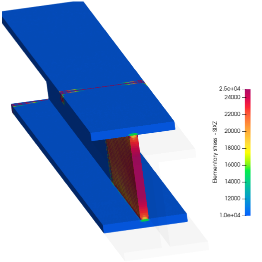

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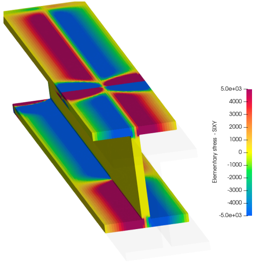
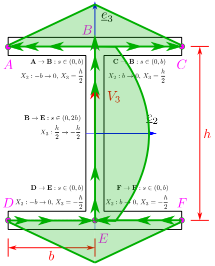
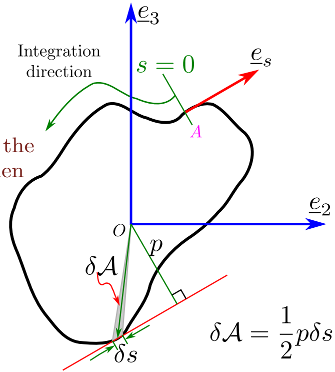


Shear Flow



2.2. Closed Sections

Shear Stress and Flow in Sections



- The shear-flow integral formula makes no reference to whether the section is open or closed.
- Considering the generic closed section shown, we start the integral at some **arbitrary** point A. The integral is then written as,

$$q(s) - q_A = \underbrace{-\int_0^s t\sigma_{11,1} ds}_{q_b(s)}.$$

- When no twisting is expected at the section, the moment along $\underline{e}_1$ about O has to be zero. This is computed as,

$$\int_{A^-}^{A^+} dM_1 = \oint pq(s)ds = q_A \overbrace{\oint pq(s)ds}^{2\mathcal{A}} + \oint pq_b(s)ds = 0 \implies \boxed{q_A = -\frac{1}{2\mathcal{A}} \oint pq_b(s)ds}.$$

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Shear Stress and Flow in Sections

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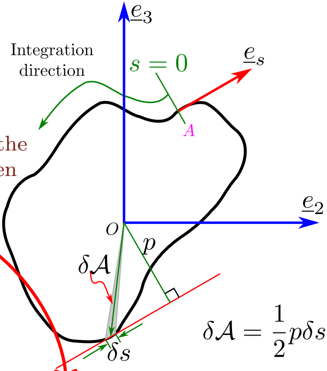
Are we already assuming that O is the shear center with this?!

$$\underbrace{\int_0 \cdots}_{q_b(s)}$$



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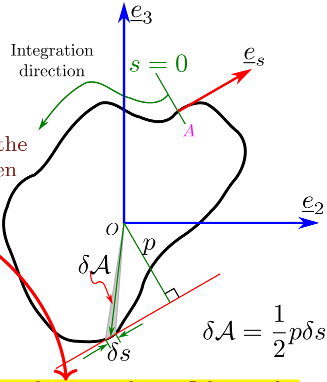
2.2. Closed Sections

Shear Stress and Flow in Sections



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- Considering the generic closed section shown, we start the integral at some **arbitrary** point A. The integral is then

Are we already assuming that O is the shear center with this?!

Yes. We need to add an extra term when resultants V_2, V_3 are acting with an offset.

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2.2. Closed Sections: Shear Center

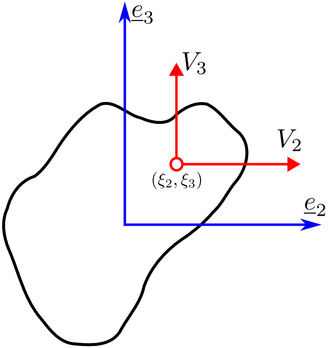
Shear Stress and Flow in Sections

- If we suppose that the resultants are acting along some (ξ_2, ξ_3), the applied moment is: $(\xi_2 \underline{e}_2 + \xi_3 \underline{e}_3) \times (V_2 \underline{e}_2 + V_3 \underline{e}_3)$:

$$M_1 = \xi_2 V_3 - \xi_3 V_2.$$

- Equating this to the moment developed from the shear flow distribution, we get:

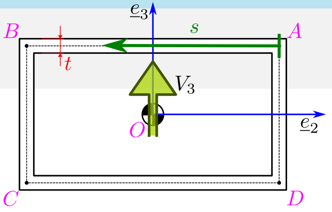
$$\xi_2 V_3 - \xi_3 V_2 = 2\mathcal{A} q_{A^-} + \oint p q_b(s) ds$$



- For shear center determination (the point around which resultants act), this is not enough.
- We will additionally invoke an argument of **zero twist** ($\theta_{1,1} = 0$) in the deflection field to get an additional relationship ($\oint \frac{q}{Gt} ds = 0$). This will be covered in the next module (Torsion).
- **For symmetric closed sections**, however, the shear center coincides with the centroid (through symmetry arguments).

2.2.1. The Rectangular "Box" Section

Closed Sections



To avoid confusion choose a clock-wise integration direction always!

- Let us consider the rectangular beam for V_3 alone.
- We start the integration from point A arbitrarily.
- The shear flow equation is $q(s) - q_A = -\frac{tV_3}{I_{22}} \int_0^s X_3 ds = -\frac{tV3}{I_{22}} \underbrace{(z_3(s) - z_A)}_{z_{3b}(s)}$.
- We write down the table as follows:

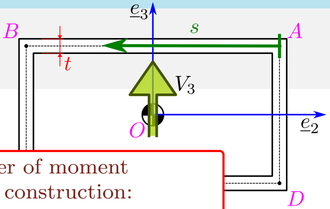
	$s(X_2, X_3)$	X_3	$z_3 - z_{30}$	$z_{31} - z_{30}$	$z_{3b}(X_2, X_3)$
A→B	$\frac{b}{2} - X_2$	$\frac{h}{2}$	$-\frac{h}{2}(X_2 - \frac{b}{2})$	$\frac{hb}{2}$	$-\frac{h}{2}(X_2 - \frac{b}{2})$
B→C	$b + \frac{h}{2} - X_3$	X_3	$-\frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$	0	$\frac{hb}{2} - \frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$
C→D	$\frac{3b}{2} + h + X_2$	$-\frac{h}{2}$	$-\frac{h}{2}(X_2 + \frac{b}{2})$	$-\frac{hb}{2}$	$-\frac{h}{2}(X_2 - \frac{b}{2})$
D→A	$2b + \frac{3h}{2} + X_3$	X_3	$\frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$	0	$\frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$

- The zero twist condition can be interpreted as saying that the moment about O is zero:

$$M_O = \oint p q(s) ds = 0 \implies z_A(-2bh) + \oint p \cdot z_3(s) ds = 0$$

2.2.1. The Rectangular "Box" Section

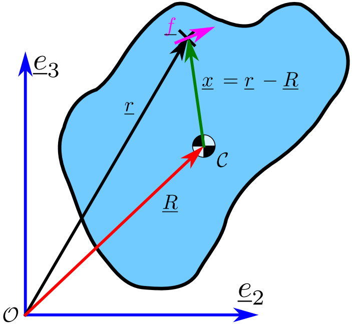
Closed Sections



To avoid confusion choose a clock-wise integration direction always!

- Le…
- W…
- Th…
- W…

Choosing a non-centroidal point as the center of moment is sometimes helpful. Always remember this construction:



Moment about Centroid C

$$\begin{aligned} \underline{M_{\mathcal{C}}} &= \int_\Omega \underline{x} \times \underline{f}\, d\Omega \\ &= \int_\Omega (\underline{r} - \underline{R}) \times \underline{f}\, d\Omega \\ &= \int_\Omega \underline{r} \times \underline{f}\, d\Omega - \underline{R} \times \int_\Omega \underline{f}\, d\Omega \\ &= \underline{M_{\mathcal{O}}} - \underline{R} \times \underline{F} \end{aligned}$$

$X_3)$ … $-\frac{b}{2})$ … $-(\frac{h}{2})^2)$ … $-\frac{b}{2})$ … $(\frac{h}{2})^2)$

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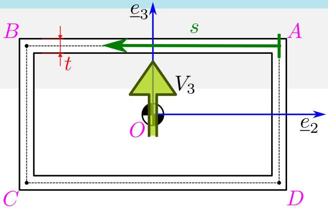


- Based on the above construction, we have

$$M_{\mathcal{O}} = M_{\mathcal{C}} - \frac{V_3 b}{2} = 0.$$

- The moment w.r.t. $\mathcal{C}$ is written as

$$M_{\mathcal{C}} = -\frac{tV_3}{I_{22}} \oint p(z_A + z_{3b}(s))ds = -\frac{tV_3}{I_{22}} \left(2bhz_A + \oint pz_{3b}(s)ds\right).$$

Let us evaluate the last integral using the table (you can just add columns):

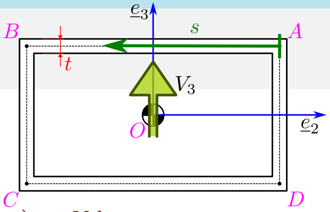
	$s(X_2, X_3)$	p	$z_{3b}(X_2, X_3)$	$\int_{s_0}^{s_1} pz_{3b}(s)ds$
A→B	$\frac{b}{2} - X_2$	h	$-\frac{h}{2}(X_2 - \frac{b}{2})$	$\frac{h^2b^2}{4}$
B→C	$b + \frac{h}{2} - X_3$	0	$\frac{hb}{2} - \frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$	0
C→D	$\frac{3b}{2} + h + X_2$	0	$-\frac{h}{2}(X_2 - \frac{b}{2})$	0
D→A	$2b + \frac{3h}{2} + X_3$	b	$\frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$	$-\frac{bh^3}{12}$

- Summing up the last column we can write

$$M_{\mathcal{C}} = -V_3 \left(\frac{2bht}{I_{22}} z_A + \frac{h^2bt}{12I_{22}}(3b - h)\right)$$

2.2.1. The Rectangular “Box” Section

Closed Sections



- The zero twist condition reads:

$$M_{\mathcal{C}} = -V_3\left(\frac{2bht}{I_{22}}z_A + \frac{h^2bt}{12I_{22}}(3b-h)\right) = \frac{V_3 b}{2}.$$

- Solving this for z_A we get

$$z_A = -\frac{hb}{8} + \frac{h^2}{24} - \frac{I_{22}}{4ht}.$$

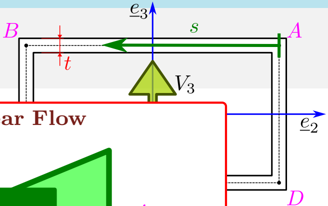
- The second area moment is $I_{22} = \frac{h^2 t}{6}(3b+h)$. Substituting this we get $\boxed{z_A = -\frac{hb}{4}}$.

- Adding this to $z_{3b}(s)$ we can write $z_3(s)$ s.t. $q(s) = -\frac{tV_3}{I_{22}} z_3(s)$:

	$A \to B$	$B \to C$	$C \to D$	$D \to A$
$z_3(s)$	$-\frac{h}{2}X_2$	$\frac{hb}{4} - \frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$	$-\frac{h}{2}X_2$	$-\frac{hb}{4} + \frac{1}{2}(X_3^2 - (\frac{h}{2})^2)$

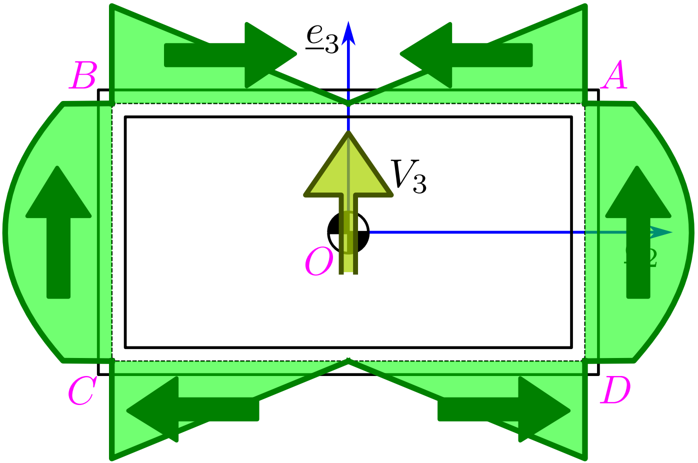
2.2.1. The Rectangular "Box" Section

Closed Sections

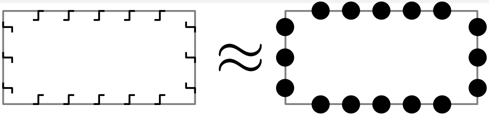


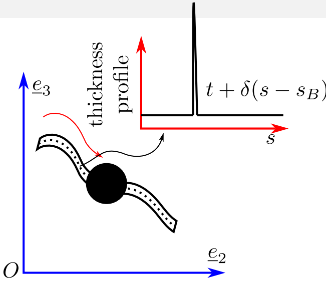
- The zer
- Solving
- The sec $-\frac{hb}{4}$.
- Adding

Graphical Visualization of Shear Flow



3. Stringer-Web Idealization





- A stringer, when looked at from a section, looks like a feature with a sudden increase in thickness.
- Consider the r^{th} "Boom" to be located at (X_{r_2}, X_{r_3}), with area A_r.
  - So the shear flow integral can be generalized to,

$$q(s) - q(0) = -\frac{\begin{bmatrix}\int_0^s tX_2 ds + \sum_r A_r X_{r_2} & \int_0^s tX_3 ds + \sum_r A_r X_{r_3}\end{bmatrix}}{I_{22}I_{33} - I_{23}^2}\begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{22}\end{bmatrix}\begin{bmatrix} V_2 \\ V_3 \end{bmatrix}$$

  - It is sometimes possible to also "lump" the effects of the skins (with thickness t) into the boom areas to simplify the above

$$q(s) - q(0) = -\frac{\begin{bmatrix}\sum_r A_r X_{r_2} & \sum_r A_r X_{r_3}\end{bmatrix}}{I_{22}I_{33} - I_{23}^2}\begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33}\end{bmatrix}\begin{bmatrix} V_2 \\ V_3 \end{bmatrix}$$

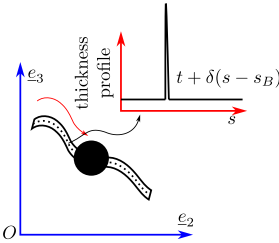
  - This is known as **Stringer-Web Idealization**.

3. Stringer-Web Idealization

- Considering the integral right across the boom, we have,

$$q^+ - q^- = -\frac{\begin{bmatrix} A_r X_{r_3} & -A_r X_{r_2} \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}.$$

- For the sections without a boom, there is **no change in the shear flow**.
- Therefore, the shear flow is constant in the webs that join two booms.

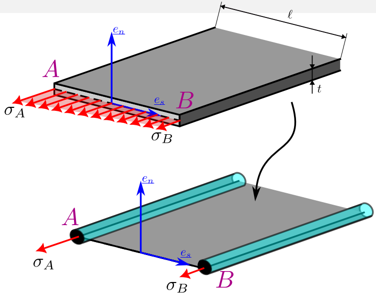


General Design Principle

- As a general principle, the stringers/booms are added to support bending.
- The web thicknesses are chosen to ensure the shear stresses don't exceed failure threshold (we should like to have $t = 0$ to minimize weight!).

3.1. Idealizing a Thin Rectangular Section

Stringer-Web Idealization



- Beam theory predicts that the axial stress σ_{11} varies linearly in the section. So let us consider a panel with ends A and B under stresses $\sigma_{11}(\underline{x_A}) = \sigma_A$ and $\sigma_{11}(\underline{x_B}) = \sigma_B$.
- While the complete panel responds to the σ_{11} distribution in reality, in the idealized panel, the "booms" alone bear the stresses (the webs have zero cross section). Since stress is proportional to the section moment, **we need to ensure that the moment along the e_n direction is conserved in the idealization process**.
- Taking moment about point A

$$M_n = -\int_0^\ell x\left(\sigma_A + (\sigma_B - \sigma_A)\frac{x}{\ell}\right) tdx := -\sigma_B A r_B = M_{n,i} \implies \boxed{Ar_B = \frac{t\ell^2}{6}\left(2 + \frac{\sigma_A}{\sigma_B}\right)}$$

- We also require the overall load to be conserved:

$$\int_0^\ell \sigma_A + (\sigma_B - \sigma_A)\frac{x}{\ell} tdx := \sigma_A A r_A + \sigma_B A r_B \implies \boxed{Ar_A = \frac{t\ell^2}{6}\left(2 + \frac{\sigma_B}{\sigma_A}\right)}.$$

4. Shear Lag

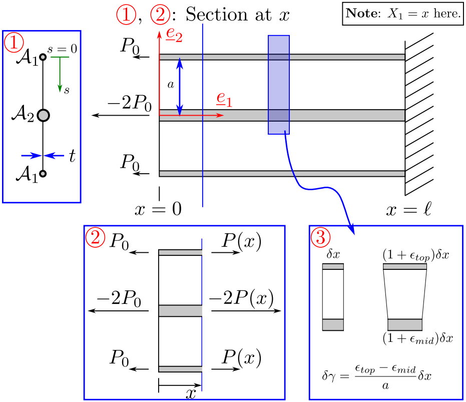
Example from Sun (2006) (Section 3.1)

- We already saw the St. Venant's principle. Can we say something more specific about "how fast" the end effects start getting smoothed out in the stress field? Consider:

- Applying equilibrium on the section at some $X_1 = x$ ① leads to top and bottom booms experiencing $\boxed{\int_{\mathcal{A}_1} \sigma_{11} dA = P(x)}$, and the central boom, $\boxed{\int_{\mathcal{A}_2} \sigma_{11} dA = -2P(x)}$.

- Shear flow in top web is given as

$$q_{top} = -\sigma_{11,1}\mathcal{A}_1 = -P_{,1}$$
$$\text{or,} \; -t\sigma_{12} = -P_{,1}.$$



4. Shear Lag

Example from Sun (2006) (Section 3.1)

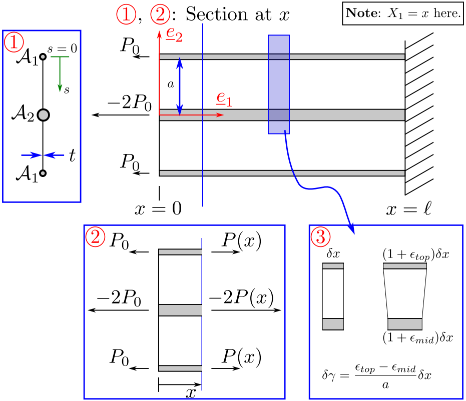
- Considering the shear differential in an infinitesimal section (③) we have,

$$\frac{\partial \gamma_{12}}{\partial x} = \frac{1}{a}(\epsilon_{top} - \epsilon_{mid}) = \frac{1}{E_Y a}\left(\frac{P(x)}{\mathcal{A}_1} - \frac{-2P(x)}{\mathcal{A}_2}\right)$$

- Since $\gamma_{12} = \frac{1}{G}\sigma_{12}$, this becomes,

$$\sigma_{12,1} = \frac{G}{E_Y a}\left(\frac{1}{\mathcal{A}_1} + \frac{2}{\mathcal{A}_2}\right) P(x).$$

- Using the force balance relationship $\boxed{\sigma_{12} = \frac{1}{t} P_{,1}}$ also



$$\implies \boxed{P_{,11} = \frac{Gt}{E_Y a}\left(\frac{1}{\mathcal{A}_1} + \frac{2}{\mathcal{A}_2}\right) P}$$

4. Shear Lag

Example from Sun (2006) (Section 3.1)

- The equation governing the **boom restoring axial force** is of the form

$$P_{,11} - \lambda^2 P = 0, \qquad \lambda = \sqrt{\frac{Gt}{E_Y a}\left(\frac{1}{\mathcal{A}_1} + \frac{2}{\mathcal{A}_2}\right)}.$$

- This is solved by

$$P(X_1) = C_1 e^{-\lambda X_1} + C_2 e^{\lambda X_1}.$$

- Solving it over $X_1 \in (0, \infty)$, it is easy to see that C_2 must be 0 for $P(X_1)$ to be bounded. So we have an **exponentially decaying** restoring force on the booms:

$$P(X_1) = P_0 e^{-\lambda X_1}.$$

- Substituting for σ_{12} we have (for the top web),

$$\sigma_{12} = -\frac{\lambda P_0}{t} e^{-\lambda X_1},$$

which also decays exponentially in X_1.

4. Shear Lag: Decay Rate

Example from Sun (2006) (Section 3.1)

- The shear lag factor λ controls how quickly the effects “diffuse out”.
  - Large λ implies “fast” diffusion and potentially high concentration around the ends..
  - Small λ implies “slow” diffusion and potentially low concentrations.
- In general for stringer-web structures,

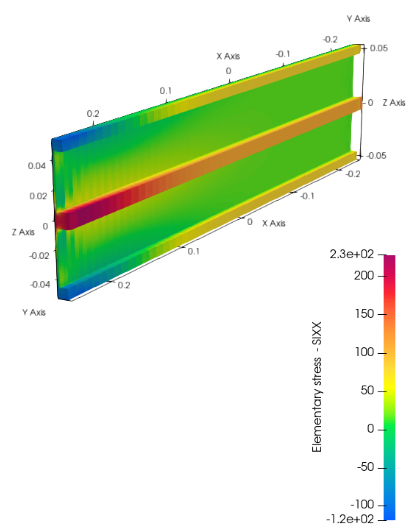
$$\lambda \propto \sqrt{\frac{G}{E_Y}}.$$

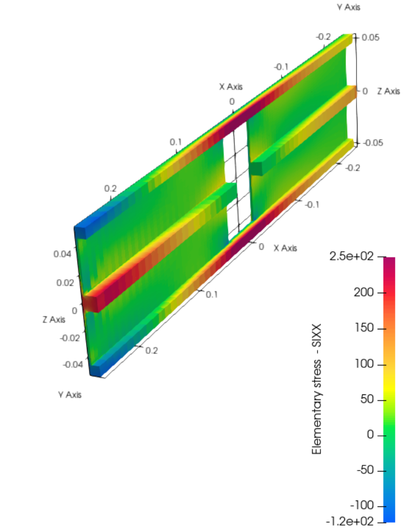
  - $\uparrow G$ (stiffer web), $\uparrow \lambda$ (”faster” diffusion).
  - $\uparrow E_Y$ (stiffer boom), $\downarrow \lambda$ (”slower” diffusion).

The terms “faster” and “slower” are used in the sense that “slower” implies that the stress distribution needs a longer axial distance to get averaged out (vise versa for “faster”).

4. Shear Lag

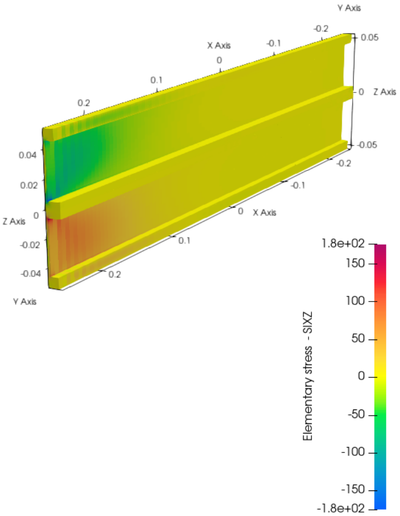
Example from Sun (2006) (Section 3.1): FE Results

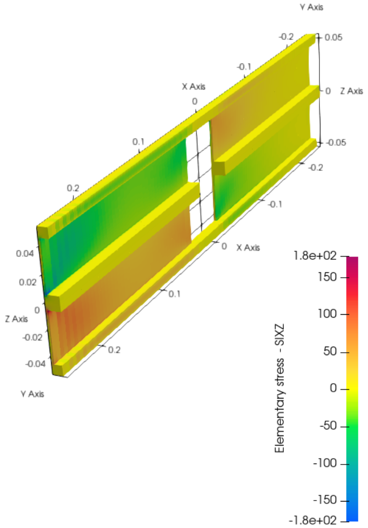




4. Shear Lag

Example from Sun (2006) (Section 3.1): FE Results





References I

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