AS3020: Aerospace Structures

Module 4: Bending of Beam-Like Structures

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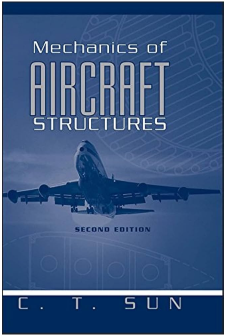
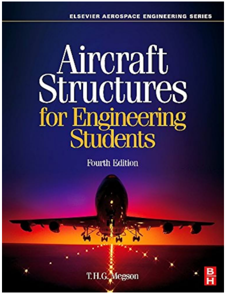


Chapters 4-5 in Sun (2006)



Chapters 16-20 in Megson (2013)

1. Unsymmetrical Bending

Assumptions

1. Plane sections remain planar.
2. Sections remain perpendicular to neutral axis: $\gamma_{12} = \gamma_{13} = 0$.
3. Plane Stress: $\sigma_{22} = \sigma_{33} = 0$.

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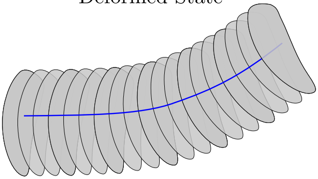
Deformed State



1. Rigid Section Displacement Field

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ v(X_1) \\ w(X_1) \end{bmatrix} + \underbrace{\begin{bmatrix} X_3\theta_2 - X_2\theta_3 \\ 0 \\ 0 \end{bmatrix}}_{\underset{\sim}{\underline{\theta} \times \underline{X}}}$$

2. Zero Shear Strain Simplification

$$\gamma_{12} = \gamma_{13} = 0 \implies \boxed{\theta_2 = -w', \quad \theta_3 = v'}$$

3. Plane Stress Constitution

$$\boxed{\sigma_{11} = E_Y \underline{E}_{11}}$$

$$\implies \mathcal{E}_{11} = u_{1,1} = \begin{bmatrix} X_3 & -X_2 \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

We shall develop the theory without the zero strain simplification first.

1.1. Axial Stress and its Moments

Unsymmetrical Bending

- The axial stress distribution is $\boxed{\sigma_{11} = E_Y \begin{bmatrix} X_3 & -X_2 \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}}$. The traction vector in the section is $\underline{t} = \sigma_{11}\underline{e_1} + \sigma_{12}\underline{e_2} + \sigma_{13}\underline{e_3}$.
- Considering just the axial component ($\sigma_{11}\underline{e_1}$), we write the overall axial force as the area integral (**zeroth moment**):

$$N_1 = \int_{\mathcal{A}} \sigma_{11} = E_Y \begin{bmatrix} \int_{\mathcal{A}} X_3 dA & -\int_{\mathcal{A}} X_2 dA \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

- Recall that we have already chosen then origin as the section centroid for expressing the rigid rotation displacement field, s.t. $\int_{\mathcal{A}} \underline{X}\, dA = \underline{0}$. Therefore $N_1 = 0$ for pure bending.
- Considering the moment due to the axial component ($d\underline{m} = (X_k \underline{e_k}) \times (\sigma_{11}\underline{e_1}\, dA)$ we have (**first moment**):

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = \int_{\mathcal{A}} \begin{bmatrix} X_3 \\ -X_2 \end{bmatrix} \sigma_{11} dA = \int_{\mathcal{A}} E_Y \begin{bmatrix} X_3 \\ -X_2 \end{bmatrix} \begin{bmatrix} X_3 & -X_2 \end{bmatrix} dA \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}$$
$$= \int_{\mathcal{A}} E_Y \begin{bmatrix} X_3^2 & -X_2X_3 \\ -X_2X_3 & X_2^2 \end{bmatrix} dA \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

For constant E_Y through section,

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = E_Y \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

Second Moments of Area

$$I_{22} = \int_{\mathcal{A}} X_3^2 dA$$
$$I_{33} = \int_{\mathcal{A}} X_2^2 dA$$
$$I_{23} = \int_{\mathcal{A}} X_2 X_3 dA$$

1.2. Axial Stress In Terms of Moments and Forces

Unsymmetrical Bending

- It is sometimes convenient to have the stress σ_{11} expressed in terms of its resultant moments instead of kinematic quantities like θ_2 and θ_3. So we will invert the relationship that we have to first get:

$$\begin{bmatrix}\theta_2' \\ \theta_3'\end{bmatrix} = \frac{1}{E_Y}\frac{1}{I_{22}I_{33}-I_{23}^2}\begin{bmatrix}I_{33} & I_{23} \\ I_{23} & I_{22}\end{bmatrix}\begin{bmatrix}M_2 \\ M_3\end{bmatrix}.$$

- Stress simplifies as

$$\begin{aligned}\sigma_{11} &= E_Y\begin{bmatrix}X_3 & -X_2\end{bmatrix}\begin{bmatrix}\theta_2' \\ \theta_3'\end{bmatrix} \\ &= \frac{\begin{bmatrix}X_3 & -X_2\end{bmatrix}}{I_{22}I_{33}-I_{23}^2}\begin{bmatrix}I_{33} & I_{23} \\ I_{23} & I_{22}\end{bmatrix}\begin{bmatrix}M_2 \\ M_3\end{bmatrix}.\end{aligned}$$

- **Observe that we have gotten to the above without requiring shear strains to be zero.**

1.3. Equilibrium Equations

Unsymmetrical Bending

- We shall invoke and simplify the equilibrium equations in an integral sense in the presence of transverse forces only (**stress assumptions**: $\sigma_{22} = \sigma_{33} = \sigma_{23} = 0$).

$$\sigma_{1j,j} = 0 \implies \int_{\mathcal{A}} \sigma_{1j,j} dA = 0$$
$$\sigma_{12,1} + f_2 = 0 \implies \int_{\mathcal{A}} \sigma_{12,1} dA + \int_{\mathcal{A}} f_2 dA = 0$$
$$\sigma_{13,1} + f_3 = 0 \implies \int_{\mathcal{A}} \sigma_{13,1} dA + \int_{\mathcal{A}} f_3 dA = 0$$

- $\int_{\mathcal{A}} \sigma_{1j,j} dA$ is simplified as

$$\int_{\mathcal{A}} \sigma_{1j,j} dA = \int_{\mathcal{A}} \sigma_{11,1} dA + \overbrace{\int_{\mathcal{A}} \sigma_{12,2} + \sigma_{13,3} dA}^{\text{Gauss divergence in 2D: } \cancel{\int_{\mathcal{A}} \sigma_{1j} n_j dA}} = N_{1,1}$$

where $\underline{n} = n_j \underline{e_j}$ is the **outward pointing normal** on the boundary of the section $(n_1 = 0)$.

- $\sigma_{1j} n_j$ is the $\underline{e_1}$ component of the traction vector on the free surface. By definition this has to be zero, so we have $N_{1,1} = 0$.
- Defining the shearing forces as $V_2 = \int_{\mathcal{A}} \sigma_{12} dA$ and $V_3 = \int_{\mathcal{A}} \sigma_{13} dA$, the second two equations can be read as:

$$\boxed{V_{2,1} + F_2 = 0}, \qquad \boxed{V_{3,1} + F_3 = 0}.$$

1.3. Equilibrium Equations

Unsymmetrical Bending

- In order to relate the different stresses, we invoke $M_2 = X_3\sigma_{11}$ and $M_3 = -X_2\sigma_{11}$ now.
- We first pre-multiply $\sigma_{1j,j}$ by X_3 **and then integrate** over the section:

$$\int_{\mathcal{A}} X_3\sigma_{11,1}dA + \int_{\mathcal{A}} X_3\sigma_{12,2} + X_3\sigma_{13,3}dA = M_{2,1} + \int_{\mathcal{A}} (X_3\sigma_{12})_{,2} + (X_3\sigma_{13})_{,3} - \sigma_{13}dA$$

$$\int_{\mathcal{A}} (X_3\sigma_{12})_{,2} + (X_3\sigma_{13})_{,3}dA = \cancel{\int_{\partial\mathcal{A}} X_3\sigma_{1k}n_k d\ell} \implies M_{2,1} - \int_{\mathcal{A}} \sigma_{13}dA = \boxed{M_{2,1} - V_3 = 0}.$$

- Next we pre-multiply $\sigma_{1j,j}$ by X_2 and repeat the same:

$$\int_{\mathcal{A}} X_2\sigma_{11,1}dA + \int_{\mathcal{A}} X_2\sigma_{12,2} + X_3\sigma_{13,3}dA = -M_{3,1} + \int_{\mathcal{A}} (X_2\sigma_{12})_{,2} - \sigma_{12} + (X_2\sigma_{13})_{,3}dA$$

$$\int_{\mathcal{A}} (X_2\sigma_{12})_{,2} + (X_2\sigma_{13})_{,3}dA = \cancel{\int_{\partial\mathcal{A}} X_2\sigma_{1k}n_k d\ell} \implies M_{3,1} + \int_{\mathcal{A}} \sigma_{12}dA = \boxed{M_{3,1} + V_2 = 0}.$$

- We are finally left with 4 equilibrium equations applicable for beam theory:

$$\boxed{V_{2,1} + F_2 = 0}, \quad \boxed{V_{3,1} + F_3 = 0}, \quad \boxed{M_{2,1} - V_3 = 0}, \quad \boxed{M_{3,1} + V_2 = 0}.$$

These are independent of any kinematic assumptions that we may make.

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Unsymmetrical Bending

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- We first pre-multiply $\sigma_{1j,j}$ by X_3 **and then integrate** over the section:

$$\int X_3\sigma_{11,1}dA + \int X_3\sigma_{12,2} + X_3\sigma_{13,3}dA = M_{2,1} + \int (X_3\sigma_{12})_{,2} + (X_3\sigma_{13})_{,3} - \sigma_{13}dA$$

$$\int_{\mathcal{A}} (X \quad \ldots \quad \boxed{- V_3 = 0}.$$

Transverse Force-Bending Moment Relationship

$$V_{2,1} + F_2 = 0, \quad V_3,1 + F_3 = 0, \quad M_{2,1} - V_3 = 0, \quad M_{3,1} + V_2 = 0$$

$$\implies \begin{bmatrix} M_{3,11} \\ -M_{2,11} \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}.$$

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$$\int_{\mathcal{A}} X_2\sigma_{11,1}dA + \int_{\mathcal{A}} X_2\sigma_{12,2} + X_3\sigma_{13,3}dA = -M_{3,1} + \int_{\mathcal{A}} (X_2\sigma_{12})_{,2} - \sigma_{12} + (X_2\sigma_{13})_{,3}dA$$

$$\int_{\mathcal{A}} (X_2\sigma_{12})_{,2} + (X_2\sigma_{13})_{,3}dA = \cancel{\int_{\partial\mathcal{A}} X_2\sigma_{1k}n_k d\ell} \implies M_{3,1} + \int_{\mathcal{A}} \sigma_{12}dA = \boxed{M_{3,1} + V_2 = 0}.$$

- We are finally left with 4 equilibrium equations applicable for beam theory:

$$\boxed{V_{2,1} + F_2 = 0}, \quad \boxed{V_{3,1} + F_3 = 0}, \quad \boxed{M_{2,1} - V_3 = 0}, \quad \boxed{M_{3,1} + V_2 = 0}.$$

These are independent of any kinematic assumptions that we may make.

1.4. Equations of Motion in Terms of Displacement

Unsymmetrical Bending

- The moments are related to the kinematics through

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = E_Y \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix}.$$

- For the zero shear strain case ($\theta_2 = -w'$, $\theta_3 = v'$) the equilibrium equations simplify in the following manner:

$$\begin{bmatrix} M_{3,11} \\ -M_{2,11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} M_{2,11} \\ M_{3,11} \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix},$$

$$E_Y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v'''' \\ w'''' \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix},$$

$$\implies E_Y \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix} \begin{bmatrix} v'''' \\ w'''' \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix},$$

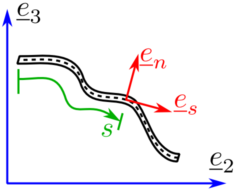
or in more compact notation,

$$\boxed{E_Y \underset{\approx}{I} \underset{\sim}{V}'''' = \underset{\sim}{F}}, \quad \underset{\approx}{I} = \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix}, \quad \underset{\sim}{V} = \begin{bmatrix} v \\ w \end{bmatrix}.$$

(Recall that the planar symmetric bending equation is $EIv'''' = F$)

2. Shear Stress and Flow in Sections

Thin Section: Plane Stress Assumption



- Now we shall pursue the equilibrium equations for thin-walled sections.
- We define the above section-local coordinate system and transform the elasticity equations to $\sigma_{11,1} + \sigma_{1s,s} + \sigma_{1n,n} = 0$. We integrate this along the thickness:

$$\int_{X_n - \frac{t}{2}}^{X_n + \frac{t}{2}} \sigma_{11,1} dX_n + \int \sigma_{1s,s} dX_n + \cancelto{0}{\int \sigma_{1n,n} dX_n} = 0$$

- σ_{1n} has to be zero on the surfaces with normal $\underline{e_n}$ since these are "free" surfaces; so the last integral goes to zero. The integral above simplifies (for constant thickness along s) to:

$$t\sigma_{11,1} + \int \sigma_{1s,s} dX_n = 0 \implies \boxed{t\sigma_{11,1} + q_{,s} = 0},$$

  where we define **shear flow** q, a new quantity that is basically the integral of the transverse shear stress along the thickness: $\boxed{q(s) = \int \sigma_{1s} dX_n}$.

2.1. Shear Flow Distribution

Shear Stress and Flow in Sections

- The stress distribution is written as:

$$\sigma_{11} = \frac{\begin{bmatrix} X_3 & -X_2 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \end{bmatrix}.$$

- Differentiating this we get:

$$\sigma_{11,1} = \frac{\begin{bmatrix} X_3 & -X_2 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix} \begin{bmatrix} \cancelto{V_3}{M_{2,1}} \\ \cancelto{-V_2}{M_{3,1}} \end{bmatrix}$$

$$\implies \sigma_{11,1} = \frac{\begin{bmatrix} X_2 & X_3 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}.$$

- Substituting this in $t\sigma_{11,1} + q_{,s} = 0$ we have,

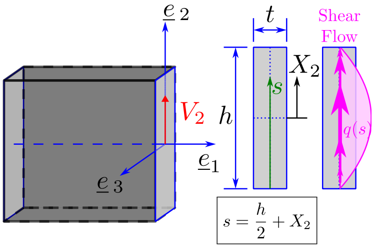
$$\frac{dq}{ds} = -\frac{\begin{bmatrix} tX_2 & tX_3 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}.$$

- Integrating this from some point we designate as $s = 0$, we have

$$\boxed{q(s) - q_0 = -\frac{\begin{bmatrix} \int_0^s tX_2 ds & \int_0^s tX_3 ds \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}}.$$

2. Shear Stress and Flow in Sections

- Consider the rectangular section with height h and thickness t:



$$q(s) = -\frac{V_2}{I_{33}} \int_0^s t X_2 ds = -\frac{tV_2}{I_{33}} \int_{-\frac{h}{2}}^{X_2} X_2 dX_2$$

$$= -\frac{tV_2}{2I_{33}} (X_2^2 - \frac{h^2}{4})$$

- Remember that V_2 is NOT any externally **applied force**. It is merely **the resultant of all the shear stresses in the section**.
- We are asking the question: **what SHOULD be the distribution of shear stresses (flow) so that their resultant is V_2?**
- So V_2 and $q(s)$ point in the same direction in this example. It is incorrect to think that $q(s)$ is balancing out V_2.

2. Shear Stress and Flow in Sections

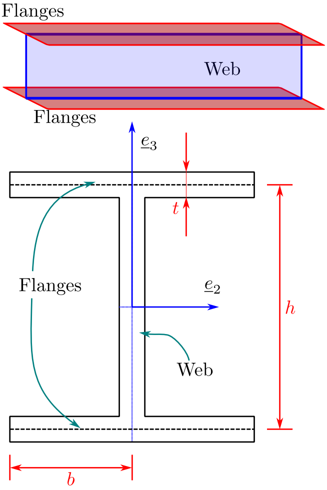
The "I" section

- Consider the shear distribution through an I-section as shown here
- The shear distribution looks like it is "flowing", with more "flow" occurring in the thin vertical web and less in the flanges.
- The second moment of area I_{22} sums up as,

$$I_{22} = \overbrace{\frac{h^3 t}{12}}^{web} + 2 \times \overbrace{\left(\frac{2bt^3}{12} + 2bt \times \frac{h^2}{4}\right)}^{flange} \approx (\frac{h^3}{12} + bh^2)t.$$

- I_{33} sums up as,

$$I_{33} = \underbrace{\frac{ht^3}{12}}_{web} + 2 \times \overbrace{\left(\frac{2b^3 t}{3}\right)}^{flange} \approx \underbrace{\frac{4b^3 t}{3}}_{\approx 0 \text{ for small b}}.$$

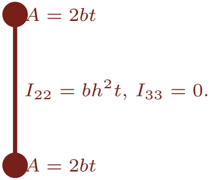


2. Shear Stress and Flow in Sections

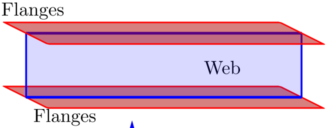
The "I" section

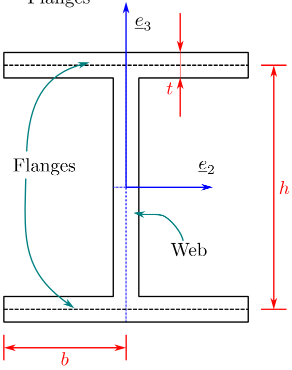
Idealization

- Both I_{22} and I_{33} are dominated by flange contributions, implying that **bending is supported primarily by the flanges.**
- This motivates the following idealization for the I-section:



- The lumped area elements denoted ● are sometimes referred to as **"Booms"** in the section.
- Thickness in the web (denoted ▬) is taken to be zero for bending-stress calculations.

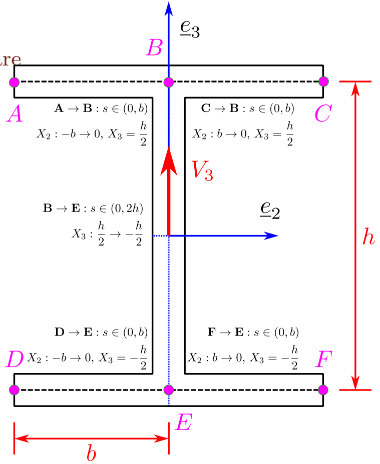




2. Shear Stress and Flow in Sections

The "I" section

- Let us consider the case with $V_2 = 0$, $V_3 \neq 0$.
- The segments $A \to B$, $C \to B$, $D \to E$, and $F \to E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).



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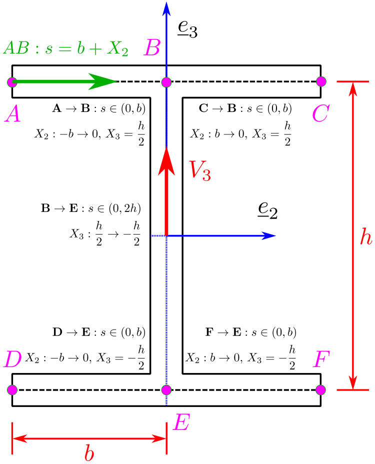
$$\mathbf{A} \to \mathbf{B} \;:\; q_{AB}(s) \equiv q_{AB}(X_2) =$$

$$-\frac{tV_3}{I_{22}} \int_0^s \cancelto{\frac{h}{2}}{X_3} ds = -\frac{htV_3}{2I_{22}}(b + X_2)$$



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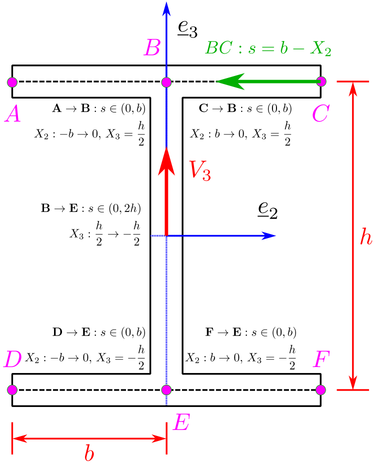
$$\mathbf{C} \to \mathbf{B} \ : \ q_{CB}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$



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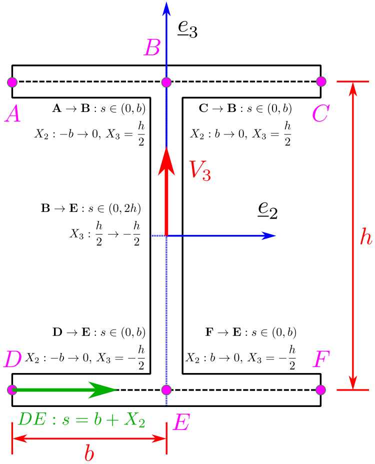
$$\mathbf{D} \to \mathbf{E} \; : \; q_{DE}(X_2) = \frac{htV_3}{2I_{22}}(b + X_2)$$



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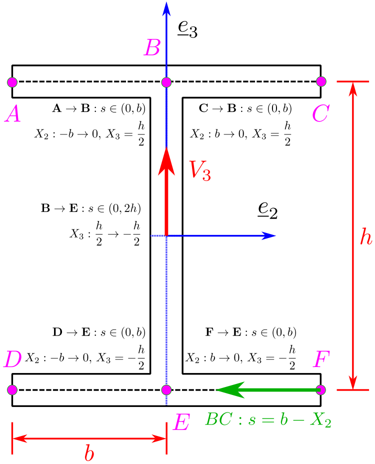
$$\mathbf{D} \to \mathbf{E} \ : \ q_{DE}(X_2) = \frac{htV_3}{2I_{22}}(b + X_2)$$

$$\mathbf{F} \to \mathbf{E} \ : \ q_{FE}(X_2) = \frac{htV_3}{2I_{22}}(b - X_2)$$



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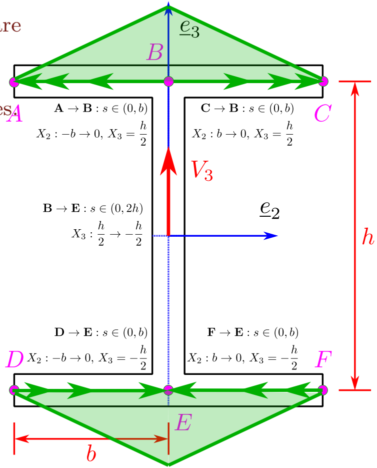
- In summary we have linear relationships at the flanges.



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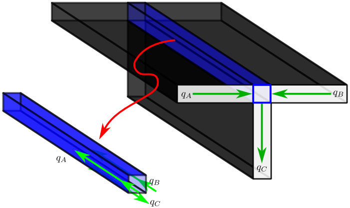
- In summary we have linear relationships at the flanges.
- Before looking at the web ($B \to E$), we have to observe the balance at the "T" junction.

Balance at the T-junction



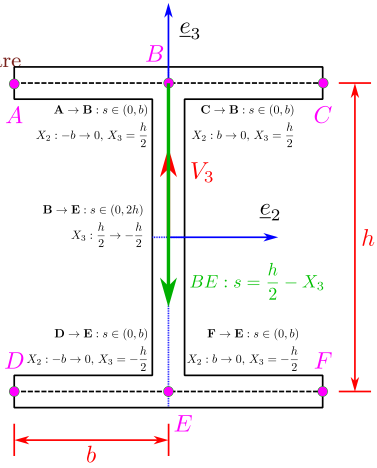
$\sum F = q_A + qB - qC = 0 => q_C = q_A + q_B$

2. Shear Stress and Flow in Sections

The "I" section

- Let us consider the case with $V_2 = 0$, $V_3 \neq 0$.
- The segments $A \to B$, $C \to B$, $D \to E$, and $F \to E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).
- On $\mathbf{B} \to \mathbf{E}$, we have $q_{BE}(0) = q_{AB}(0) + q_{CB}(0) = -\frac{bhtV_3}{I_{22}}$.
- The integration evaluates as,

$$q_{BE}(X_3) = -\frac{bhtV_3}{I_{22}} + \frac{tV_3}{2I_{22}}(X_3^2 - \frac{h^2}{4})$$
$$= -\frac{htV_3}{I_{22}}(b + \frac{h}{8}) + \frac{tV_3}{2I_{22}}X_3^2.$$



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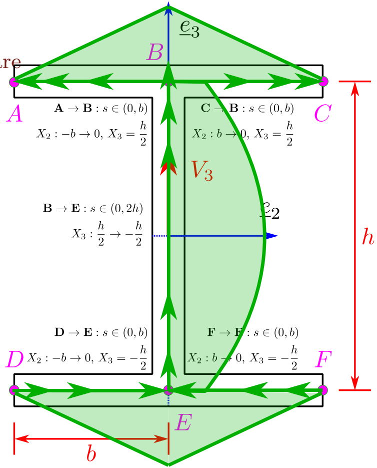
- We now have the complete shear flow in the section.



2. Shear Stress and Flow in Sections

The “I” section: Order of Magnitude Analysis

- Let us consider the “total” shear forces experienced by each member.
- **Flange AB**

$$V_{AB} = -\frac{htV_3}{2I_{22}}\int_{-b}^{0}(b+X_2)dX_2 = -\frac{b^2htV_3}{4I_{22}}$$

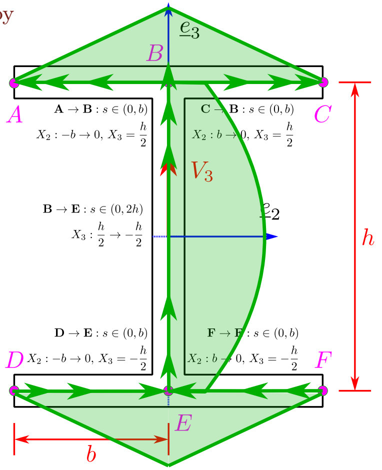
- **Web BE**

$$V_{BE} = \frac{h^2(h+12b)tV_3}{12I_{22}} = V_3$$

- For $b=\frac{h}{2}$, we have,

$$V_{AB} = -\frac{h^3tV_3}{16I_{22}} \approx -\frac{V_3}{8}$$

$$V_{BE} = V_3$$



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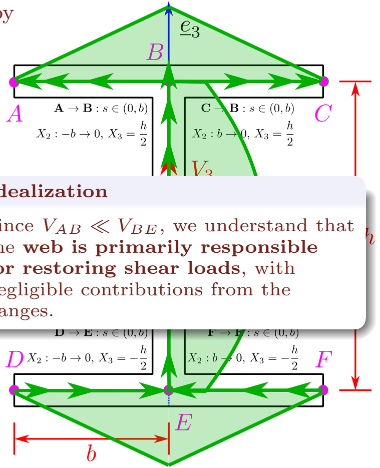
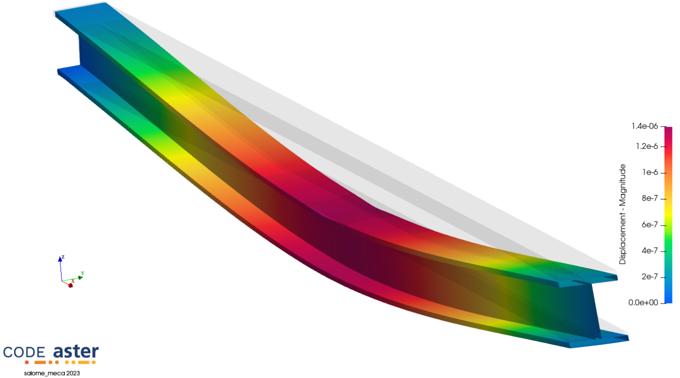


Idealization

Since $V_{AB} \ll V_{BE}$, we understand that the **web is primarily responsible for restoring shear loads**, with negligible contributions from the flanges.

2. Shear Stress and Flow in Sections

An "I" section beam subjected to 3-point bending: Finite Element Results

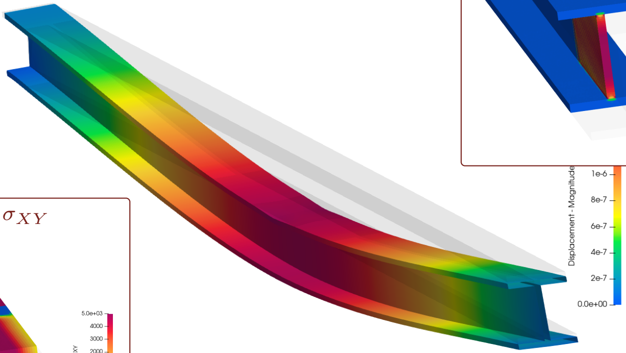


Code_Aster on the Salome Platform

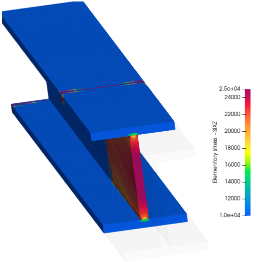
Free and Open Source (FOSS) FE solver that comes with a fully functional frontend (Salome)! Please Do Explore!

2. Shear Stress and Flow in Sections

An "I" section beam subjected to 3-point bending: Finite Element Results



Stress σ_{XZ}



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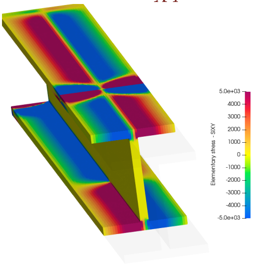

2. Shear Stress and Flow in Sections

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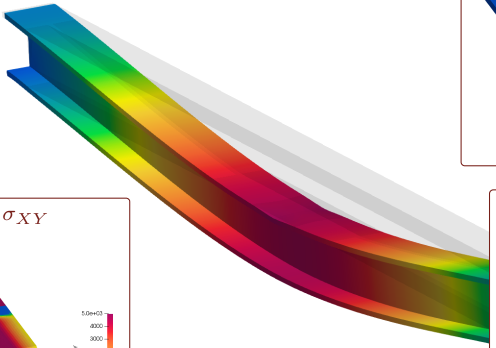

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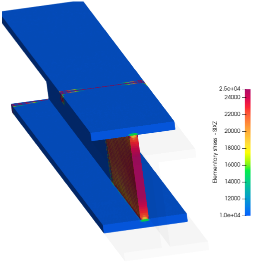

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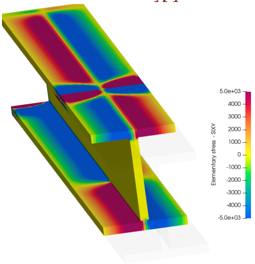
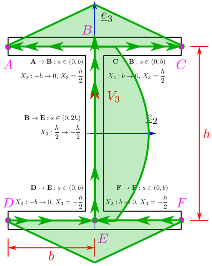


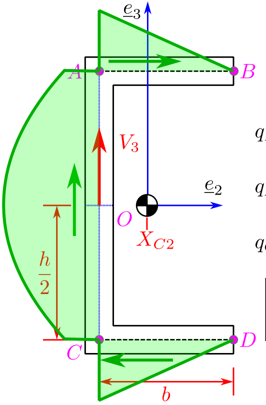
Shear Flow



2.2. Shear Center

Shear Stress and Flow in Sections

- Shear Center is the point of shear load such that $\sum m_1 = 0$. Although in a lot of symmetric sections this is coincident with the centroid, **this is NOT always the case**.
- Consider the "C" section beam:



$$\boxed{I_{22} \approxeq \frac{(h^3 + 6bh^2)t}{12}} + \mathcal{O}(t^2)$$

$$q_{BA}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

$$q_{AC}(X_3) = -\frac{htV_3}{2I_{22}}\left(b + \frac{h}{4}\right) + \frac{tV_3}{2I_{22}}X_3^2$$

$$q_{CD}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

$$\boxed{M_1 = \oint pq ds = -\frac{b^2h^2tV_3}{4I_{22}} := V_3\xi_s}$$

$$\boxed{\xi_s \approxeq -\frac{3b^2}{h + 6b} + \mathcal{O}(t)}$$

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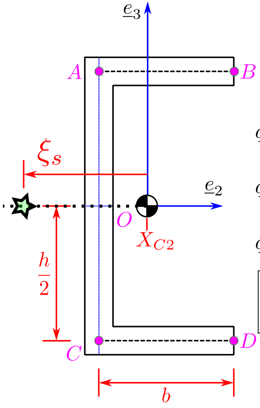

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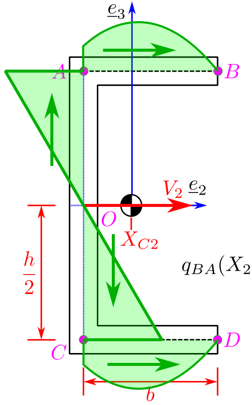


$$XC_2 = \frac{(bt)\frac{b}{2}\times 2}{2bt+ht} = \frac{b}{2+h/b}$$

$$q(s) - q(0) = -\frac{tV_2}{I_{33}}\int_0^s X_2 ds$$

Shear Flow

$$q_{BA}(X_2) = \frac{tV_2}{2I_{33}}\frac{(X_2^2(h+2b)^2 - b^2(h+b)^2)}{(h+2b)^2}$$

$$q_{AC}(X_3) = -\frac{tb^2V_2}{(2b+h)I_{33}}X_3$$

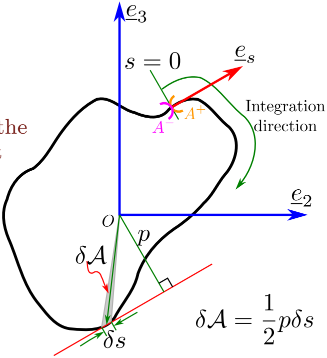
$$q_{CD}(X_2) = -q_{BA}(X_2)$$

2.3. Closed Sections

Shear Stress and Flow in Sections

- The shear-flow integral formula makes no reference to whether the section is open or closed.
- Considering the generic closed section shown, we start the integral at some **arbitrary** point A, denoting the point right before it as A^-. The integral is then written as,

$$q(s) - q_{A^-} = \underbrace{-\int_0^s t\sigma_{11,1} ds}_{q_b(s)}.$$



- When no twisting is expected at the section, the moment along $\underline{e}_1$ about O has to be zero. This is computed as,

$$\int_{A^-}^{A^+} dM_1 = \oint pq(s)ds = q_{A^-}\overbrace{\oint pq(s)ds}^{2\mathcal{A}} + \oint pq_b(s)ds \implies \boxed{q_{A^-} = -\frac{1}{2\mathcal{A}}\oint pq_b(s)ds}.$$

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Are we already assuming that O is the shear center with this?!

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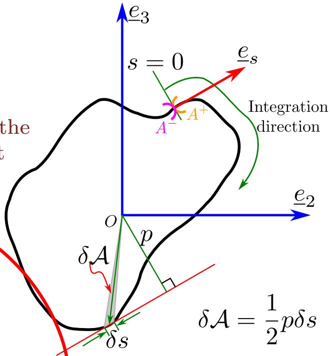

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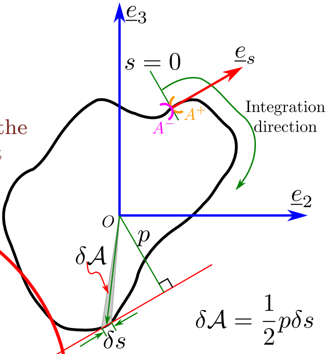
Are we already assuming that
O is the shear center with this?!

Yes. We need to add an extra term when resultants V_2, V_3 are acting with an offset.



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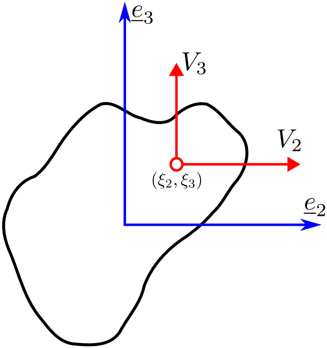
2.3. Closed Sections

Shear Stress and Flow in Sections



- If we suppose that the resultants are acting along some (ξ_2, ξ_3), the applied moment is: $(\xi_2 \underline{e}_2 + \xi_3 \underline{e}_3) \times (V_2 \underline{e}_2 + V_3 \underline{e}_3)$:

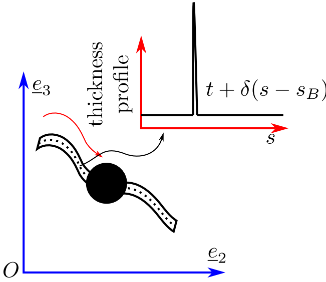
$$M_1 = \xi_2 V_3 - \xi_3 V_2.$$

- Equating this to the moment developed from the shear flow distribution, we get:

$$\xi_2 V_3 - \xi_3 V_2 = 2\mathcal{A} q_{A^-} + \oint p q_b(s) ds$$

- For shear center determination (the point around which resultants act), this is not enough.
- We will additionally invoke an argument of **zero twist** ($\theta_{1,1} = 0$) in the deflection field to get an additional relationship. This will be covered in the next module (Torsion).
- **For symmetric closed sections**, however, the shear center coincides with the centroid (through symmetry arguments).

3. Stringer-Web Idealization



- A stringer, when looked at from a section, looks like a feature with a sudden increase in thickness.
- Consider the r^{th} "Boom" to be located at (X_{r_2}, X_{r_3}), with area A_r.

- So the shear flow integral can be generalized to,

$$q(s) - q(0) = -\frac{\begin{bmatrix} t\int_0^s X_3 ds + \sum_r A_r X_{r_3} & -t\int_0^s X_2 ds - \sum_r A_r X_{r_2}\end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

- When thickness t is negligible in comparison to the boom sections, this further simplifies to,

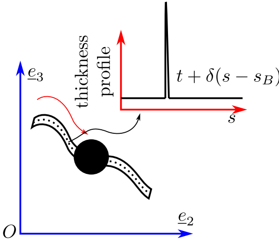
$$q(s) - q(0) = -\frac{\begin{bmatrix} \sum_r A_r X_{r_3} & -\sum_r A_r X_{r_2}\end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

3. Stringer-Web Idealization

- Considering the integral right across the boom, we have,

$$q^+ - q^- = -\frac{\begin{bmatrix} A_r X_{r_3} & -A_r X_{r_2} \end{bmatrix}}{I_{22} I_{33} - I_{23}^2} \begin{bmatrix} I_{33} V_3 - I_{23} V_2 \\ I_{23} V_3 - I_{22} V_2 \end{bmatrix}.$$

- For the sections without a boom, there is **no change in the shear flow**.
- Therefore, the shear flow is constant in the webs that join two booms.



General Design Principle

- As a general principle, the stringers/booms are added to support bending.
- The web thicknesses are chosen to ensure the shear stresses don't exceed failure threshold (we should like to have $t = 0$ to minimize weight!).

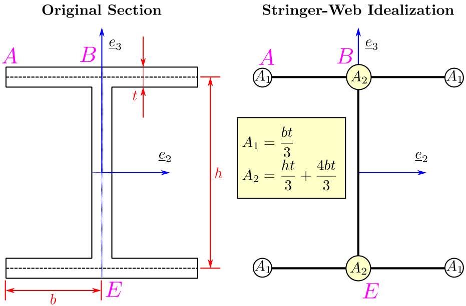
3. Stringer-Web Idealization

Idealization of the I-Section

- We idealize the I-section by lumping area elements A_1 and A_2. A_1 and A_2 are estimated by matching the second area moments:

$$\overbrace{4 \times (\frac{A_1 h^2}{4}) + 2 \times (\frac{A_2 h^2}{4})}^{I_{22,ideal}} = \overbrace{\left(\frac{h^3}{12} + bh^2\right) t + \mathcal{O}(t^2)}^{I_{22,orig}}$$

$$\overbrace{4 \times (A_1 b^2)}^{I_{33,ideal}} = \overbrace{\frac{4b^3 t}{3} + \mathcal{O}(t^2)}^{I_{33,ideal}}$$



- The total sectional area of the original section is $ht + 4bt$.
- The total area of the new section (assuming the web thickness is drastically reduced) is $4A_1 + 2A_2 = \frac{2ht}{3} + 4bt$, which is a **slight reduction**.
- Looking at this from a manufacturing standpoint, this shows that **a web-stringer construction can achieve similar bending stiffness with lesser material expenditure**.

3. Stringer-Web Idealization

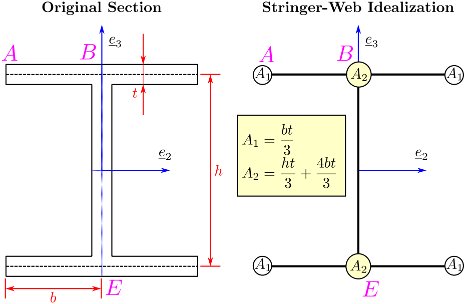
Idealization of the I-Section: Shear Flow Comparisons

- For the idealized section AB, the shear flow is given by,

$$q_{AB,ideal} = -\frac{V_3}{I_{22,ideal}} A_1 \frac{h}{2} = -\frac{V_3 b}{h^2 + 12bh}.$$

The average flow for the original section is,

$$q_{AB,avg} = \frac{1}{b}\left(-\frac{b^2 h t V_3}{4 I_{22}}\right) = \frac{3 V_3 b}{h^2 + 12bh}.$$



- On BE, the idealized flow is

$$q_{BE,ideal} = 2q_{AB,ideal} - \frac{V_3}{I_{22,ideal}} A_2 \frac{h}{2} = -\frac{V_3}{h},$$

which is the same for the original section also.

- In the stringer-web section, therefore, the flanges carry lesser average shear than the original section, and the web carries the same shear.

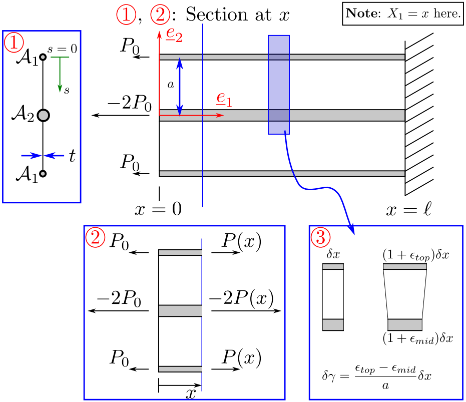
4. Shear Lag

Example from Sun (2006) (Section 3.1)

- We already saw the St. Venant's principle. Can we say something more specific about "how fast" the end effects start getting smoothed out in the stress field? Consider:

- Applying equilibrium on the section at some $X_1 = x$ 1 leads to top and bottom booms experiencing $\boxed{\int_{\mathcal{A}_1} \sigma_{11} dA = P(x)}$, and the central boom, $\boxed{\int_{\mathcal{A}_2} \sigma_{11} dA = -2P(x)}$.
- Shear flow in top web is given as

$$q_{top} = -\sigma_{11,1}\mathcal{A}_1 = -P_{,1}$$
$$\text{or,} \ -t\sigma_{12} = -P_{,1}.$$



4. Shear Lag

Example from Sun (2006) (Section 3.1)

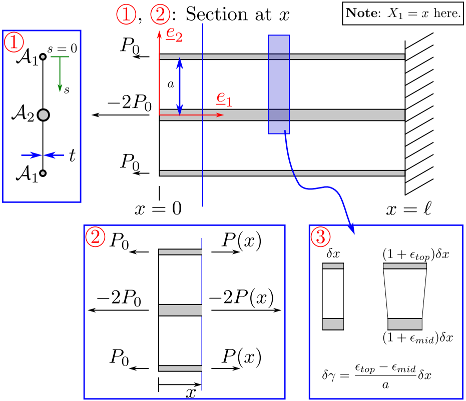
- Considering the shear differential in an infinitesimal section (3) we have,

$$\frac{\partial \gamma_{12}}{\partial x} = \frac{1}{a}(\epsilon_{top} - \epsilon_{mid})$$
$$= \frac{1}{E_Y a}\left(\frac{P(x)}{\mathcal{A}_1} - \frac{-2P(x)}{\mathcal{A}_2}\right)$$

- Since $\gamma_{12} = \frac{1}{G}\sigma_{12}$, this becomes,

$$\sigma_{12,1} = \frac{G}{E_Y a}\left(\frac{1}{\mathcal{A}_1} + \frac{2}{\mathcal{A}_2}\right) P(x).$$

- Using the force balance relationship $\boxed{\sigma_{12} = \frac{1}{t} P_{,1}}$ also



$$\implies \boxed{P_{,11} = \frac{Gt}{E_Y a}\left(\frac{1}{\mathcal{A}_1} + \frac{2}{\mathcal{A}_2}\right) P}$$

4. Shear Lag

Example from Sun (2006) (Section 3.1)

- The equation governing the **boom restoring axial force** is of the form

$$P_{,11} - \lambda^2 P = 0, \qquad \lambda = \sqrt{\frac{Gt}{E_Y a}\left(\frac{1}{\mathcal{A}_1} + \frac{2}{\mathcal{A}_2}\right)}.$$

- This is solved by

$$P(X_1) = C_1 e^{-\lambda X_1} + C_2 e^{\lambda X_1}.$$

- Solving it over $X_1 \in (0, \infty)$, it is easy to see that C_2 must be 0 for $P(X_1)$ to be bounded. So we have an **exponentially decaying** restoring force on the booms:

$$P(X_1) = P_0 e^{-\lambda X_1}.$$

- Substituting for σ_{12} we have (for the top web),

$$\sigma_{12} = -\frac{\lambda P_0}{t} e^{-\lambda X_1},$$

which also decays exponentially in X_1.

4. Shear Lag: Decay Rate

Example from Sun (2006) (Section 3.1)

- The shear lag factor λ controls how quickly the effects "diffuse out".
  - Large λ implies "fast" diffusion and potentially high concentration around the ends..
  - Small λ implies "slow" diffusion and potentially low concentrations.
- In general for stringer-web structures,

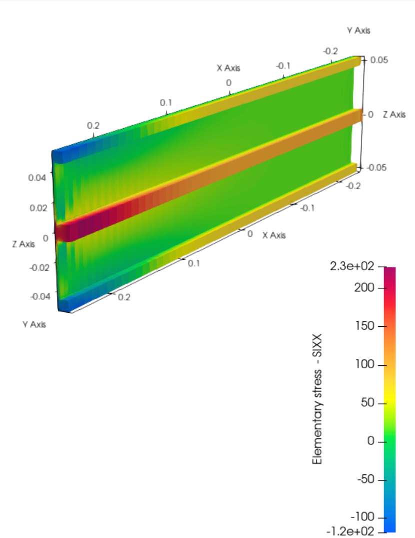
$$\lambda \propto \sqrt{\frac{G}{E_Y}}.$$

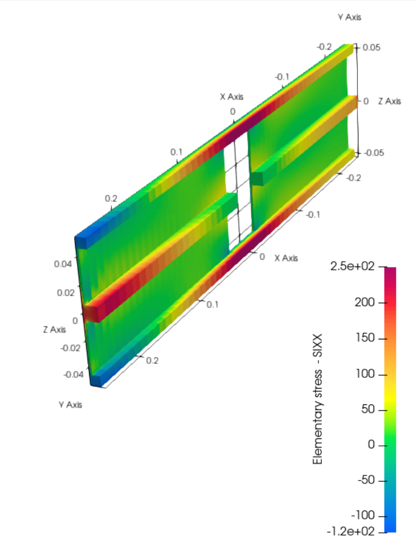
  - $\uparrow G$ (stiffer web), $\uparrow \lambda$ ("faster" diffusion).
  - $\uparrow E_Y$ (stiffer boom), $\downarrow \lambda$ ("slower" diffusion).

The terms "faster" and "slower" are used in the sense that "slower" implies that the stress distribution needs a longer axial distance to get averaged out (vise versa for "faster").

4. Shear Lag

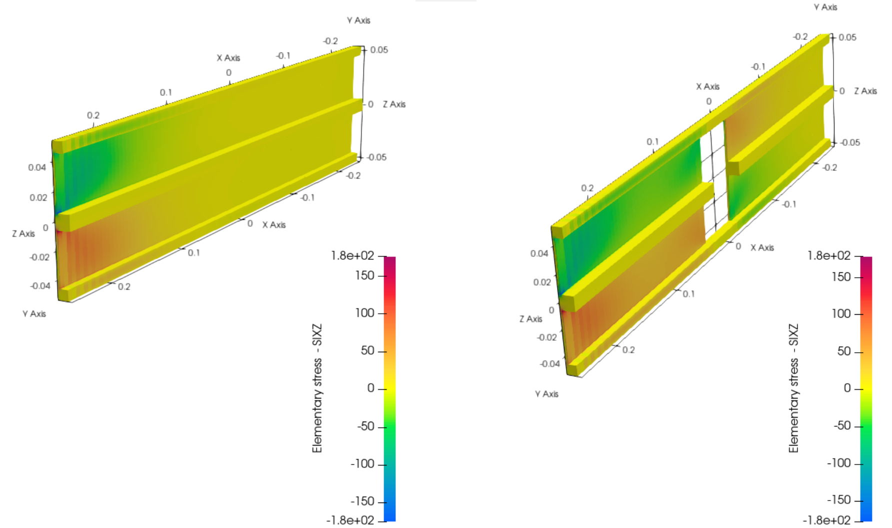
Example from Sun (2006) (Section 3.1): FE Results





4. Shear Lag

Example from Sun (2006) (Section 3.1): FE Results



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