

The Michell Solution

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ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$	$2\mu u_r$	$2\mu u_\theta$
r^2	2	0	2	$(\kappa - 1)r$	0
$r^2 \ln r$	$2 \ln(r) + 1$	0	$2 \ln(r) + 3$	$(\kappa - 1)r \ln(r) - r$	$(\kappa + 1)r\theta$
$\ln r$	$1/r^2$	0	$-1/r^2$	$-1/r$	0
θ	0	$1/r^2$	0	0	$-1/r$
$r\theta \sin \theta$	$2 \cos \theta / r$	0	0	$\frac{1}{2} \left\{ (\kappa - 1)\theta \sin \theta - \cos \theta + (\kappa + 1) \ln(r) \cos \theta \right\}$	$\frac{1}{2} \left\{ (\kappa - 1)\theta \cos \theta - \sin \theta - (\kappa + 1) \ln(r) \sin \theta \right\}$
$r \ln(r) \cos \theta$	$\cos \theta / r$	$\sin \theta / r$	$\cos \theta / r$	$\frac{1}{2} \left\{ (\kappa + 1)\theta \sin \theta - \cos \theta + (\kappa - 1) \ln(r) \cos \theta \right\}$	$\frac{1}{2} \left\{ (\kappa + 1)\theta \cos \theta - \sin \theta - (\kappa - 1) \ln(r) \sin \theta \right\}$
$r\theta \cos \theta$	$-2 \sin \theta / r$	0	0	$\frac{1}{2} \left\{ (\kappa - 1)\theta \cos \theta + \sin \theta - (\kappa + 1) \ln(r) \sin \theta \right\}$	$\frac{1}{2} \left\{ -(\kappa - 1)\theta \sin \theta - \cos \theta - (\kappa + 1) \ln(r) \cos \theta \right\}$
$r \ln(r) \sin \theta$	$\sin \theta / r$	$-\cos \theta / r$	$\sin \theta / r$	$\frac{1}{2} \left\{ -(\kappa + 1)\theta \cos \theta - \sin \theta + (\kappa - 1) \ln(r) \sin \theta \right\}$	$\frac{1}{2} \left\{ (\kappa + 1)\theta \sin \theta + \cos \theta + (\kappa - 1) \ln(r) \cos \theta \right\}$
$r^{n+2} \cos n\theta$	$-(n+1)(n-2)r^n \cos n\theta$	$n(n+1)r^n \sin n\theta$	$(n+1)(n+2)r^n \cos n\theta$	$(\kappa - n - 1)r^{n+1} \cos n\theta$	$(\kappa + n + 1)r^{n+1} \sin n\theta$
$r^{n+2} \cos(n+2)\theta$	$-(n+1)(n+2)r^n \cos(n+2)\theta$	$(n+1)(n+2)r^n \sin(n+2)\theta$	$(n+1)(n+2)r^n \cos(n+2)\theta$	$-(n+2)r^{n+1} \cos(n+2)\theta$	$(n+2)r^{n+1} \sin(n+2)\theta$
$r^{n+2} \sin n\theta$	$-(n+1)(n-2)r^n \sin n\theta$	$-n(n+1)r^n \cos n\theta$	$(n+1)(n+2)r^n \sin n\theta$	$(\kappa - n - 1)r^{n+1} \sin n\theta$	$-(\kappa + n + 1)r^{n+1} \cos n\theta$
$r^{n+2} \sin(n+2)\theta$	$-(n+1)(n+2)r^n \sin(n+2)\theta$	$-(n+1)(n+2)r^n \cos(n+2)\theta$	$(n+1)(n+2)r^n \sin(n+2)\theta$	$-(n+2)r^{n+1} \sin(n+2)\theta$	$-(n+2)r^{n+1} \cos(n+2)\theta$

with $\mu = \frac{E_y}{2(1+\nu)}$, $\kappa = \begin{cases} 3-4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$. Rigid Body Motion: $\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix} C_3 = \begin{bmatrix} C_1 \cos \theta + C_2 \sin \theta \\ -C_1 \sin \theta + C_2 \cos \theta + rC_3 \end{bmatrix}$.

Transformation: $\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix}$. Calculus: $\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta}$, $\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$.

Notch Crack Solution

$$\begin{aligned}\sigma_{rr} &= \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right) \\ \sigma_{\theta\theta} &= \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right) \\ \sigma_{r\theta} &= \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right) \\ 2\mu u_r &= K_I \sqrt{\frac{r}{2\pi}} \left(\left(\kappa - \frac{1}{2}\right) \cos \frac{\theta}{2} - \frac{1}{2} \cos \frac{3\theta}{2} \right) - K_{II} \sqrt{\frac{r}{2\pi}} \left(\left(\kappa - \frac{1}{2}\right) \sin \frac{\theta}{2} - \frac{3}{2} \sin \frac{3\theta}{2} \right) \\ 2\mu u_\theta &= K_I \sqrt{\frac{r}{2\pi}} \left(-\left(\kappa + \frac{1}{2}\right) \sin \frac{\theta}{2} + \frac{1}{2} \sin \frac{3\theta}{2} \right) - K_{II} \sqrt{\frac{r}{2\pi}} \left(\left(\kappa + \frac{1}{2}\right) \cos \frac{\theta}{2} - \frac{3}{2} \cos \frac{3\theta}{2} \right)\end{aligned}$$

Curvilinear Stresses and Strains

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{r} \left(\frac{\phi_{,\theta\theta}}{r} + \phi_{,r} \right) \\ \phi_{,rr} \\ -\left(\frac{\phi_{,\theta}}{r} \right)_{,r} \end{bmatrix}, \quad \begin{bmatrix} \mathcal{E}_{rr} \\ \mathcal{E}_{\theta\theta} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} u_{r,r} \\ \frac{u_{\theta,\theta}}{r} + \frac{u_r}{r} \\ \frac{u_{r,\theta}}{r} + r \left(\frac{u_\theta}{r} \right)_{,r} \end{bmatrix}$$