AS2070: Aerospace Structural Mechanics
Module 3: Introduction to Fatigue and Failure (V1)

Instructor: Nidish Narayanaa Balaji
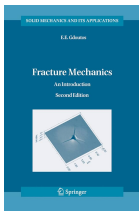
Department of Aerospace Engineering, IIT Madras
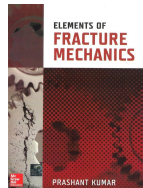
January 21, 2026

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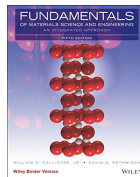
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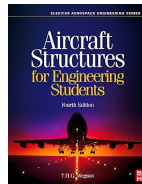
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1.1. Structure of Materials
Introduction
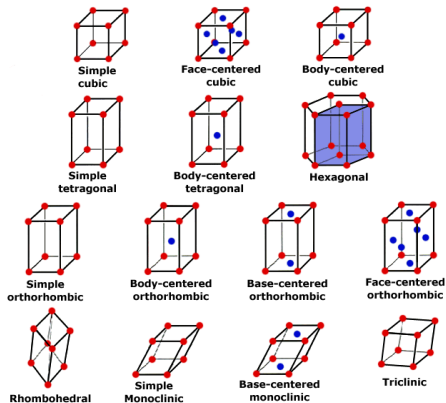
Types of crystal structures in metals Sparky (2013)

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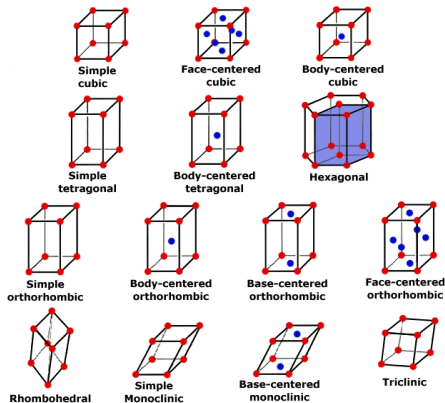
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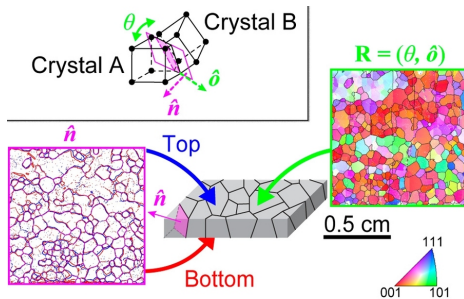
Crystal and Grain Structures New Technique Provides Detailed Views of Metals' Crystal Structure (2016). "Polycrystallinity"

1.1. Structure of Materials
Introduction

Dislocations and their "motion" under load

Applied shear

Figure from Sparky (2013)

$R = (\theta, \hat{o})$

0.5 cm

111

001   101

... que Provides Detailed Views of Metals' Crystal Structure (2016). "Polycrystallinity"

Simple cubic

Simple tetragonal

Simple orthorhombic

Rhombohedral   Monoclinic   monoclinic
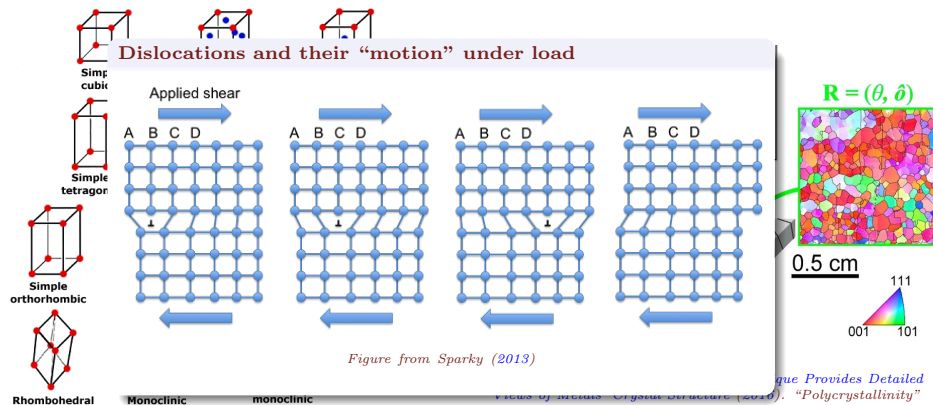
Types of crystal structures in metals Sparky (2013)

1.2. Understanding the Stress-Strain Curve
Introduction

The Uniaxial Tensile Test
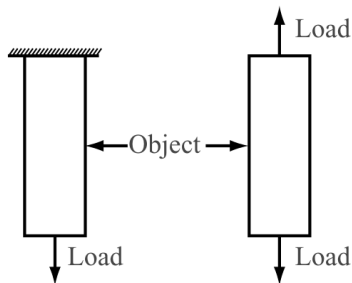
Figure from Rajendran 2011

1.2. Understanding the Stress-Strain Curve
Introduction

Terminology

1. Proportionality Limit;
2. Elastic Limit;
3. Yield Point;
4. Ultimate Strength;
5. Fracture Point;
6. Elongation at Failure;

Ductile Fracture

Figure from Rajendran 2011

Ductile Material Stress-Strain Curve
low carbon steel

Figure from Connor 2020

1.3. Failure Mechanisms: Fracture
1. Introduction

"Griffith Theory" of brittle fracture

- Theoretical fracture stress $\sim \frac{E}{5} - \frac{E}{30}$
  (steel $\sim \frac{E}{1000}$)
- Fracture occurs when
  $E_{strain} = E_{surface}$
- Crack propagates when
  $\frac{dE_{strain}}{dL} = \frac{dE_{surface}}{dL}$

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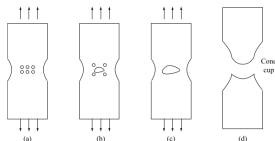- Crack propagates when $\frac{dE_{strain}}{dL} = \frac{dE_{surface}}{dL}$
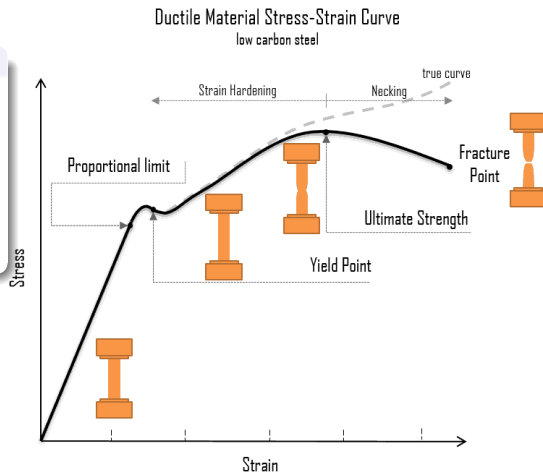
Ductile Fracture

Ductile Fracture Rajendran 2011

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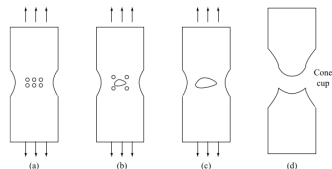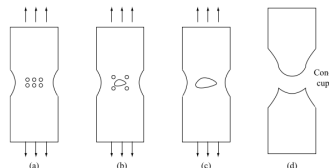- Crack propagates when $\frac{dE_{strain}}{dL} = \frac{dE_{surface}}{dL}$

Ductile Fracture

Cone cup

Ductile Fracture Rajendran 2011

Sr. No	Brittle Fracture	Ductile Fracture
1.	It occurs with no or little plastic deformation.	It occurs with large plastic deformation.
2.	The rate of propagation of the crack is fast.	The rate of propagation of the crack is slow.
3.	It occurs suddenly without any warning.	It occurs slowly.
4.	The fractured surface is flat.	The fractured surface has rough contour and the shape is similar to cup and cone arrangement.
5.	The fractured surface appears shiny.	The fractured surface is dull when viewed with naked eye and the surface has dimpled appearance when viewed with scanning electron microscope.
6.	It occurs where micro crack is larger.	It occurs in localised region where the deformation is larger.

Ductile vs Brittle Fracture Rajendran 2011
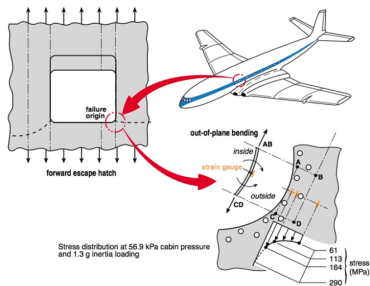
1.3. Failure Mechanisms: Fatigue
1. Introduction

..over 90% of mechanical failures are caused because of metal fatigue *What Is Metal Fatigue? 2021...*

The De Havilland Comet The deHavilland Comet Disaster 2019 [lecture]

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A more recent example (2021 United Airlines Boeing 777) DCA21FA085.Aspx 2024. [video]

The De Havilland Comet The deHavilland Comet Disaster 2019 [lecture]

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Fatigue Crack Propagation: Beech Marks

Figure from Fatigue Physics 2024
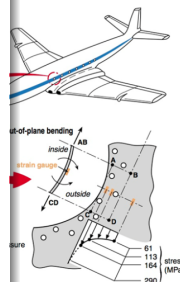
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Fatigue variables Megson 2013
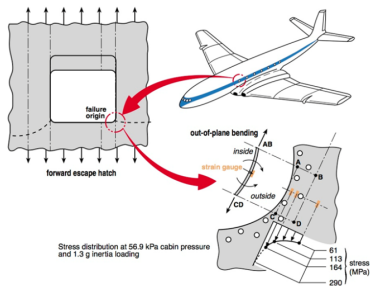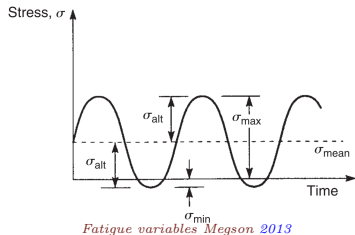
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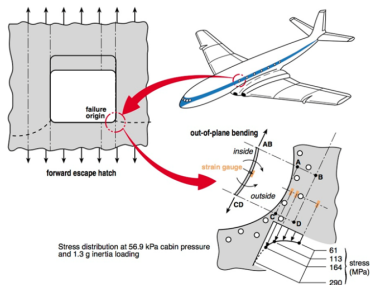*2021...*

Fatigue variables Megson 2013

The S-n Diagram Megson 2013

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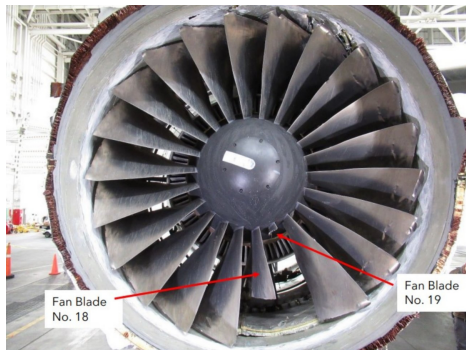*2021*...

S-N Curves for Common Metals (*Jr and Rethwisch* **2012**)

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S-N Curves for Common Metals (Jr and Rethwisch **2012**)

Homework: Watch this video on Fatigue testing.

1.4. Energy Release Rate: Griffith's Analysis
Introduction

Simplistic picture of the introduction of a crack in a stretched specimen(Figure from sec 2.5 in Kumar 2009)

- Because of the crack, we assume $\sigma \approx 0$ in the triangles.

- Corresponding energy loss:

$$E_R = V_\Delta \times (\frac{\sigma^2}{2E}) = \frac{2a^2\lambda t\sigma^2}{E}.$$

1.4. Energy Release Rate: Griffith's Analysis
Introduction
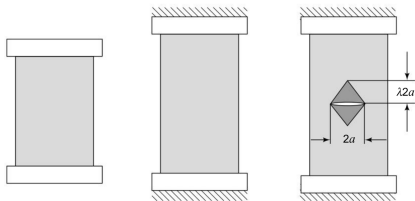
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- For thin plates, $\lambda = \frac{\pi}{2}$. So,

$$E_R = \frac{\pi a^2 t\sigma^2}{E}.$$

- The "creation" of a surface takes energy. We write this as,

$$E_S = 2(2at)\gamma = 4at\gamma.$$

(Figure from Kumar 2009)
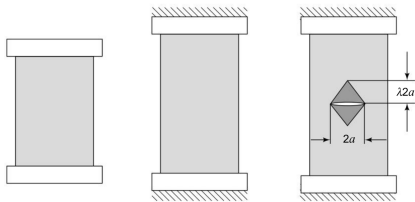
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Food For Thought

- What would a "safe size" of crack be, for a given loading condition? *Hint: Think incrementally*

- Because of
the triang

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Food For Thought

- What would a "safe size" of crack be, for a given loading condition? *Hint: Think incrementally*

- What type of an experiment would be necessary to confirm this mathematical framework?

- Because o... the triang...

- Corresponding energy loss:
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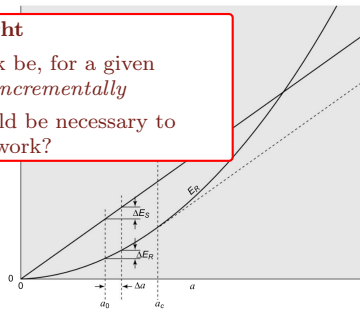
(Figure from Kumar 2009)

1.5. Linear Elastic Fracture Mechanics
Introduction

(Ref: Sec. 8.4.2 in Sadd 2009)

Consider the following two cases.
Question: Where will the point of peak stress occur? And which will have higher maximum stress?
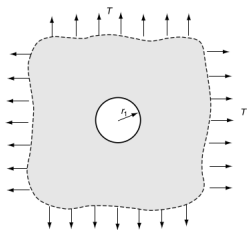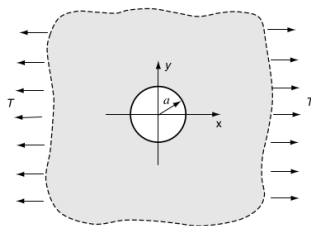
Case 1

Case 2

1.5. Linear Elastic Fracture Mechanics
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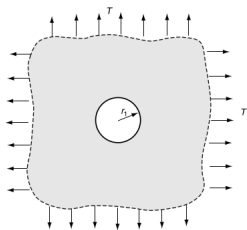
Case 2
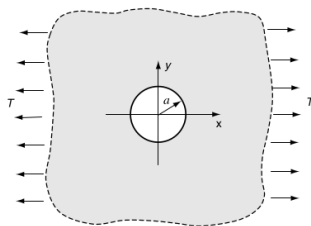
Analytical Solution

$\sigma_r = T(1 - \frac{r_1^2}{r^2})$, $\sigma_\theta = T(1 + \frac{r_1^2}{r^2})$

$\implies$ $\boxed{\sigma_{\max} = 2T}$

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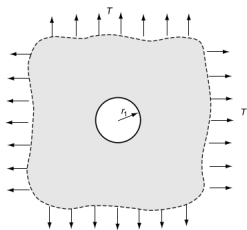
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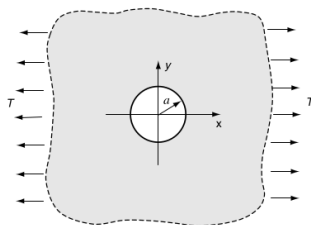
Analytical Solution

$\sigma_r = T(1 - \frac{r_1^2}{r^2})$, $\sigma_\theta = T(1 + \frac{r_1^2}{r^2})$

$\implies$ $\boxed{\sigma_{\max} = 2T}$

Analytical Solution

$\sigma_r = T(1 - \frac{r_1^2}{r^2}) + (\cdot)\cos(2\theta)$, $\sigma_\theta = \ldots$

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Introduction

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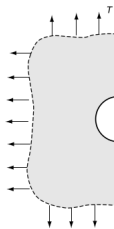
Case 3

Analytical Solution

$$\sigma_{\max} = 4T$$
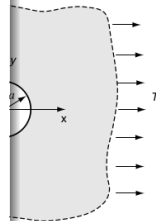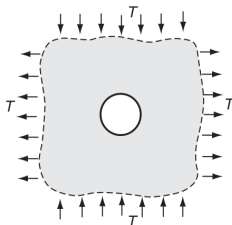
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$\implies$ $\boxed{\sigma_{\max} = 3T}$

1.6. Modes of Fracture
Introduction

Mode I
(opening mode)

Mode II
(sliding mode)

Mode III
(tearing mode)
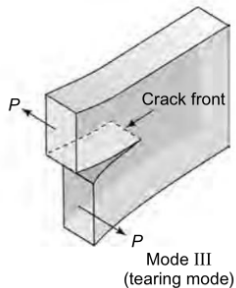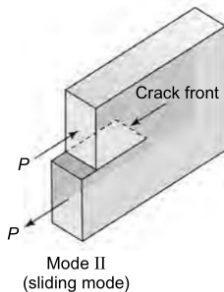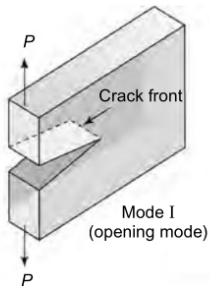
2. Introduction to Fatigue

Concepts

- Safe Life: RUL
- Fail-Safe: Redundancy

(Figure 15.1 from Megson 2013)

Tensile Stresses: The Goodman Diagram
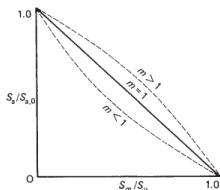
(Figure 15.2 from Megson 2013)

$$\frac{S_a}{S_{a,0}} = 1 - \left(\frac{S_m}{S_u}\right)$$

The S-N Curve
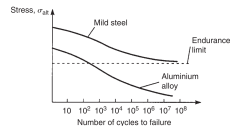
(Figure from Megson 2013)

$$\sigma_{alt} = \sigma_\infty \left(1 + \frac{C}{\sqrt{N}}\right), \quad N \propto \frac{1}{\sigma_{mean}}.$$

2.1. The deHavilland Comet
Introduction to Fatigue

No aircraft has contributed more to safety in the jet age than the Comet. The lessons it taught the world of aeronautics live in every jet airliner flying today. – D.D. Dempster, 1959, in The Tale of the Comet
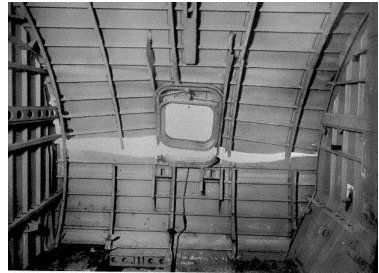
FIG. 7. VIEW FROM INSIDE OF FAILURE AT THE FORWARD ESCAPE HATCH ON THE PORT SIDE—COMET G-ALYU

(Figures from "De Havilland Comet" 2025)

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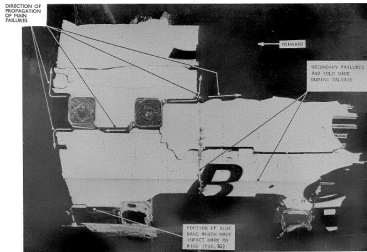
FIG. 12. PHOTOGRAPH OF WRECKAGE AROUND ADF AERIAL WINDOWS—G-ALYP.

FIG. 7. VIEW FROM INSIDE OF FAILURE AT THE FORWARD ESCAPE HATCH ON THE PORT SIDE—COMET G-ALYU

(Figures from "De Havilland Comet" 2025)

2.1. The d[...]

No aircr[...] [...]lessons it
taught th[...] [...]empster,
1959, in[...]

crack along top center line of fuselage

front fuselage separated at front spar
attachments in downward direction

rear fuselage
and tail unit
separated at
rear spar

failure probably downwards
symmetrical with starboard
wing failure

main failure
between ribs
12 and 13

frame 26

peeling off failure

frame 13a   frame 18

secondary cracking by bending
of center portion over outer portion

direction of propagation
of main cracks

top ₵ of aircraft

signs of fatigue
in skin at this corner

secondary
failure

forward   starboard

reinforcing
plates

port

peeling off failures

skin pulled over rivets on window frame   peeling off failures

DIRECTION OF
PROPAGATION
OF MAIN
FAILURES

FIG. 12. PHOTO[...]

[...]TCH ON THE

[...]t" 2025)

2.2. Miner's Rule
Introduction to Fatigue

- Suppose at an operation level of σ_m, σ_a, the fatigue life is N and the structure undergoes n cycles, Miner's rule posits that $\frac{n}{N}$ is the <u>fraction of life that has been consumed</u>.
- Suppose a structure undergoes multiple stress levels in its loading history, the total fraction of fatigue life that has been consumed is written as

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots.$$

- The structure is expected to fail when this sum becomes 1.0..

3.1. Griffith's Analysis and Energy Release Rate
Linear Elastic Fracture Mechanics

- The total energy of a loaded elastic body is written as

$$\Pi = \underbrace{U}_{\text{elastic}} - \underbrace{W}_{\text{external}}.$$

- **Griffith's principle**: The **energy lost due to the creation** of a cracked surface must be equal to the **energy required for the creation** of the cracked surface.
- Surface energy is usually expressed as $E_S = \mathcal{A}\gamma$.
- This is a general principle agnostic of the exact structure under consideration.

$$\boxed{G = -\frac{d\Pi}{d\mathcal{A}} = 2\gamma}.$$

  (note: $2\mathcal{A}$ is the effective total "new" surface area that has been created)

3.1. Griffith's Analysis and Energy Release Rate: Examples
Linear Elastic Fracture Mechanics

Crack in Stretched Specimen
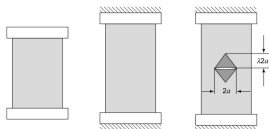
(Figure from sec 2.5 in Kumar 2009)

- Crack: $\mathcal{A} = 2at$, $\partial_{\mathcal{A}} = \frac{1}{2t}\partial_a$

- $\Pi = U = \frac{\sigma^2 t}{2E'}(\mathcal{A}_{tot} - 4\lambda a^2)$.

- $E_S = 2\mathcal{A}\gamma$, $\frac{dE_S}{d\mathcal{A}} = 2\gamma$.

- $G = -\frac{d\Pi}{d\mathcal{A}} = -\frac{1}{2t}\frac{d\Pi}{da} = \frac{\lambda a}{2E'}\sigma^2$.

- $\sigma_{cr} = \sqrt{\frac{E'\gamma}{\lambda a}} = \sqrt{\frac{2E'\gamma}{\pi a}}$.

Double Cantilever Beam (DCB)
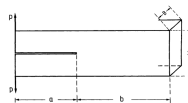
(Figure 4.14 in Gdoutos 2005)

- $u = CP = \frac{2a^3}{3EI}P$, $C = \frac{2a^3}{3EI}$.

- $U = \frac{Pu}{2} = \frac{CP^2}{2} = \frac{P^2}{3EI}a^3$,
  $W = Pu = CP^2 = \frac{2P^2}{3EI}a^3$,
  $\Pi = -\frac{P^2}{2}C = -\frac{P^2}{3EI}a^3$.

- $\mathcal{A} = aB$, $\partial_{\mathcal{A}} = \frac{1}{B}\partial_a$.

- $G = -\frac{d\Pi}{d\mathcal{A}} = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2 a^2}{EIB} = \frac{12P^2 a^2}{EB^2 h^3}$

3.1. Griffith's Analysis and Energy Release Rate: Examples
Linear Elastic Fracture Mechanics

Crack in Stre   **Additional Cases to Consider**   **er Beam (DCB)**
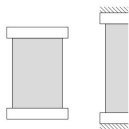
(Figure from sec

- Crack: $\mathcal{A} = 2at$, $\mathcal{E}$

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- $E_S = 2\mathcal{A}\gamma$, $\frac{dE_S}{d\mathcal{A}} =$

- $G = -\frac{d\Pi}{d\mathcal{A}} = -\frac{1}{2t}$

- $\sigma_{cr} = \sqrt{\frac{E'\gamma}{\lambda a}} = \sqrt{}$
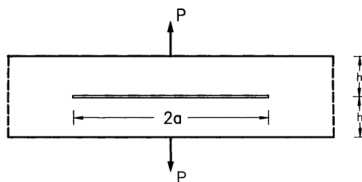
(Figure 4.23 from Gdoutos (2005))
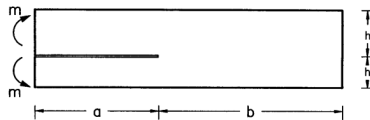
(Figure 4.20 from Gdoutos (2005))
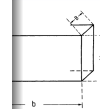
Gdoutos 2005)

$= \frac{2a^3}{3EI}$.

$\frac{P^2}{3EI}a^3$,

$\frac{2P^2}{3EI}a^3$,

a^3.

$= \frac{P^2 a^2}{EIB} = \frac{12P^2 a^2}{EB^2 h^3}$

3.2. A Primer on 2D Elasticity
Linear Elastic Fracture Mechanics

- In 2D, the governing equations of elasticity (let us assume no body loads for simplicity) are written as,

$$\sigma_{x,x} + \tau_{xy,y} = 0, \quad \tau_{xy,x} + \sigma_{y,y} = 0.$$

- If we seek to obtain **solutions expressed directly in the stresses**, 2 equations won't cut it (we have 3 unique stresses $\sigma_x, \sigma_y, \tau_{xy}$). So we invoke <u>strain compatibility</u>, which is written as

$$\boxed{\varepsilon_{x,yy} + \varepsilon_{y,xx} = \gamma_{xy,xy}}$$

- This can be expressed in terms of the stresses if we invoke the **stress-strain constitutive relationships**.

3.2. A Primer on 2D Elasticity
Linear Elastic Fracture Mechanics

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$$\varepsilon_{x,yy} + \varepsilon_{y,xx} = \gamma_{xy,xy}$$

> **Recall**: These are conditions that the strains <u>must satisfy</u> in order for them to have been generated by a continuously differentiable displacement field.

- This can be expressed in terms of the stresses if we invoke the **stress-strain constitutive relationships**.

3.2. A Primer on 2D Elasticity
Linear Elastic Fracture Mechanics

Plane Stress

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Compatibility

$$\sigma_{x,yy} + \sigma_{y,xx} - \nu(\sigma_{x,xx} + \sigma_{y,yy}) = 2(1+\nu)\tau_{xy,xy}.$$

Plane Strain

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Compatibility

$$(1-\nu)(\sigma_{x,yy} + \sigma_{y,xx}) - \nu(\sigma_{x,xx} + \sigma_{y,yy}) = 2\tau_{xy,xy}.$$

- Making the substitution $\sigma_x = \phi_{,yy}$, $\sigma_y = \phi_{,xx}$, $\tau_{xy} = -\phi_{,xy}$, it is trivial to see that the equilibrium equations are satisfied automatically.

- In both the above cases, the compatibility equation comes out to be:

$$\phi_{,xxxx} + \phi_{,yyyy} + 2\phi_{,xxyy} = (\partial_{xx} + \partial_{yy})^2 \phi = \nabla^4 \phi = 0.$$

- Since the Laplacian when set to zero ($\nabla^2 \phi = 0$) is referred to as the **harmonic equation** (recall complex analyticity), $\nabla^4 \phi = 0$ is referred to as the **bi-harmonic equation**. ϕ is the **Airy Stress Function**.

3.3. Classical Solutions
Linear Elastic Fracture Mechanics

- Restricting ourselves to 2D problems, the governing equations may be written using the Airy's stress formulation as the biharmonic equation

$$\nabla^4 \phi = 0$$

- Let us look at this with cylindrical coordinates ($x = r\cos\theta$, $y = r\sin\theta$).

$$\underline{\nabla}\phi = \begin{bmatrix} \underline{e}_r & \underline{e}_\theta \end{bmatrix} \begin{bmatrix} \phi_{,r} \\ \frac{\phi_{,\theta}}{r} \end{bmatrix}, \quad \underline{\nabla u} = \begin{bmatrix} \underline{e}_r & \underline{e}_\theta \end{bmatrix} \begin{bmatrix} u_{r,r} & \frac{u_{r,\theta} - u_\theta}{r} \\ u_{\theta,r} & \frac{u_{\theta,\theta} + u_r}{r} \end{bmatrix} \begin{bmatrix} \underline{e}_r \\ \underline{e}_\theta \end{bmatrix}$$

$$\underline{\nabla}^2 \phi = \begin{bmatrix} \underline{e}_r & \underline{e}_\theta \end{bmatrix} \begin{bmatrix} \phi_{,rr} & \partial_r(\frac{\phi_{,\theta}}{r}) \\ \partial_r(\frac{\phi_{,\theta}}{r}) & \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2} \end{bmatrix} \begin{bmatrix} \underline{e}_r \\ \underline{e}_\theta \end{bmatrix}.$$

- The stresses are expressed (to satisfy equilibrium) as

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2}, \quad \sigma_{\theta\theta} = \phi_{,rr}, \quad \tau_{r\theta} = -\partial_r(\frac{\phi_{,\theta}}{r}).$$

3.3. Classical Solutions
Linear Elastic Fracture Mechanics

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General form of the Airy's Stress Function
(Michell Solution, see Barber 2022, Ch. 8-9)

$$\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$$

$$(a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta$$

$$(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r)\cos\theta$$

$$(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r)\sin\theta$$

$$\sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n})\cos n\theta$$

$$\sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + b_{n3} r^{-n} + b_{n4} r^{2-n})\sin n\theta.$$

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2}, \quad \sigma_{\theta\theta} = \phi_{,rr}, \quad \tau_{r\theta} = -\partial_r(\frac{\phi_{,\theta}}{r}).$$

3.3.1. The Michell Solution: Tabled Expressions
Classical Solutions

Stress Components

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$r^2\ln(r)$	$2\ln(r)+1$	0	$2\ln(r)+3$
$\ln(r)$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^3\cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln(r)\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln(r)\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$n(n+1)r^n\sin n\theta$	$(n+1)(n+2)r^n\cos n\theta$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-n(n+1)r^n\cos n\theta$	$(n+1)(n+2)r^n\sin n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$

(Table 8.1 from Barber 2022)

We set rigid body motion components to zero for the displacements

Displacement Components

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$r^2\ln(r)$	$(\kappa-1)r\ln(r)-r$	$(\kappa+1)r\theta$
$\ln(r)$	$-1/r$	0
θ	0	$-1/r$
$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r\theta\sin\theta$	$\frac{1}{2}\{(\kappa-1)\theta\sin\theta-\cos\theta+(\kappa+1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta-\sin\theta-(\kappa+1)\ln(r)\sin\theta\}$
$r\ln(r)\cos\theta$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta-\cos\theta+(\kappa-1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\cos\theta-\sin\theta-(\kappa-1)\ln(r)\sin\theta\}$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa+2)r^2\cos\theta$
$r\theta\cos\theta$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta+\sin\theta-(\kappa+1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{-(\kappa-1)\theta\sin\theta-\cos\theta+(\kappa+1)\ln(r)\cos\theta\}$
$r\ln(r)\sin\theta$	$\frac{1}{2}\{-(\kappa+1)\theta\cos\theta-\sin\theta+(\kappa-1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta+\cos\theta+(\kappa-1)\ln(r)\cos\theta\}$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^{n+2}\cos n\theta$	$(\kappa-n-1)r^{n+1}\cos n\theta$	$(\kappa+n+1)r^{n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa-n-1)r^{n+1}\sin n\theta$	$-(\kappa+n+1)r^{n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$

(Table 9.1 from Barber 2022)

Plane Stress $\kappa = \dfrac{3-\nu}{1+\nu}$

Plane Strain $\kappa = 3 - 4\nu$

3.3.Plate with a Hole. Plate With a Hole Under Tension
Linear Elastic Fracture Mechanics

- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.
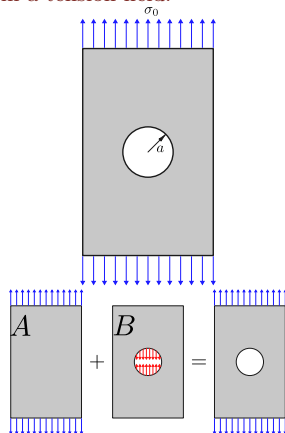
3.3.Plate with a Hole. Plate With a Hole Under Tension
Linear Elastic Fracture Mechanics

- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.

Displacement Field

$$u_r = \frac{\sigma_0}{2}(\kappa - 1)r - \frac{\sigma_0}{2}r\cos 2\theta$$

$$u_\theta = \frac{\sigma_0}{2}r\sin 2\theta$$
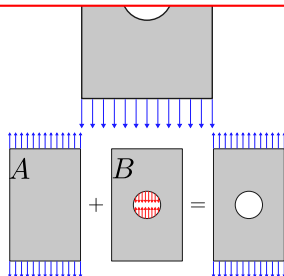
Problem A

The 2D stress field (cartesian) is

$$\underline{\underline{\sigma}}_{cart} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix}.$$

- Transforming to cylindrical coordinates,

$$\underline{\underline{\sigma}}_{cyl} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \sigma_0 \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

The components can be written as

$$\sigma_{rr} = \sigma_0\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right), \quad \sigma_{r\theta} = \sigma_0\frac{\sin 2\theta}{2},$$

$$\sigma_{\theta\theta} = \sigma_0\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right).$$

A + B = $\bigcirc$

3.3.Plate with a Hole. Plate With a Hole Under Tension
Linear Elastic Fracture Mechanics

- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.

Displacement Field

$$u_r = \frac{\sigma_0 a^2}{2r} \left(1 - (\kappa + 1 - (\frac{a}{r})^2) \cos 2\theta \right)$$

$$u_\theta = \frac{\sigma_0 a^2}{2r} \left(\kappa - 1 + (\frac{a}{r})^2 \right) \sin 2\theta.$$

Problem B

At $r = a$ we want

$$\sigma_{rr} = \sigma_0 \left(\frac{1}{2} - \frac{\cos 2\theta}{2} \right), \quad \sigma_{r\theta} = \sigma_0 \frac{\sin 2\theta}{2}.$$

(no hoop component specified)

As $r \to \infty$, we want $\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta} \to 0$ to match the far-field.

- Based on inspection (shown in class), we find the following Airy stress function to be a good starting point: $\phi = A \log r + B\theta + C \cos 2\theta + D\frac{\cos 2\theta}{r^2}$.

- Solving for A, B, C, D based on the B.C.s we get,

$$\sigma_{rr} = -\frac{\sigma_0}{2} \left(\frac{a}{r} \right)^2 + 2\sigma_0 \left(\frac{a}{r} \right)^2 \left(1 - \frac{3}{4} \left(\frac{a}{r} \right)^2 \right) \cos 2\theta,$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2} \left(\frac{a}{r} \right)^2 + \frac{3\sigma_0}{4} \left(\frac{a}{r} \right)^4 \cos 2\theta,$$

$$\sigma_{r\theta} = \sigma_0 \left(\frac{a}{r} \right)^2 \left(1 - \frac{3}{2} \left(\frac{a}{r} \right)^2 \right) \sin 2\theta$$
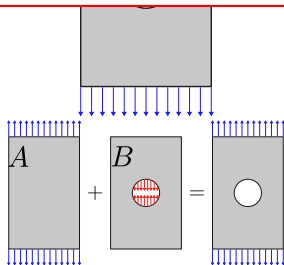
3.3.Plate with a Hole. Plate With a Hole Under Tension

Linear Elastic Fracture Mechanics

3.3.3. Notch Crack
Classical Solutions

Traction-Free, $\theta \in \{-\alpha, \alpha\}$

$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_{r\theta} \\ \sigma_{\theta\theta} \end{bmatrix}$$
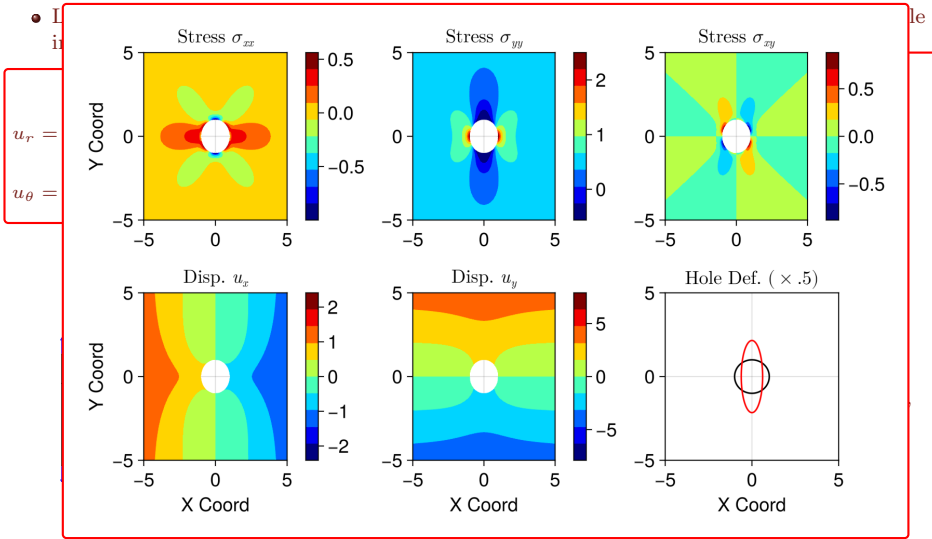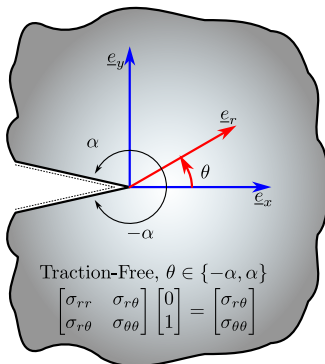
- We seek an analytical solution for this problem setting **very close to the crack**.
- While we may intuitively expect stress to be singular at the crack tip, the strain energy has to be finite.
- Suppose $\sigma \sim \mathcal{O}(r^\lambda)$, $\varepsilon \sim \mathcal{O}(r^\lambda)$ necessarily.
  - So $\mathcal{U} = \int \int \frac{\sigma \varepsilon}{2} r dr d\theta \sim \mathcal{O}(r^{2\lambda+1})$.
  - For this to be finite, $2\lambda + 1 \geq 0 \implies \lambda \geq -\frac{1}{2}$.
- We list out **the only Airy stress functions** that can show this in the following table (refer sl. 18).

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
$r^{n+2}\cos n\theta$	$(..)r^n \cos n\theta$	$(..)r^n \sin n\theta$	$(..)r^n \cos n\theta$
$r^n \cos n\theta$	$(..)r^{n-2}\cos n\theta$	$(..)r^{n-2}\sin n\theta$	$(..)r^{n-2}\cos n\theta$
$r^{n+2}\sin n\theta$	$(..)r^n \sin n\theta$	$(..)r^n \cos n\theta$	$(..)r^n \sin n\theta$
$r^n \sin n\theta$	$(..)r^{n-2}\sin n\theta$	$(..)r^{n-2}\cos n\theta$	$(..)r^{n-2}\sin n\theta$

3.3.3. Singularity Close to Notch Crack
Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\phi = r^{\lambda+1} \left(A_1 \cos((\lambda-1)\theta) + A_2 \cos((\lambda+1)\theta) + B_1 \sin((\lambda-1)\theta) + B_2 \sin((\lambda+1)\theta) \right).$$

3.3.3. Singularity Close to Notch Crack
Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right).$$

(Figure 11.7 from Barber 2022)

3.3.3. Singularity Close to Notch Crack
Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\phi = r^{\lambda+1} \left(A_1 \cos((\lambda-1)\theta) + A_2 \cos((\lambda+1)\theta) + B_1 \sin((\lambda-1)\theta) + B_2 \sin((\lambda+1)\theta) \right)$$

- Applying the boundary conditions (along with $\alpha = \pi$), we get a nonlinear eigenvalue problem that has the following solutions:

λ	Eigenfunction
$\frac{1}{2}$	$A_2 = \frac{A_1}{3}$, $B_2 = -B_1$
1	$A_2 = -A_1$, $B_2 = 0$ $(B_1 = 0)$
$\frac{3}{2}$	$A_2 = -\frac{A_1}{5}$, $B_2 = -B_1$
$\vdots$	

- $\lambda = \frac{1}{2}$ corresponds to the near-field singular stress field, given by

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right)$$

3.3.3. Singularity Close to Notch Crack
Classical Solutions

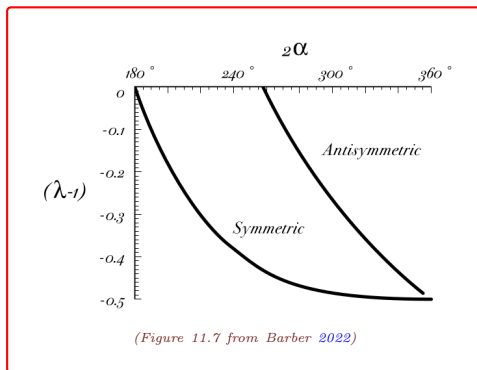- For the Notch crack problem, we posit the Airy stress function

$$\phi = r^{\lambda+1} \left(A_1 \cos((\lambda-1)\theta) + A_2 \cos((\lambda+1)\theta) + B_1 \sin((\lambda-1)\theta) + B_2 \sin((\lambda+1)\theta) \right).$$

- Applying the boundary conditions (along with $\alpha = \pi$), we get a nonlinear eigenvalue problem that has the following solutions:

$$\lambda \mid \text{Eigenfunction}$$

Displacement Field

$$2\mu u_r = K_I \sqrt{\frac{r}{2\pi}} \left((\kappa - \frac{1}{2}) \cos \frac{\theta}{2} - \frac{1}{2} \cos \frac{3\theta}{2} \right) - K_{II} \sqrt{\frac{r}{2\pi}} \left((\kappa - \frac{1}{2}) \sin \frac{\theta}{2} - \frac{3}{2} \sin \frac{3\theta}{2} \right)$$

$$2\mu u_\theta = K_I \sqrt{\frac{r}{2\pi}} \left(-(\kappa + \frac{1}{2}) \sin \frac{\theta}{2} + \frac{1}{2} \sin \frac{3\theta}{2} \right) - K_{II} \sqrt{\frac{r}{2\pi}} \left((\kappa + \frac{1}{2}) \cos \frac{\theta}{2} - \frac{3}{2} \cos \frac{3\theta}{2} \right)$$

- $\lambda = \frac{1}{2}$ corresponds to the near-field singular stress field, given by

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right)$$

3.3.3. Singularity Close to Notch Crack
Classical Solutions

Mode 1 Loading (unit K_I)
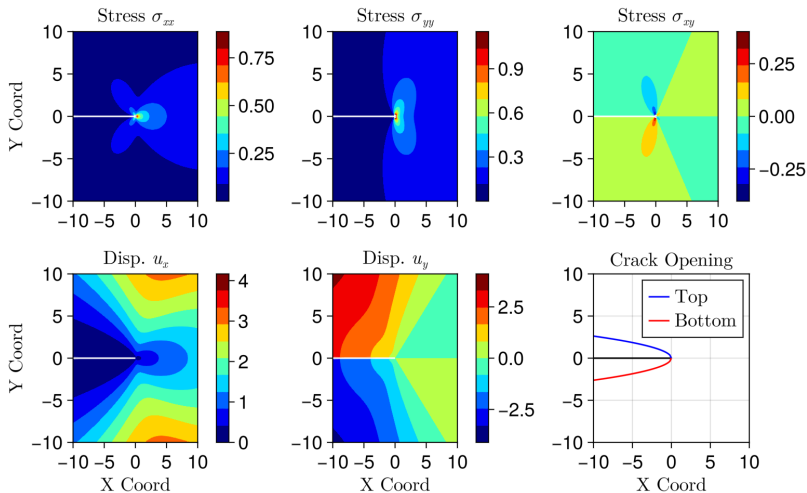
3.3.3. Singularity Close to Notch Crack
Classical Solutions

Mode 2 Loading (unit K_{II})
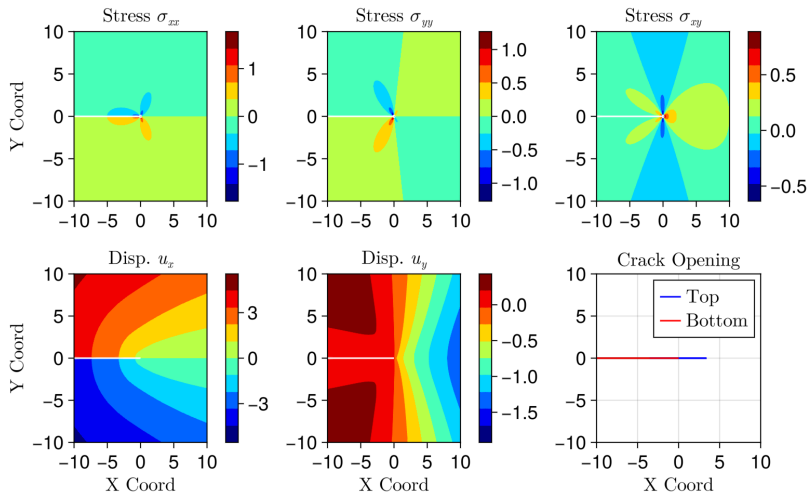
3.3.3. Energy Release Rate
Classical Solutions

- Let us think of how much energy will be necessary to "close" a crack.

3.3.3. Energy Release Rate
Classical Solutions

- Let us think of how much energy will be necessary to "close" a crack.
- We observe that (all quantities in cylindrical):

$$@\,\theta = 0, \quad \underline{\underline{\sigma}} = \frac{K_I}{\sqrt{2\pi r}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi x}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix} \kappa - 1 \\ 0 \end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix} 0 \\ \kappa - 1 \end{bmatrix}$$

$$@\,\theta = \pi, \quad \underline{\underline{\sigma}} = \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, \quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix} 0 \\ -(\kappa + 1) \end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix} \kappa + 1 \\ 0 \end{bmatrix}.$$

3.3.3. Energy Release Rate

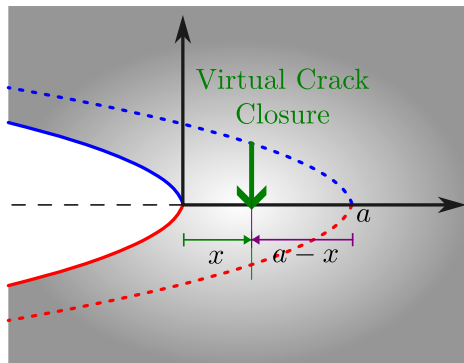Classical Solutions

- Let us think of how much energy will be necessary to "close" a crack.
- We observe that (all quantities in cylindrical):

$$@\,\theta = 0, \quad \underline{\underline{\sigma}} = \frac{K_I}{\sqrt{2\pi r}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi x}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix} \kappa - 1 \\ 0 \end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix} 0 \\ \kappa - 1 \end{bmatrix}$$

$$@\,\theta = \pi, \quad \underline{\underline{\sigma}} = \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, \quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix} 0 \\ -(\kappa+1) \end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix} \kappa + 1 \\ 0 \end{bmatrix}.$$

- For virtual crack closure, the work done can be written as,

$$W(a) = 2\int_0^a \frac{1}{2}\left(\sigma_{\theta\theta}\Big|_{\theta=0}(-u_\theta)\Big|_{\theta=\pi} + \sigma_{r\theta}\big|_{\theta=0}(-u_r)\big|_{\theta=\pi}\right)dx$$

$$= \int_0^a \frac{K_I}{\sqrt{2\pi x}}K_I\sqrt{\frac{a-x}{2\pi}}\frac{\kappa+1}{2\mu} + \frac{K_{II}}{\sqrt{2\pi x}}K_{II}\sqrt{\frac{a-x}{2\pi}}\frac{\kappa+1}{2\mu}dx$$

$$= \frac{K_I^2 + K_{II}^2}{2\pi}\frac{\kappa+1}{2\mu}\underbrace{\int_0^a \sqrt{\frac{a-x}{x}}dx}_{\frac{a\pi}{2}} = \frac{K_I^2 + K_{II}^2}{8\mu^2}(\kappa+1)^2 a = \begin{cases} \frac{K_I^2 + K_{II}^2}{E}a & \text{Plane-}\sigma \\ \frac{K_I^2 + K_{II}^2}{E}(1-\nu^2)a & \text{Plane-}\varepsilon \end{cases}.$$

- The Griffith Energy Release Rate is the derivative $\lim_{a\to0}\frac{1}{B}\frac{dW}{da}$, which evaluates as

$$G = \frac{1}{B}\begin{cases} \frac{K_I^2}{E} + \frac{K_{II}^2}{E} & \text{Plane Stress} \\ \frac{K_I^2}{E}(1-\nu^2) + \frac{K_{II}^2}{E}(1-\nu^2) & \text{Plane Strain} \end{cases}.$$

3.3.3. Stress Intensity Factor
Classical Solutions

- A crack is said to propagate when G exceeds G_{cr}.
- Therefore, under "pure" mode 1 loading, the *Critical Stress Intensity Factor* ($K_{I,cr}$) is

$$K_{I,cr} = \begin{cases} \sqrt{BG_{cr}E} & \text{Plane Stress} \\ \sqrt{\frac{BG_{cr}E}{1-\nu^2}} & \text{Plane Strain} \end{cases}.$$

- **But how do we relate K_I, K_{II} with far-field applied stresses?**

3.3.3. Stress Intensity Factor
Classical Solutions

- A crack is said to propagate when G exceeds G_{cr}.
- Therefore, under "pure" mode 1 loading, the *Critical Stress Intensity Factor* ($K_{I,cr}$) is

$$K_{I,cr} = \begin{cases} \sqrt{BG_{cr}E} & \text{Plane Stress} \\ \sqrt{\dfrac{BG_{cr}E}{1-\nu^2}} & \text{Plane Strain} \end{cases}.$$

- **But how do we relate K_I, K_{II} with far-field applied stresses?** The answer is very closely tied in to the exact geometry, loading conditions, etc.

3.3.3. Griffith-Inglis Crack Revisited
Classical Solutions

- For the flat crack of length $2a$ (aka the Griffith-Inglis crack), the SIF is related to tensile stresses by
$$K_I = \sigma_0 \sqrt{\pi a}.$$

- Note that this is why we chose $\lambda = \frac{\pi}{2}$ in sl. 7. If we left it in, we'll have to satisfy (plane stress considered here):
$$\frac{4\lambda a}{E} \sigma_0^2 = \frac{2K_I^2}{E} = \frac{2\pi a}{E} \sigma_0^2.$$

3.4. Crack Propagation and the Paris Law
Linear Elastic Fracture Mechanics

- **Paris Law**: $\frac{da}{dN} = C(\Delta K)^m$.
- Usually a_f is specified and we are interested in finding how many cycles until a crack of size a_i grows to a_f. This is the "life" of the material.

Values for common engineering materials, from Kumar 2009

Material	C	m
Ferrite-Pearlite (S)	$6.8{\times}10^{-12}$	3.0
Martensite (S)	$1.33{\times}10^{-10}$	2.25
Austenite (S)	$5.5{\times}10^{-12}$	3.25
Cast Iron (S)	$5.5{\times}10^{-12}$	3.25
Al-Alloy	$1.1{\times}10^{-11}$	3.89

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