AS2070: Aerospace Structural Mechanics

Module 2: Composite Material Mechanics (V7)

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Department of Aerospace Engineering, IIT Madras, Chennai

March 17, 2026

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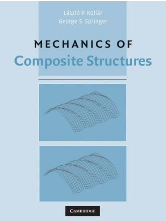
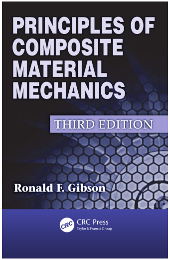
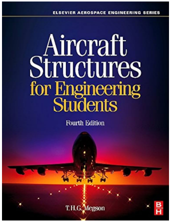
(Also see Daniel and Ishai 2006)



Chapters 1-3, 11 in Kollár and Springer (2003).



Chapters 1-3 in Gibson (2012).

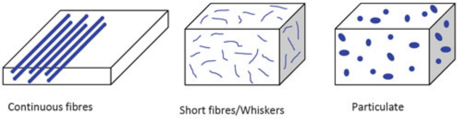


Chapter 25 in Megson (2013)

1.1. What are Composites?

Introduction

- Structural material consisting of multiple non-soluble macro-constituents.
- Main motivation: material properties tailored to applications.
- Both stiffness and strength comes from the fibers/particles, and the matrix holdes everything together.



Types of composite materials (Figure from NPTEL Online-IIT KANPUR (2025))

Examples

- Reinforced concrete
- Wood (lignin matrix reinforced by cellulose fibers)
- Carbon-Fiber Reinforced Plastics (CFRP)

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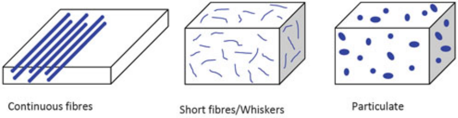

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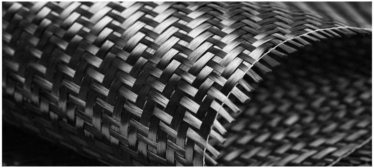
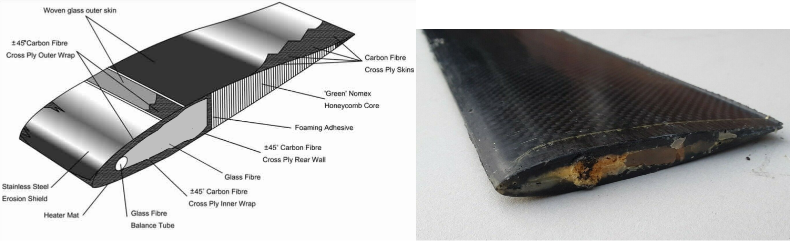


1.1. What are Composites?

Introduction

- Structural material consisting of multiple non-soluble macro-constituents.

CFRP Helicopter Blades



(Figures from Carbon Fiber Top Helicopter Blades 2025)

Exa

- Wood (lignin matrix reinforced by cellulose fibers)
- Carbon-Fiber Reinforced Plastics (CFRP)

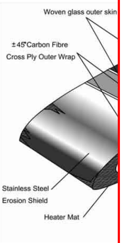
AA.

- High fatigue resistance. But quite brittle.
- Main- and tail-planes, fuselages, etc. Helicopter blades.

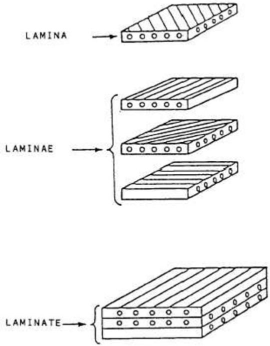
1.1. What are Composites?

Introduction

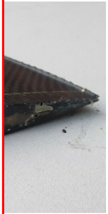
- Structural ma ts.



Laminated Composites



(Figure from Kalkan 2017)



Exa

AA.

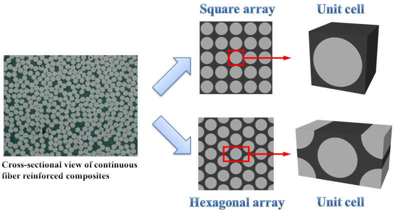
- Wood (lignin ma fibers) quite brittle.
- Carbon-Fiber Re es, etc. Helicopter

1.2. Modeling Composite Material

Introduction

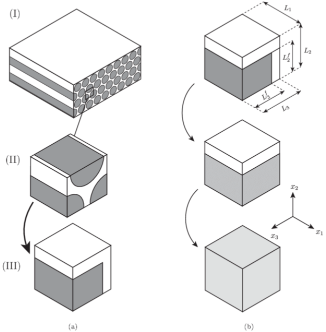
Two main approaches:

Micro-Mechanics



(Figure from “Micro-Mechanics of Failure” 2024)

Macro-Mechanics

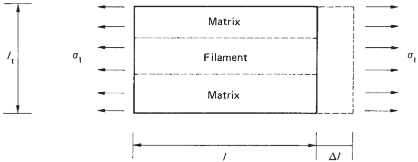


Homogenization of micro-structure (Figure from Skovsgaard and Heide-Jørgensen 2021)

1.3. Constitutive Modeling for Composites

Introduction

Axial Elongation



- Strain is fixed, but stress experienced by media differ.

$$\sigma_l = E_l \varepsilon_l$$

- Stress-strain relationship simplifies as,

$$\sigma_m = E_m \varepsilon_l, \quad \sigma_f = E_f \varepsilon_l$$

$$\sigma_l A = \sigma_m A_m + \sigma_f A_f$$

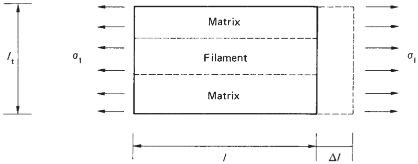
$$\implies \boxed{E_l = \frac{A_f}{A} E_f + \frac{A_m}{A} E_m}.$$

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction

Axial Elongation



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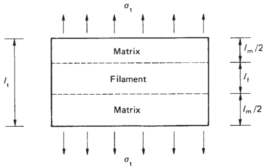
Transverse Elongation



- Stress is fixed, strains differ:

$$\varepsilon_t l_t = \varepsilon_m l_m + \varepsilon_f l_f$$

$$\implies \frac{\sigma_t}{E_t} l_t = \frac{\sigma_t}{E_m} l_m + \frac{\sigma_t}{E_f} l_f$$

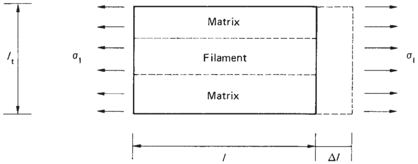
$$\implies \boxed{\frac{1}{E_t} = \frac{1}{E_m}\frac{l_m}{l_t} + \frac{1}{E_f}\frac{l_f}{l_t}}.$$

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction: Poisson Effects

Axial-Transverse Coupling



- Transverse displacement written as

$$\Delta_t = \nu_m \varepsilon_l l_m + \nu_f \varepsilon_l l_f := \nu_{lt} \varepsilon_l l_t$$

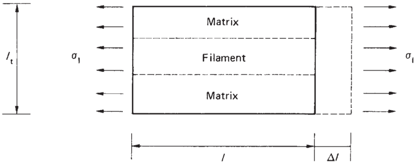
$$\implies \boxed{\nu_{lt} = \frac{l_m}{l_t}\varepsilon_l + \frac{l_f}{l_t}\varepsilon_f}.$$

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction: Poisson Effects

Axial-Transverse Coupling



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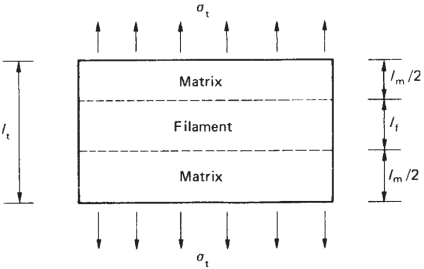
Transverse-Axial Coupling



- Axial displacement written as

$$\nu_m \frac{\sigma_t}{E_m} = \nu_f \frac{\sigma_t}{E_f} := \nu_{tl} \frac{\sigma_t}{E_t},$$

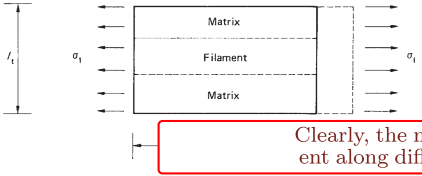
$$\implies \boxed{\nu_{tl} = \frac{E_t}{E_l}\nu_{lt}}.$$

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction: Poisson Effects

Axial-Transverse Coupling



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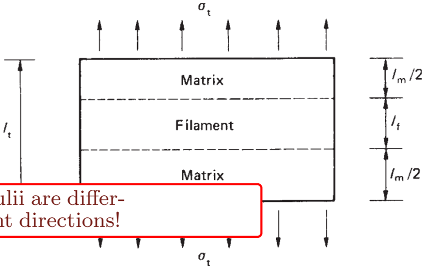

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$$\implies \boxed{\nu_{tl} = \frac{E_t}{E_l}\nu_{lt}}.$$

Clearly, the modulii are different along different directions!

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction: Anisotropy

General Anisotropy (aka "Triclinic")

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46}\\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56}\\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

1.3. Constitutive Modeling for Composites

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Monoclinic: Single Plane of Symmetry

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0\\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0\\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0\\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & C_{56}\\ 0 & 0 & 0 & 0 & C_{56} & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

1.3. Constitutive Modeling for Composites

Introduction: Anisotropy

Orthotropic: Three Orthogonal Planes of Symmetry

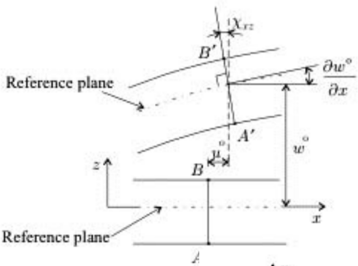
$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

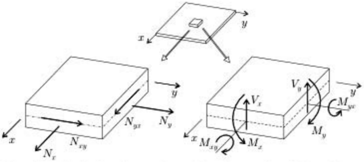
Transversely Isotropic

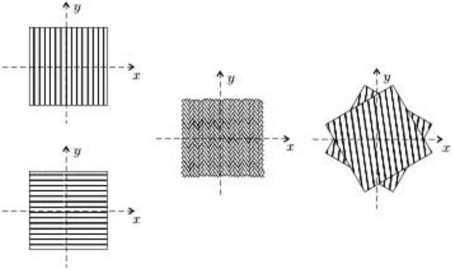
$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2}\end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

1.4. Classical Laminate Theory

Introduction

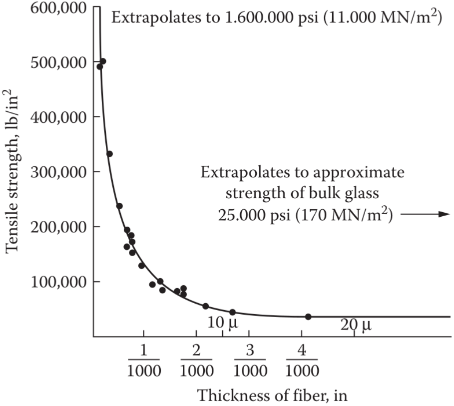






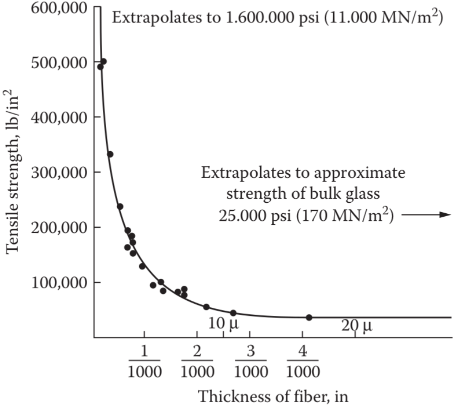
Figures from Kollár and Springer 2003

2. Composite Materials

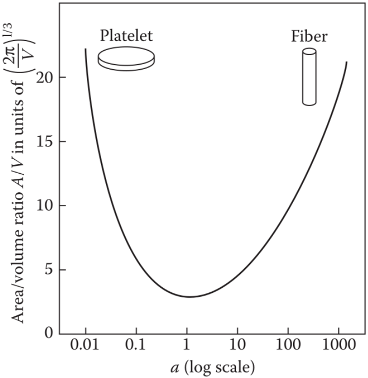


Griffith's experiments with glass fibres (1920) (Figure from Gibson 2012)

2. Composite Materials



Griffith's experiments with glass fibres (1920) (Figure from Gibson 2012)



(Figure from Gibson 2012)

2.1. Types of Composite Materials

Composite Materials

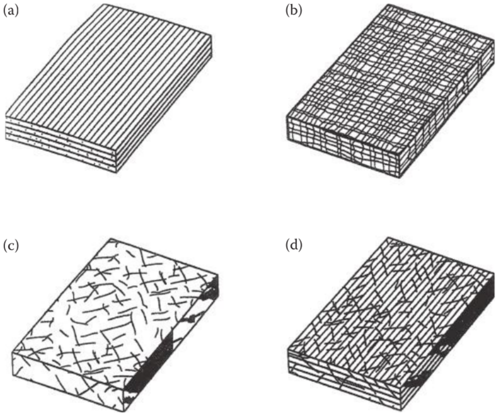
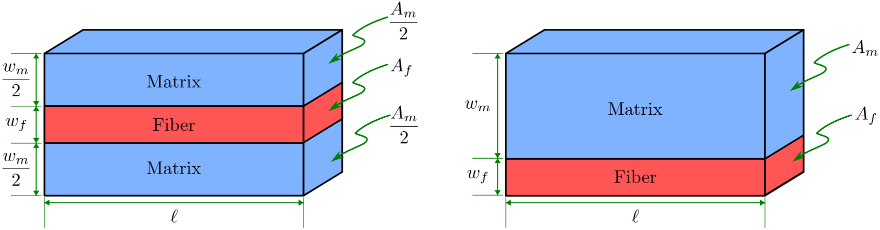


FIGURE 1.4
Types of fiber-reinforced composites. (a) Continuous fiber composite, (b) woven composite, (c) chopped fiber composite, and (d) hybrid composite.

(Figure from Gibson 2012)

3. Micro-Mechanics Descriptions

- We shall now seek a “homogenized” understanding of the mechanics of a composite material.
- By homogenization, we shall abstract away all the spatial information to such an extent that we shall claim that the following two abstractions of a composite material behave identically.



(a) Abstraction 1 of a composite material

(b) Abstraction 2 of a composite material

- We shall analyze the mechanics of abstraction 2 to derive an “equivalent” elastic description of the material as a whole. (We will restrict ourselves to the 2D plane for this)

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

The rule of mixtures is a very simple framework for developing expressions for homogenized mechanical properties.

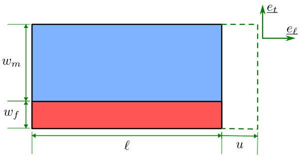
Basic Definitions

Subscripts $(\cdot)_f$, $(\cdot)_m$, $(\cdot)_v$, and $(\cdot)_c$ denote quantities corresponding to the fiber, matrix, void, and composite (as a whole).

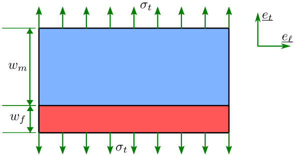
Volume Fraction $v_f = \frac{V_f}{V_c}, v_m = \frac{V_m}{V_c}, v_v = \frac{V_v}{V_c}$ such that $v_f + v_m + v_v = 1$.
Note that composite density $\rho_c = \rho_f v_f + \rho_m v_m$.

Weight Fraction $w_f = \frac{\rho_f}{\rho_c} v_f$

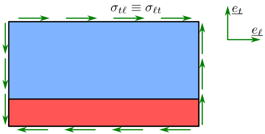
- We shall consider the behavior under the following 3 fundamental cases.



(a) The abstract material under longitudinal extension (iso-strain)



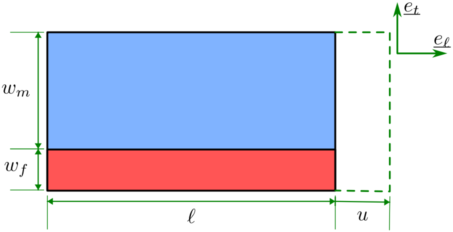
(b) The abstract material under transverse extension (iso-stress)



(c) The abstract material under in-plane shear (iso-stress)

3.1. The Rule of Mixtures: Case 1: Longitudinal Iso-Strain Extension

Micro-Mechanics Descriptions



- Under longitudinal extension, the matrix as well as the fiber are under the same strain, i.e., $\varepsilon_\ell = \frac{u}{\ell}$. The stresses, respectively, are

$$\sigma_m = E_m \varepsilon_\ell = E_m \frac{u}{\ell}; \quad \sigma_f = E_f \varepsilon_\ell = E_f \frac{u}{\ell}.$$

- Subscripts m and f denote matrix and fiber properties.
- Converting the above stresses to forces (by multiplying with A_m and A_f respectively) and summing them up leads to the overall reaction force:

$$F_{tot} = \left(A_m E_m + A_f E_f\right) \frac{u}{\ell} := A_{tot} E_\ell \frac{u}{\ell}.$$

- Here we introduce E_ℓ as the effective longitudinal modulus, so that we have:

$$E_\ell = \frac{A_m}{A_{tot}} E_m + \frac{A_f}{A_{tot}} E_f \implies \boxed{E_\ell = v_m E_m + v_f E_f}.$$

- This is closely related to the springs-in-parallel formula that you may already be familiar with. The jargon name for this is *Voigt Model*.

3.1. The Rule of Mixtures: Case 1: Longitudinal Iso-Strain Extension

Micro-Mechanics Descriptions

- Under longitudinal extension, the matrix as well as the fiber are under the same

Poisson's Effect

- The transverse displacements due to Poisson's effect can be written as

$$u_{t,m} = -\nu_m \frac{u}{\ell} w_m, \quad u_{t,f} = -\nu_f \frac{u}{\ell} w_f.$$

- The overall transverse displacement is:

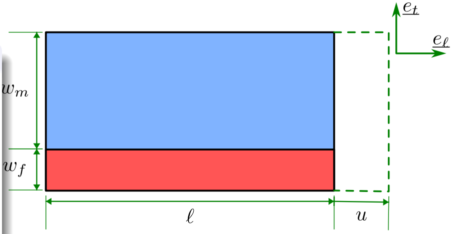
$$u_t = -\left(\nu_m w_m + \nu_f w_f\right) \frac{u}{\ell} := -\nu_{\ell t} \frac{u}{\ell} w_{tot},$$

where we have substituted the homogenized formula in the end.

- So the effective Poisson's ratio is:

$$\boxed{\nu_{\ell t} = v_m \nu_m + v_f \nu_f}$$

- Here we have used $\frac{w_m}{w_{tot}} = v_m$ and $\frac{w_f}{w_{tot}} = v_f$.



multiplying with A_m and A_f respectively)

eaction force:

$_f E_f$

udin

$\Longrightarrow$

Notation for Poisson's ratio subscripts: $\nu_{<cause><effect>}$

For example: $\nu_{\ell t}$ encodes the transverse strain caused by axial strains such that:

$$\boxed{\varepsilon_t = -\nu_{\ell t} \varepsilon_\ell}$$

- This is closely related to the springs-in-parallel formula that you may already be familiar with. The jargon name for this is *Voigt Model*.

3.1. The Rule of Mixtures: Case 2: Transverse Iso-Stress Extension

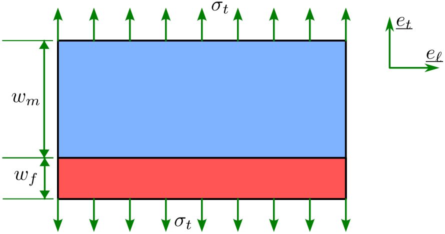
Micro-Mechanics Descriptions

- Under transverse loading, the iso-stress condition is very naturally considered.
- The elongations in the transverse directions are written as:

$$u_{t,m} = \frac{\sigma_t}{E_m} w_m, \quad u_{t,f} = \frac{\sigma_t}{E_f} w_f.$$

- The overall elongation can be expressed as

$$u_t = \left(\frac{w_m}{E_m} + \frac{w_f}{E_f}\right)\sigma_t := \frac{w_{tot}}{E_t}\sigma_t.$$



- By equating the coefficients, we get the effective transverse modulus as

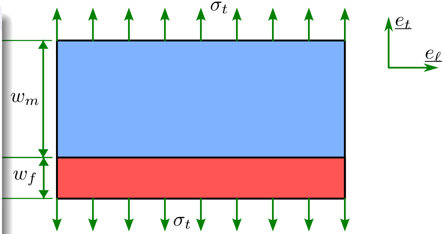
$$\boxed{E_t = \left(\frac{v_m}{E_m} + \frac{v_f}{E_f}\right)^{-1}},$$

which closely resembles the springs-in-series formula. The jargon name for this is *Reuß model*.

3.1. The Rule of Mixtures: Case 2: Transverse Iso-Stress Extension

Micro-Mechanics Descriptions

- Under transverse loading, the iso-stress



Poisson's Effect

- The **longitudinal displacements** due to Poisson's effect can be written as

$$u_{\ell,m} = -\frac{\nu_m}{E_m}\sigma_t \ell, \quad u_{\ell,f} = -\frac{\nu_f}{E_f}\sigma_t \ell.$$

- Since the fiber and matrix get displaced differently, we write down the overall averaged (effective homogenized) longitudinal displacement as:

$$\begin{aligned} u_\ell &= v_m u_{\ell,m} + v_f u_{\ell,f} \\ := -\frac{\nu_{t\ell}}{E_t}\sigma_t &= -\left(\nu_m \frac{v_m}{E_m} + \nu_f \frac{v_f}{E_f}\right)\sigma_t \ell \end{aligned}$$

- So the effective Poisson's ratio is:

$$\boxed{\nu_{t\ell} = \frac{E_f v_m}{E_f v_m + E_m v_f}\nu_m + \frac{E_m v_f}{E_f v_m + E_m v_f}\nu_f}$$

ctive

$+$

forn

Reuß

model.

Notation for Poisson's ratio subscripts: $\nu_{<cause><effect>}$

For example: $\nu_{\ell t}$ encodes the transverse strain caused by axial strains such that:

$$\boxed{\varepsilon_t = -\nu_{\ell t}\varepsilon_\ell}$$

3.1. The Rule of Mixtures: Case 3: In-plane Iso-Stress Shearing

- Under shear loading, the iso-stress condition is very naturally considered.
- The shear strains (angles) are:

$$\gamma_{t\ell,m} = \frac{\sigma_{t\ell}}{G_m}, \quad \gamma_{t\ell,f} = \frac{\sigma_{t\ell}}{G_f}.$$

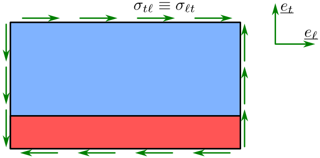
- The overall shear strain (deflection angle) is:

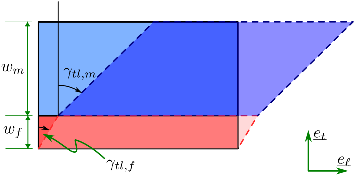
$$\gamma_{t\ell} = \gamma_{t\ell,m}\frac{w_m}{w_{tot}} + \gamma_{t\ell,f}\frac{w_f}{w_{tot}}$$
$$:= \frac{\sigma_{t\ell}}{G_{t\ell}} = \left(\frac{v_m}{G_m} + \frac{v_f}{G_f}\right)\sigma_{t\ell}.$$

- From this, we write down the overall shear modulus as

$$G_{t\ell} = \left(\frac{v_m}{G_m} + \frac{v_f}{G_f}\right)^{-1}.$$

- This is also a Reuß model (springs-in-series model).



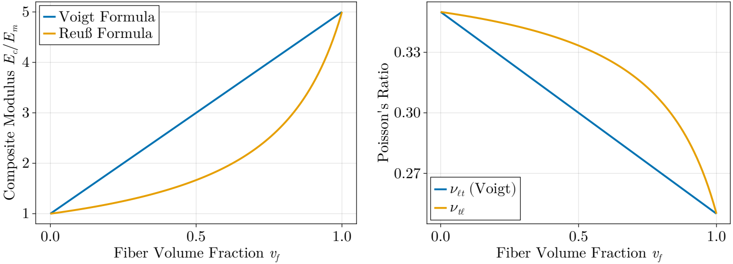


3.1. The Rule of Mixtures: Case 3: In-plane Iso-Stress Shearing

Micro-Mechanics Descriptions

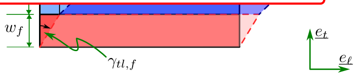
- Under shear loading, the iso-stress condition is very naturally considered.

The rule of mixtures, visualized.



- modulus as

$$G_{t\ell} = \left(\frac{v_m}{G_m} + \frac{v_f}{G_f}\right)^{-1}.$$



- This is also a Reuß model (springs-in-series model).

3.1. The Rule of Mixtures: Case 3: In-plane Iso-Stress Shearing

Micro-Mechanics Descriptions

- Under
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-
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  modul

RoM is not always satisfactory!

NOTE: The Reuß formula has been observed to consistently under-predict the modulii that are experimentally measured. This affects the E_t, $G_{t\ell}$, and the $\nu_{t\ell}$ estimates shown above.

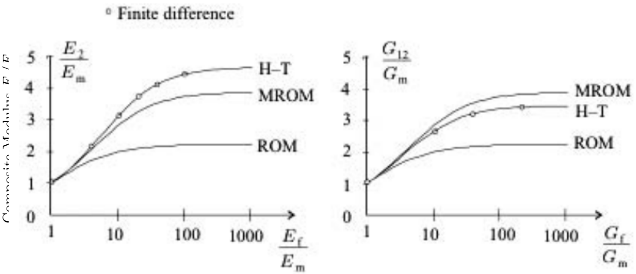
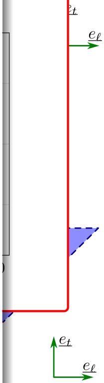


Figure 11.8: The transverse Young and shear moduli calculated by the rule of mixtures (ROM), the modified rule of mixtures (MROM), the Halpin–Tsai (H–T) equations, and the finite difference solutions (circles) of Adams and Doner ($v_f = 0.55$).

(Figure 11.8 from Kollár and Springer 2003)

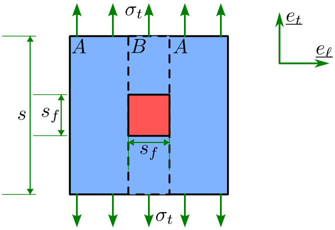


- This is also a Reuß model (springs-in-series model).

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

- The mismatch is related to the fact that our idealized picture was a poor representation of reality to begin with. Let us see how far we get with added geometrical detailing in our abstraction.



An abstraction with slightly more geometrical details: a square fiber embedded in a square region of the matrix

$$V_f = s_f^2, \quad V_{tot} = s^2, \quad v_f = \frac{V_f}{V_{tot}} = \frac{s_f^2}{s^2}$$

- Since the Voigt model is so successful, we look at the cross-section and divide it into regions A and B, which are acting as springs in parallel.
- For region B, the transverse modulus is written using the Reuß formula as:

$$E_{Bt} = \left(\frac{s_f/s}{E_f} + \frac{1 - s_f/s}{E_m}\right)^{-1}$$
$$= \left(\frac{\sqrt{v_f}}{E_f} + \frac{1 - \sqrt{v_f}}{E_m}\right)^{-1}.$$

- Now we apply the Voigt formula to obtain the overall transverse modulus:

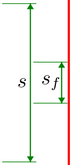
$$E_t = \left(1 - \frac{s_f}{s}\right) E_m + \frac{s_f}{s} E_{Bt}$$
$$= E_m \left[\left(1 - \sqrt{v_f}\right) + \frac{\sqrt{v_f}}{1 - \sqrt{v_f}\left(1 - \frac{E_m}{E_f}\right)}\right]$$

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

- The r resentation of rea tailing in our a

ccessful, we ivide it into ting as



odulus is a as:

$\Big)^{-1}$

$\Big)^{-1}.$

la to obtain :

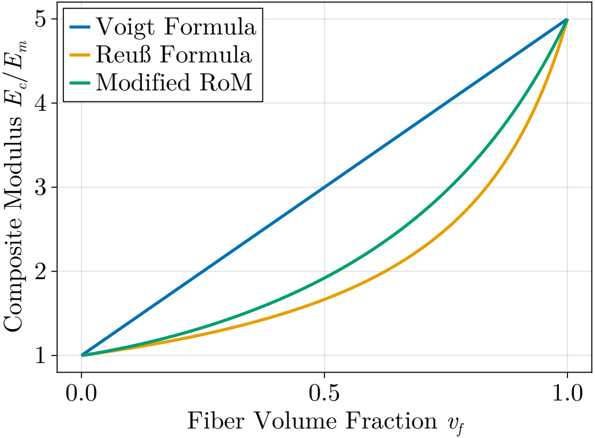
An abstraction
fiber embedded i

$V_f = s_f^2$

$\sqrt{v_f}$

$\left(1-\frac{E_m}{E_f}\right)$

This "Modified" Rule of Mixtures (mRoM) presents just one modification to RoM, but it can be seen to pro-



The same formula may also be used for the shear modulus.

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

- Even the mRoM is found insufficient in a lot of cases, so a heuristic formula known as the **Halpin-Tsai Equation** is quite popular.

(Recommended reading: Sec. 3.2.3 in Daniel and Ishai 2006)

The Halpin-Tsai Equation

$$E_t = E_m \frac{1+\xi\eta v_f}{1-\eta v_f}, \quad \eta = \frac{E_f - E_m}{E_f + \xi E_m}$$
$$= E_m \frac{E_f + \xi E_m + \xi v_f(E_f - E_m)}{E_f + \xi E_m - v_f(E_f - E_m)}$$

Note: $\xi = 2$ for circular section fibers. $\xi = \frac{2a}{b}$ for rectangular fibers (b being loaded side).

- This is parameterized by ξ, which allows us to recover both the Voigt and Reuß models.

Case 1: $\xi \to 0$

$$E_t = \left(\frac{v_f}{E_f} + \frac{1-v_f}{E_m}\right)^{-1}$$

Series, *Reuss* model.

Case 2: $\xi \to \infty$

$$E_2 = E_f v_f + E_m(1-v_f)$$

Parallel, *Voigt* model.

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

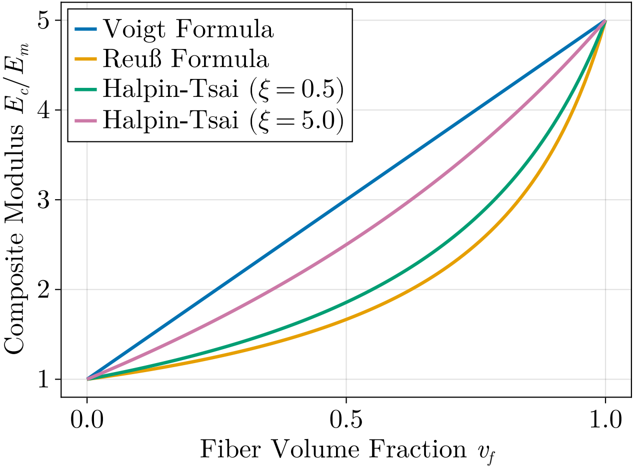
- Eve own as
  the

Ishai 2006)

Note: ξ .

- Thi models.

Graphical Comparison for varying ξ



3.2. Numerical Example

Micro-Mechanics Descriptions

(from Kollár and Springer 2003)

Consider a Graphite/Epoxy unidirectional ply. Matrix properties are given with subscript m in the table below. Nominal properties with fiber volume fraction $v_f = 60\,\%$ are also given. Assume that the fibers show anisotropy ($E_{f1} \neq E_{f2}$). Use the modified Rule of Mixtures.

	E_1	E_2	G_{12}	ν_{12}	E_m	G_m	ν_m
Value	148	9.65	4.55	0.3	4.1	1.5	0.35

All modulii in GPa.

Estimate the following:

- Fiber modulus properties
- Composite material modulii for volume fraction $v_f = 0.55$.

Note: It is common to denote the fiber-longitudinal direction as 1 and the transverse direction as 2.

3.2. Numerical Example

Micro-Mechanics Descriptions

(from Kollár and Springer 2003)

Consider a Graphite/Epoxy unidirectional ply. Matrix properties are given with subscript m in the table below. Nominal properties with fiber volume fraction $v_f = 60\,\%$ are also given. Assume that the fibers show anisotropy ($E_{f1} \neq E_{f2}$). Use the modified Rule of Mixtures.

	E_1	E_2	G_{12}	ν_{12}	E_m	G_m	ν_m
Value	148	9.65	4.55	0.3	4.1	1.5	0.35

All modulii in GPa.

Estimate the following:

- Fiber modulus properties
- Compo

Note: It is c ection as 2.

Answers:

- $E_{f_1} \approx 240\,\text{GPa}$, $\nu_{f_{12}} = 0.27$, $E_{f_2} \approx 23\,\text{GPa}$, $G \approx 23\,\text{GPa}$.
- For $v_f = 0.55$, we have $E_1 \approx 130\,\text{GPa}$, $\nu_{12} = 0.31$, $E_2 \approx 8.8\,\text{GPa}$, $G_{12} \approx 4\,\text{GPa}$.

4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

Material Symmetry

The study of material symmetry is concerned with finding answers to the question:
If the strain field on a deformable object is changed, how does the stress field change?

4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

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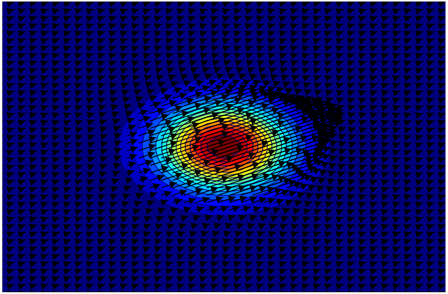
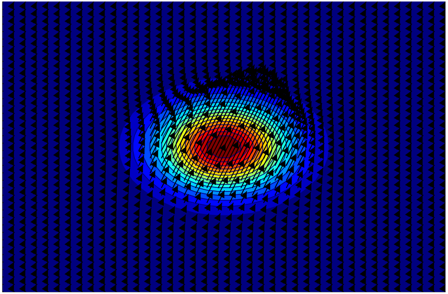
Consider the following Deformation Fields



Deformation Case 1



Deformation Case 2 (Case 1 Rotated)

4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

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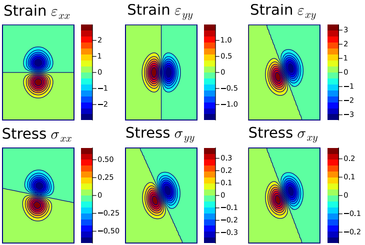
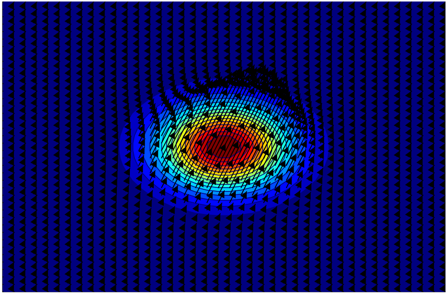
Stress and Strain Field



Isotropic Stress-Strain Relationship



Deformation Case 2 (Case 1 Rotated)

4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

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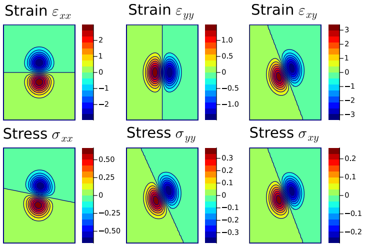
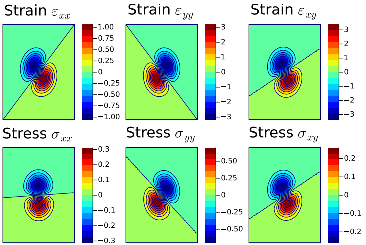
Stress and Strain Field



Isotropic Stress-Strain Relationship

Stress and Strain Field



Isotropic Stress-Strain Relationship

4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

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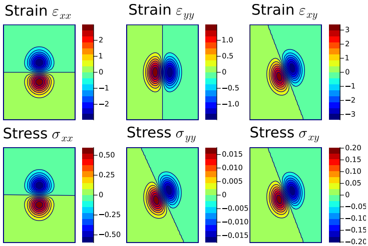
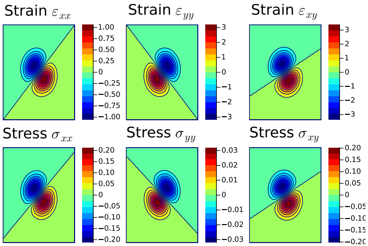
Stress and Strain Field



Anisotropic Case

Stress and Strain Field



Anisotropic Case

4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

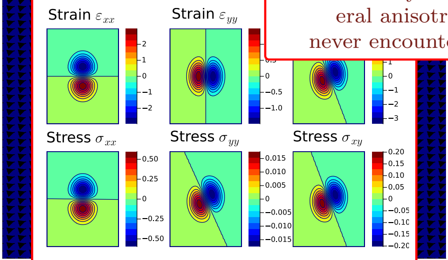
Material Symmetry

The study of material symmetry is concerned with finding answers to the question:
If the strain field on a deformable object is changed, how does the stress field change?

Consider the following Deformation Fields

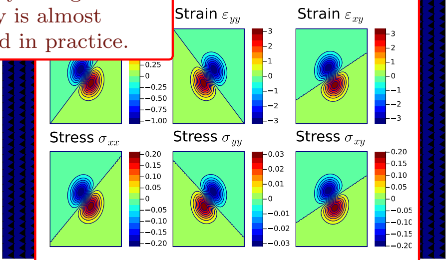
Most materials exhibit some sort of symmetry and general anisotropy is almost never encountered in practice.

Stress and Stra



Anisotropic Case

and Strain Field



Anisotropic Case

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

How do stresses and strains transform under coordinate change?

- Suppose $\underline{x} \in \mathbb{R}^3$ are the coordinates of a point in 3D space.
- Let $\underline{x}' \in \mathbb{R}^3$ be the coordinates under transformation.
- We will write: $\boxed{\underline{x}' = \underline{\underline{Q}}\,\underline{x}}$, with $\underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^T$.

Strains

- $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\nabla}_{\underline{x}}\underline{u} + \underline{\nabla}_{\underline{x}}\underline{u}^T\right)$
- $\underline{\nabla}_{\underline{x}'}\underline{u}' = \underline{\underline{Q}}\,\underline{\nabla}_{\underline{x}}\underline{u}\underline{\underline{Q}}^{-1} \implies \boxed{\underline{\underline{\varepsilon}}' = \underline{\underline{Q}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{Q}}^T}$.

Stresses

- Cauchy Stress Definition: $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$
- $\underline{\underline{Q}}\,\underline{t} = \underline{t}' = \underline{\underline{\sigma}}'\underline{n}' = \underline{\underline{\sigma}}'\underline{\underline{Q}}\,\underline{n} = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{n}$
  $\implies \boxed{\underline{\underline{\sigma}}' = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{\underline{Q}}^T}$

Reflections

Note that reflections may be expressed as a coordinate change with $\underline{\underline{Q}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (reflection about the xy plane).

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- Under reflection about the xy plane, the strain transforms as,

$$\begin{bmatrix} \varepsilon_x' & \frac{\gamma_{xy}'}{2} & \frac{\gamma_{xz}'}{2} \\ & \varepsilon_y' & \frac{\gamma_{yz}'}{2} \\ \text{sym} & & \varepsilon_z' \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \text{sym} & & \varepsilon_z \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & -\frac{\gamma_{xz}}{2} \\ & \varepsilon_y & -\frac{\gamma_{yz}}{2} \\ \text{sym} & & \varepsilon_z \end{bmatrix}$$

- So in Voigt notation we have,

$$\begin{bmatrix} \varepsilon_x' \\ \varepsilon_y' \\ \varepsilon_y' \\ \gamma_{xy}' \\ \gamma_{xz}' \\ \gamma_{yz}' \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \qquad \begin{bmatrix} \sigma_x' \\ \sigma_y' \\ \sigma_y' \\ \tau_{xy}' \\ \tau_{xz}' \\ \tau_{yz}' \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

Similarly for Stress

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- Under reflection about the xy plane, the strain transforms as,

$$\begin{bmatrix} \varepsilon' & \underline{\gamma'_{xy}} & \underline{\gamma'_{xz}} \end{bmatrix} \quad \begin{bmatrix} 1 & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \varepsilon_x & \underline{\gamma_{xy}} & \underline{\gamma_{xz}} \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \end{bmatrix}$$

If a material were symmetric about the xy plane, then reflecting the strain field about the xy plane will result in a stress field that is reflected about the same xy plane.

Note

- Strain field reflection is a kinematic operation/configuration change.
- Change in the Stress field is the effect that the above kinematic change results in.
- If the material happens to be symmetric about the reflection plane, then this change will be a reflection.

- So

$$\begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_y \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \gamma_{yz} \end{bmatrix} \quad \begin{bmatrix} \tau'_{yz} \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \tau_{yz} \end{bmatrix}$$

Similarly for Stress

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- We have said the following :

$$\boxed{\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & \text{sym} & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}}$$

Recall that this symmetry follows from strain energy existence

$$\boxed{\begin{bmatrix}\sigma_x'\\ \sigma_y'\\ \sigma_z'\\ \tau_{xy}'\\ \tau_{xz}'\\ \tau_{yz}'\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & \text{sym} & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x'\\ \varepsilon_y'\\ \varepsilon_z'\\ \gamma_{xy}'\\ \gamma_{xz}'\\ \gamma_{yz}'\end{bmatrix}}$$

(The $\underline{\underline{C}}$ matrix has to be the same in both the original and the reflected coordinate systems, if the material shows reflection symmetry)

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- We have said the following :

This leads to

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ & & C_{33} & C_{34} & -C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & -C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

Recall that this symmetry follows from strain energy existence

$$\begin{bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{xz} \\ \tau'_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_z \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{bmatrix}$$

(The $\underline{\underline{C}}$ matrix has to be the same in both the original and the reflected coordinate systems, if the material shows reflection symmetry)

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Macro-Mechanics Descriptions

- We have said the following :

This leads to

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ & & C_{33} & C_{34} & -C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & -C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

Recall that this symmetry follows from strain energy existence

$$\begin{bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{xz} \\ \tau'_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_z \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{bmatrix}$$

(The $\underline{\underline{C}}$ matrix has to be the same in both the original and the reflected coordinate systems, if the material shows reflection symmetry)

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- We have said the following :

This leads to

$$\underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ & & C_{33} & C_{34} & -C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & -C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}}_{\text{Recall that this symmetry follows from strain energy existence}}$$

Finally we see that material symmetry about the xz plane implies the following simplification to the constitutive relationship.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

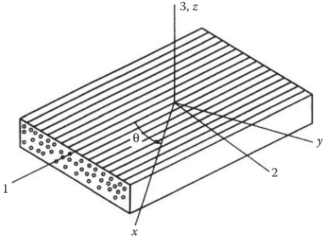
This is known as a **Monoclinic Material** (13 constants). This is also quite rare to encounter in practice.

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

Suppose all the three fundamental planes are planes of symmetry, we have an **Orthotropic Material** (9 constants).

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix}$$



(Figure 2.5 from Gibson 2012)

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

Suppose all the three fundamental planes are planes of symmetry, we have an **Orthotropic Material** (9 constants).

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix}$$

Notice that $(\sigma_{(1,2,3)}, \varepsilon_{(1,2,3)})$ and $(\tau_{(12,13,23)}, \gamma_{(12,13,23)})$ are naturally decoupled as a consequence of symmetry in this coordinate system.

Also note,

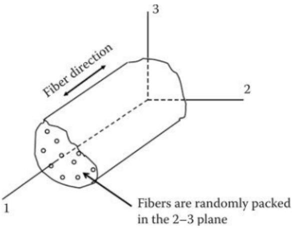
- Specially orthotropic
- Generally orthotropic

(Figure 2.5 from Gibson 2012)

4.1. Material Symmetry and Anisotropy: Transverse Isotropy

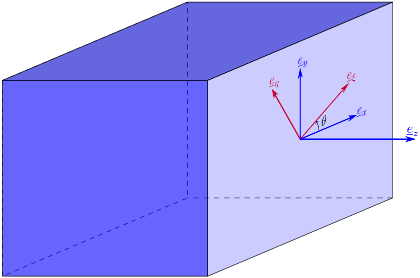
Macro-Mechanics Descriptions

- In continuous fiber reinforced composites, it is often the case that the fibers are randomly distributed on a plane. This leads to planar isotropy in the plane perpendicular to the fiber stacking direction.



(Figure 2.6 from Gibson 2012)

- How do the stresses and strains transform on the plane?



$$(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}) \to (\sigma_\xi, \sigma_\eta, \sigma_z, \tau_{\xi\eta}, \tau_{\xi z}, \tau_{\eta z})$$

$$(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \to (\varepsilon_\xi, \varepsilon_\eta, \varepsilon_z, \gamma_{\xi\eta}, \gamma_{\xi z}, \gamma_{\eta z})$$

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- Here, $\underline{\underline{Q}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\underline{\underline{\sigma}}' = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{\underline{Q}}^T$. So we get

$$\begin{bmatrix} \sigma_{\xi\xi} & \sigma_{\xi\eta} & \sigma_{\xi z} \\ \sigma_{\xi\eta} & \sigma_{\eta\eta} & \sigma_{\eta z} \\ \sigma_{\xi z} & \sigma_{\eta z} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta & & \\ \sigma_{xy}\cos 2\theta - \frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta & \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \sigma_{xy}\sin 2\theta & \\ \sigma_{xz}\cos\theta + \sigma_{yz}\sin\theta & \sigma_{yz}\cos\theta - \sigma_{xz}\sin\theta & \sigma_{zz} \end{bmatrix}$$

In Voigt notation, we can write the same as

$$\begin{bmatrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \sigma_{zz} \\ \sigma_{\xi\eta} \\ \sigma_{\xi z} \\ \sigma_{\eta z} \end{bmatrix} = \begin{bmatrix} \frac{1+\cos 2\theta}{2} & \frac{1-\cos 2\theta}{2} & 0 & \sin 2\theta & 0 & 0 \\ \frac{1-\cos 2\theta}{2} & \frac{1+\cos 2\theta}{2} & 0 & -\sin 2\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\sin 2\theta}{2} & \frac{\sin 2\theta}{2} & 0 & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- H $\quad \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix}$

Strain Transformation in Voigt Notation

$$\begin{bmatrix} \varepsilon_{\xi\xi} \\ \varepsilon_{\eta\eta} \\ \varepsilon_{zz} \\ \gamma_{\xi\eta} \\ \gamma_{\xi z} \\ \gamma_{\eta z} \end{bmatrix} = \begin{bmatrix} \frac{1+\cos 2\theta}{2} & \frac{1-\cos 2\theta}{2} & 0 & \frac{\sin 2\theta}{2} & 0 & 0 \\ \frac{1-\cos 2\theta}{2} & \frac{1+\cos 2\theta}{2} & 0 & -\frac{\sin 2\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\sin 2\theta & \sin 2\theta & 0 & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

Note that we have used $\gamma_{xy} = 2\varepsilon_{xy}$ for the shear strains so some terms will be different from the stress transformation.

In Vo

$$\begin{bmatrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \sigma_{zz} \\ \sigma_{\xi\eta} \\ \sigma_{\xi z} \\ \sigma_{\eta z} \end{bmatrix} = \begin{bmatrix} \frac{1+\cos 2\theta}{2} & \frac{1-\cos 2\theta}{2} & 0 & \sin 2\theta & 0 & 0 \\ \frac{1-\cos 2\theta}{2} & \frac{1+\cos 2\theta}{2} & 0 & -\sin 2\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\sin 2\theta}{2} & \frac{\sin 2\theta}{2} & 0 & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- The stresses and strains transform as follows on the plane:

$$\begin{aligned}
\sigma_\xi &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\\
\sigma_\eta &= \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta\\
(\sigma_z &= \sigma_z)\\
\tau_{\xi\eta} &= -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\\
\tau_{\xi z} &= \tau_{xz}\cos\theta + \tau_{yz}\sin\theta\\
\tau_{\eta z} &= -\tau_{xz}\sin\theta + \tau_{yz}\cos\theta
\end{aligned}$$

$$\begin{aligned}
\varepsilon_\xi &= \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta\\
\varepsilon_\eta &= \frac{\varepsilon_x+\varepsilon_y}{2} - \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta\\
(\varepsilon_z &= \varepsilon_z)\\
\gamma_{\xi\eta} &= -(\varepsilon_x-\varepsilon_y)\sin 2\theta + \gamma_{xy}\cos 2\theta\\
\gamma_{\xi z} &= \gamma_{xz}\cos\theta + \gamma_{yz}\sin\theta\\
\gamma_{\eta z} &= -\gamma_{xz}\sin\theta + \gamma_{yz}\cos\theta
\end{aligned}$$

- For an orthotropic material, the straight stresses/strains and shear stresses/strains are fully decoupled.
- So we will consider different cases of kinematic deformation fields to see if more can be said.

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- The

$\sigma_\xi = $ … $\sin 2\theta$

$\sigma_\eta = $ … $\sin 2\theta$

$(\sigma_z = \sigma$

$\tau_{\xi\eta} = -$

$\tau_{\xi z} = \tau$

$\tau_{\eta z} = -$

- For … ins are fully
- So w … can be said

1. Pure Out-Of-Plane Shear ($\gamma_{xz} \neq 0$)

- The stresses and strains are,

$$\begin{aligned} \sigma_\xi &= 0\\ \sigma_\eta &= 0\\ (\sigma_z &= 0)\\ \tau_{\xi\eta} &= 0\\ \begin{bmatrix}\tau_{\xi z}\\ \tau_{\eta z}\end{bmatrix} &= \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\tau_{xz}\\ \tau_{yz}\end{bmatrix}\\ &= \begin{bmatrix}C_{55}\gamma_{xz}\cos\theta\\ -C_{55}\gamma_{xz}\sin\theta\end{bmatrix} := \begin{bmatrix}C_{55}\gamma_{\xi\eta}\\ C_{66}\gamma_{\eta z}\end{bmatrix}.\end{aligned}$$

$$\begin{aligned} \varepsilon_\xi &= 0\\ \varepsilon_\eta &= 0\\ (\varepsilon_z &= 0)\\ \gamma_{\xi\eta} &= 0\\ \begin{bmatrix}\gamma_{\xi z}\\ \gamma_{\eta z}\end{bmatrix} &= \begin{bmatrix}\gamma_{xz}\cos\theta\\ -\gamma_{xz}\sin\theta\end{bmatrix}\end{aligned}$$

- Under symmetry, $(\tau_{\xi z}, \tau_{\eta z})$ is related to $(\gamma_{\xi z}, \gamma_{\eta z})$ in the same way that (τ_{xz}, τ_{yz}) is related to $(\gamma_{xz}, \gamma_{yz})$.
- So we have,

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & \cancelto{C_{55}}{C_{66}} \end{bmatrix}$$

4.1. Material Symmetry and Anisotropy

2. Pure Out-Of-Plane Stretch ($\varepsilon_z \neq 0$)

- We have straight stresses $\sigma_x = C_{13}\varepsilon_z, \sigma_y = C_{23}\varepsilon_z$.
- Upon transformation we have,

$$\begin{aligned}
\sigma_\xi &= \left(\frac{C_{13}+C_{23}}{2} + \frac{C_{13}-C_{23}}{2}\cos 2\theta\right)\varepsilon_z \\
\sigma_\eta &= \left(\frac{C_{13}+C_{23}}{2} - \frac{C_{13}-C_{23}}{2}\cos 2\theta\right)\varepsilon_z \\
\sigma_z &= \sigma_z \\
\tau_{\xi\eta} &= -\frac{C_{13}-C_{23}}{2}\sin 2\theta \\
\tau_{\xi z} &= \tau_{\eta z} = 0
\end{aligned}
\qquad
\begin{aligned}
\varepsilon_\xi &= 0 \\
\varepsilon_\eta &= 0 \\
\varepsilon_z &= \varepsilon_z \\
\gamma_{\xi\eta} &= 0 \\
\gamma_{\xi z} &= \gamma_{\eta z} = 0
\end{aligned}$$

- For planar isotropy, the relationship between $(\sigma_\xi, \sigma_\eta)$ and σ_z must be independent of θ. This is only possible for $C_{13} = C_{23}$.
- So we have,

$$\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
 & C_{22} & \overset{C_{13}}{\cancel{C_{23}}} & 0 & 0 & 0 \\
 & & C_{33} & 0 & 0 & 0 \\
 & & & C_{44} & 0 & 0 \\
 & \text{sym} & & & C_{55} & 0 \\
 & & & & & C_{55}
\end{bmatrix}$$

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

3. Pure In-Plane Stretch $(\varepsilon_x \neq 0, \varepsilon_y = 0)$

- From the constitutive properties we have $\sigma_x = C_{11}\varepsilon_x$ and $\sigma_y = C_{12}\varepsilon_x$.
- Using this all the other components can be written as

$$\begin{aligned}
\sigma_\xi &= \left(\frac{C_{11}+C_{12}}{2} + \frac{C_{11}-C_{12}}{2}\cos 2\theta\right)\varepsilon_x & \varepsilon_\xi &= \frac{1+\cos 2\theta}{2}\varepsilon_x\\
\sigma_\eta &= \left(\frac{C_{11}+C_{12}}{2} + \frac{C_{11}-C_{12}}{2}\cos 2\theta\right)\varepsilon_x & \varepsilon_\eta &= \frac{1-\cos 2\theta}{2}\varepsilon_x\\
&= C_{12}\varepsilon_x + C_{22}\varepsilon_y & &\\
\sigma_z &= 0 & \varepsilon_z &= 0\\
\tau_{\xi\eta} &= 0 & \gamma_{\xi\eta} &= 0\\
\tau_{\xi z} &= \tau_{\eta z} = 0. & \gamma_{\xi z} &= \gamma_{\eta z} = 0.
\end{aligned}$$

- For the σ_η equality to hold, we need $C_{22} = C_{11}$. So we have

$$\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\
 & \cancel{C_{22}} \nearrow C_{11} & C_{13} & 0 & 0 & 0\\
 & & C_{33} & 0 & 0 & 0\\
 & & & C_{44} & 0 & 0\\
 & \text{sym} & & & C_{55} & 0\\
 & & & & & C_{55}
\end{bmatrix}$$

- Th

$\sigma_\xi =$ … $\frac{xy}{2}\sin 2\theta$

$\sigma_\eta =$ … $\frac{xy}{2}\sin 2\theta$

$(\sigma_z =$

$\tau_{\xi\eta} =$

$\tau_{\xi z} =$

$\tau_{\eta z} =$

- For … ains are full
- So … can be said

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

4. Pure In-Plane Shear ($\gamma_{xy} \neq 0$)

- From the constitutive properties we have $\tau_{xy} = C_{44}\gamma_{xy}$.
- Using this all the other components can be written as

$$\begin{aligned}
\sigma_\xi &= C_{44}\gamma_{xy}\sin 2\theta = C_{11}\varepsilon_\xi + C_{12}\varepsilon_\eta \\
\sigma_\eta &= -C_{44}\gamma_{xy}\sin 2\theta = C_{12}\varepsilon_\xi + C_{11}\varepsilon_\eta \\
\sigma_z &= 0 \\
\tau_{\xi\eta} &= C_{44}\gamma_{xy}\cos 2\theta \\
\tau_{\xi z} &= \tau_{\eta z} = 0.
\end{aligned}
\qquad
\begin{aligned}
\varepsilon_\xi &= \frac{\gamma_{xy}}{2}\sin 2\theta \\
\varepsilon_\eta &= -\frac{\gamma_{xy}}{2}\sin 2\theta \\
\varepsilon_z &= 0 \\
\gamma_{\xi\eta} &= \gamma_{xy}\cos 2\theta \\
\gamma_{\xi z} &= \gamma_{\eta z} = 0.
\end{aligned}$$

- So we have $C_{44}\gamma_{xy}\sin 2\theta = \frac{C_{11}-C_{12}}{2}\gamma_{xy}\sin 2\theta$. Therefore,

$$\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
 & C_{11} & C_{13} & 0 & 0 & 0 \\
 & & C_{33} & 0 & 0 & 0 \\
 & & & \cancelto{\frac{C_{11}-C_{12}}{2}}{C_{44}} & 0 & 0 \\
 & \text{sym} & & & C_{55} & 0 \\
 & & & & & C_{55}
\end{bmatrix}$$

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- Th

$\sigma_\xi =$ … $\frac{xy}{2}\sin 2\theta$

$\sigma_\eta =$ … $\frac{xy}{2}\sin 2\theta$

$(\sigma_z =$

$\tau_{\xi\eta} =$

$\tau_{\xi z} =$

$\tau_{\eta z} =$

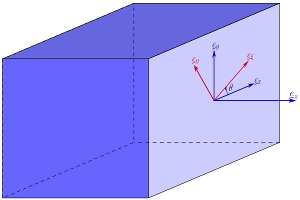
- For … ains are full

- So … can be said

To Summarize,
a **Transversely Isotropic Material**
constitution can be expressed as

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{11} & C_{13} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{55}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

The material is fully characterized by five engineering constants.



4.1. Material Symmetry and Anisotropy: Engineering Constants

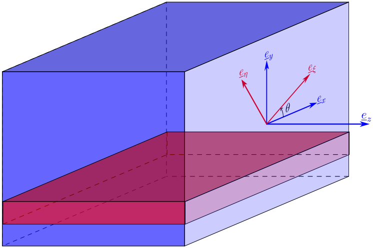
Macro-Mechanics Descriptions

- In engineering practice, the constants are usually written easier in terms of compliance.
- For a specially orthotropic material the strain-stress relationship are usually expressed as,

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & -\frac{\nu_{31}}{E_{33}} & 0 & 0 & 0\\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{33}} & 0 & 0 & 0\\ -\frac{\nu_{13}}{E_{11}} & -\frac{\nu_{23}}{E_{22}} & \frac{1}{E_{33}} & 0 & 0 & 0\\ & & & \frac{1}{G_{12}} & 0 & 0\\ & \text{sym} & & & \frac{1}{G_{13}} & 0\\ & & & & & \frac{1}{G_{23}} \end{bmatrix} \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{bmatrix}$$

5. Analysis of Planar Laminates

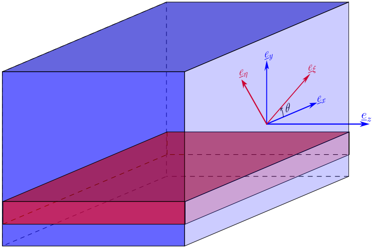
- Let us just consider one thin layer of a transversely isotropic material (continuously reinforced composite along a single direction).



$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{11} & C_{13} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{55}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

5. Analysis of Planar Laminates

- Let us just consider one thin layer of a transversely isotropic material (continuously reinforced composite along a single direction).



$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{11} & C_{13} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{55}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

- We invoke plane stress assumptions, setting $\sigma_y = 0$. Let us also assume small shears, $\tau_{xy} = 0, \tau_{yz} = 0$.
  (Note: ε_z is not zero, and is implicitly defined)

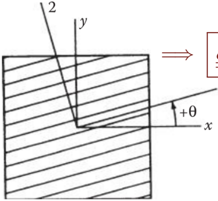
$$\boxed{\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{33}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix}} \text{(4 constants)}$$

(Note change in notation in C_{ij})

5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



Positive θ

(Figure 2.11 from Gibson 2012)

$$\Longrightarrow \boxed{\underline{\underline{\sigma}}_{(x,y)} = \underline{\underline{Q}}\,\underline{\underline{\sigma}}_{(1,2)}\underline{\underline{Q}}^T}$$

- If the coordinate system is not aligned with the fiber axes, the stress and strain transformations need to be invoked.
- In the constitutive relationship we have,

$$\underline{\underline{\sigma}}_{(1,2)} = \underline{\underline{C}}\,\varepsilon_{(1,2)}$$

$$\underline{\underline{T}}_{\sigma}^{-1}\underline{\underline{\sigma}}_{(x,y)} = \underline{\underline{\sigma}}_{(1,2)} = \underline{\underline{C}}\,\varepsilon_{(1,2)} = \underline{\underline{C}}\,\underline{\underline{T}}_{\varepsilon}^{-1}\underline{\varepsilon}_{(x,y)}$$

$$\Longrightarrow \underline{\underline{\sigma}}_{(x,y)} = \underbrace{\underline{\underline{T}}_{\sigma}\underline{\underline{C}}\,\underline{\underline{T}}_{\varepsilon}^{-1}}_{\underline{\underline{C}}'}\varepsilon_{(x,y)}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underline{\underline{T}}_{\sigma}} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

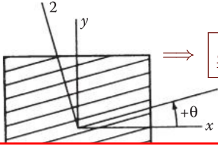
$$\underline{\underline{T}}_{\sigma}^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$\text{where } \underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}.$$

5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\Longrightarrow \boxed{\underline{\underline{\sigma}}_{(x,y)} = \underline{\underline{Q}}\,\underline{\underline{\sigma}}_{(1,2)}\underline{\underline{Q}}^T}$$

- If the coordinate system is not aligned with the fiber axes, the stress and strain ns need to be invoked.

utive relationship we have,

$$_{(1,2)} = \underline{\underline{C}}\,\varepsilon_{(1,2)}$$

$$_{(1,2)} = \underline{\underline{C}}\,\varepsilon_{(1,2)} = \underline{\underline{C}}\,\underline{\underline{T}}_{\varepsilon}^{-1}\varepsilon_{(x,y)}$$

$$_{(x,y)} = \underbrace{\underline{\underline{T}}_{\sigma}\underline{\underline{C}}\,\underline{\underline{T}}_{\varepsilon}^{-1}}_{\underline{\underline{C}}'}\varepsilon_{(x,y)}$$

Recall that Strain Transformation looks slightly different because of our definition of shear strain $\gamma_{xy} = 2\varepsilon_{xy}$.

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2\theta & \sin^2\theta & -\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & \cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underline{\underline{T}}_{\varepsilon}} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_y \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underline{\underline{T}}_{\sigma}} \begin{bmatrix} \sigma_2 \\ \tau_{12} \end{bmatrix}$$

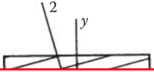
$$\underline{\underline{T}}_{\sigma}^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

where $\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}$.

5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\Longrightarrow \boxed{\underline{\underline{\sigma}}_{(x,y)} = \underline{\underline{Q}}\,\underline{\underline{\sigma}}_{(1,2)}\underline{\underline{Q}}^T}$$

Transformed $\underline{\underline{C}}$ Matrix ($\underline{\sigma} = \underline{\underline{C}}\,\underline{\varepsilon}$)

$$\underline{\underline{C}}' = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} \\ C'_{12} & C'_{22} & C'_{23} \\ C'_{13} & C'_{23} & C'_{33} \end{bmatrix}$$

$$C'_{11} = C_{11}c^4 + C_{22}s^4 + (2C_{33} + C_{12})2c^2s^2$$

$$C'_{22} = C_{11}s^4 + C_{22}c^4 + (2C_{33} + C_{12})2c^2s^2$$

$$C'_{33} = (C_{11} + C_{22} - 2C_{33} - 2C_{12})c^2s^2 + C_{33}(c^4 + s^4)$$

$$C'_{12} = (C_{11} + C_{22} - 4C_{33})c^2s^2 + C_{12}(c^4 + s^4)$$

$$C'_{13} = (C_{11} - 2C_{33} - C_{12})c^3s - (C_{22} - 2C_{33} - C_{12})cs^3$$

$$C'_{23} = (C_{11} - 2C_{33} - C_{12})cs^3 - (C_{22} - 2C_{33} - C_{12})c^3s.$$

ordinate system is not aligned fiber axes, the stress and strain ns need to be invoked.

utive relationship we have,

$$_{(1,2)} = \underline{\underline{C}}\,\underline{\varepsilon}_{(1,2)}$$

$$_{(1,2)} = \underline{\underline{C}}\,\underline{\varepsilon}_{(1,2)} = \underline{\underline{C}}\,\underline{\underline{T}}_{\varepsilon}^{-1}\underline{\varepsilon}_{(x,y)}$$

$$_{(x,y)} = \underbrace{\underline{\underline{T}}_{\sigma}\underline{\underline{C}}\,\underline{\underline{T}}_{\varepsilon}^{-1}}_{\underline{\underline{C}}'}\underline{\varepsilon}_{(x,y)}$$

$$\begin{bmatrix} \sigma_y \\ \tau_{xy} \end{bmatrix}$$

ere $\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}$.

$$\underline{\underline{T}}_{\sigma}^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

5.1. Generally Orthotropic Laminates

Analysis of Planar Laminates

- Compliance is often more convenient:

$$\underline{\varepsilon}_{(x,y)} = \underline{\underline{T}}_{\varepsilon} \underline{\underline{S}}\, \underline{\underline{T}}_{\sigma}^{-1} \underline{\sigma}_{(x,y)}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{13} \\ & S'_{22} & S'_{23} \\ & & S'_{33} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$S'_{11} = S_{11}c^4 + S_{22}s^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{22} = S_{11}s^4 + S_{22}c^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{33} = (2S_{11} + 2S_{22} - S_{33} - 4S_{12})2c^2s^2 + S_{33}(c^4 + s^4)$$

$$S'_{12} = (S_{11} + S_{22} - S_{33})c^2s^2 + S_{12}(c^4 + s^4)$$

$$S'_{13} = (2S_{11} - S_{33} - 2S_{12})c^3s - (2S_{22} - S_{33} - 2S_{12})cs^3$$

$$S'_{23} = (2S_{11} - S_{33} - 2S_{12})cs^3 - (2S_{22} - S_{33} - 2S_{12})c^3s.$$

- In the material principal directions we have,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Engineering Constants: $E_1, E_2, G_{12}, \nu_{12}$

- Based on this we can write,

$$E_x = \left[\frac{c^4}{E_1} + \frac{s^4}{E_2} + \left(\frac{1}{G_{12}} - \frac{2\nu_{21}}{E_2} \right) c^2s^2 \right]^{-1}$$

$$E_y = \left[\frac{s^4}{E_1} + \frac{c^4}{E_2} + \left(\frac{1}{G_{12}} - \frac{2\nu_{21}}{E_2} \right) c^2s^2 \right]^{-1}$$

$$\begin{aligned} G_{xy} = &\Bigg[\frac{c^4 + s^4}{G_{12}} + \Bigg(\frac{1}{E_1} + \frac{1}{E_2} \\ &- \frac{1}{2G_{12}} + 2\frac{\nu_{21}}{E_2} \Bigg) 4c^2s^2 \Bigg]^{-1} \end{aligned}$$

$$\begin{aligned} \nu_{yx} = &E_y \Bigg[\frac{\nu_{21}}{E_2}(c^4 + s^4) \\ &- \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} \right) c^2s^2 \Bigg] \end{aligned}$$

- It is customary to express the laminate constitutive relationship as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

5.1. Generally Orthotropic Laminates

Analysis of Planar Laminates

- Compliance is often more convenient:

$$\underline{\varepsilon}_{(x,y)} = \underline{\underline{T}}_{\varepsilon}\underline{\underline{S}}\,\underline{\underline{T}}_{\sigma}^{-1}\underline{\sigma}_{(x,y)}$$

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}S'_{11} & S'_{12} & S'_{13}\\ & S'_{22} & S'_{23}\\ & & S'_{33}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

$$S'_{11} = S_{11}c^4 + S_{22}s^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{22} = S_{11}s^4 + S_{22}c^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{33} = (2S_{11} + 2S_{22} - S_{33} - 4S_{12})2c^2s^2 + S_{33}(c$$

$$S'_{12} = (S_{11} + S_{22} - S_{33})c^2s^2 + S_{12}(c^4 + s^4)$$

$$S'_{13} = (2S_{11} - S_{33} - 2S_{12})c^3s - (2S_{22} - S_{33} - 2S_{12})cs^3 \quad \nu_{yx} = E_y\Big[\;\frac{}{E_2}$$

$$S'_{23} = (2S_{11} - S_{33} - 2S_{12})cs^3 - (2S_{22} - S_{33} - 2S_{12})c^3s.$$

$$- \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)c^2s^2\Big]$$

- In the material principal directions we have,

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G_{12}}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}$$

Engineering Constants: $E_1, E_2, G_{12}, \nu_{12}$

- Based on this we can write,

The Shear Constants can be written as

$$\begin{aligned}\eta_{xy,x} =& G_{xy}\Big[\left(\frac{2}{E_1} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\right)c^3s\\ &- \left(\frac{2}{E_2} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\right)cs^3\Big]\\ \eta_{xy,y} =& G_{xy}\Big[\left(\frac{2}{E_1} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\right)cs^3\\ &- \left(\frac{2}{E_2} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\right)c^3s\Big]\end{aligned}$$

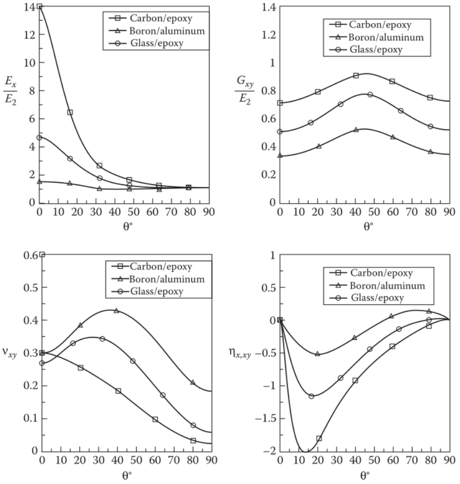
- It is customary to express the laminate constitutive relationship as

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}}\\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}}\\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

5.1. Generally Orthotropic Laminates

Analysis of Planar Laminates

Off-Axis Modulii



(Figure 2.14 from Gibson 2012)

- Compliance is often r … write,

$\underline{\varepsilon}_{(x,y)} = \underline{\underline{T}}_{\varepsilon}\underline{\underline{S}}\,\underline{\underline{T}}_{\sigma}^{-1}\underline{\sigma}_{(x,y}$

$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \Big[\begin{matrix} S'_{11} & S'_{12} \\ & S'_{22}\end{matrix}$

$S'_{11} = S_{11}c^4 + S_{22}s^4$

$S'_{22} = S_{11}s^4 + S_{22}c^4$

$S'_{33} = (2S_{11} + 2S_{22} -$

$S'_{12} = (S_{11} + S_{22} - S$

$S'_{13} = (2S_{11} - S_{33} - 2$

$S'_{23} = (2S_{11} - S_{33} - 2$

can be written as

$\frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\Big) c^3 s$

$+ \frac{2\nu_{21}}{E_2}\Big) cs^3\Big]$

$\frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\Big) cs^3$

$+ \frac{2\nu_{21}}{E_2}\Big) c^3 s\Big]$

$- \frac{1}{G_{12}}\Big) c^2 s^2\Big]$

- In the material princ … ress the laminate … ip as

$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix} = \Big[\begin{matrix}\frac{1}{E_1}\\ -\frac{\nu_1}{E}\\ 0\end{matrix}$

$\begin{matrix}\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}}\\ \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}}\\ \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}}\end{matrix}\Big]\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$

$\gamma_{xy}\Big]$

Engineering Constants: $E_1, E_2, G_{12}, \nu_{12}$

5.2. Numerical Examples: 1

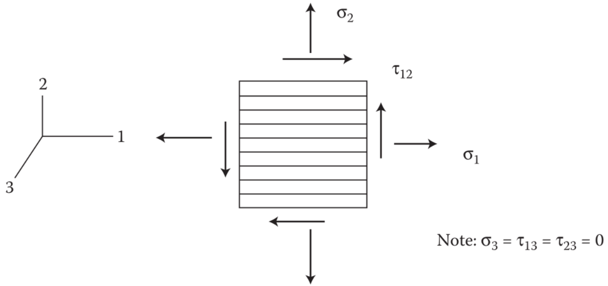
Analysis of Planar Laminates(Example 2.2 from Gibson 2012)

Consider an orthotropic laminate with the properties

$$E_1 = 140\,\text{GPa},\ E_2 = 10\,\text{GPa},\ G_{12} = 7\,\text{GPa},\ \nu_{12} = 0.3,\ \nu_{23} = 0.2.$$

Compute the strains if it is subjected to the following state of stress in the principal coordinates:

$$\sigma_1 = 70\,\text{MPa},\ \sigma_2 = 140\,\text{MPa},\ \tau_{12} = 35\,\text{MPa},\ \sigma_3 = \tau_{12} = \tau_{23} = 0.$$

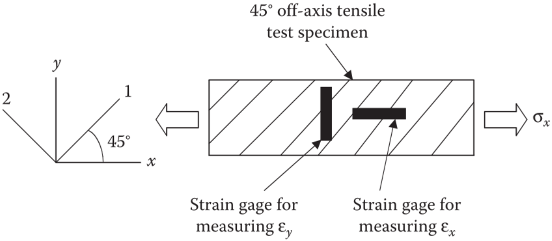


(Figure 2.10 from Gibson 2012)

5.2. Numerical Examples: 2

Analysis of Planar Laminates(Example 2.3 from Gibson 2012)

A 45° off-axis tensile test is conducted on a generally orthotropic test specimen by applying a normal stress σ_x. The specimen has strain gauges attached to measure axial and transverse strains $(\varepsilon_x, \varepsilon_y)$. How many engineering parameters can be estimated from measurements of $\sigma_x, \varepsilon_x, \varepsilon_y$?

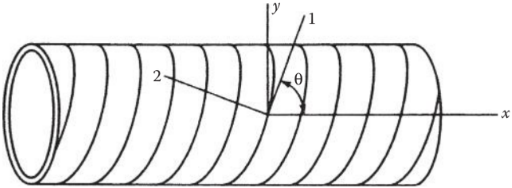


(*Figure 2.15 from Gibson 2012*)

5.2. Numerical Examples: 3

Analysis of Planar Laminates(Example 2.4 from Gibson 2012)

A filament-wound cylindrical pressure vessel of mean diameter $d = 1\,\text{m}$ and wall thickness $t = 20\,\text{mm}$ is subjected to an internal pressure, p. The filament- winding angle $\theta = 53.1^\circ$ from the longitudinal axis of the pressure vessel, and the glass/epoxy material has the following properties: $E_{11} = 40\,\text{GPa}$, $E_{22} = 10\,\text{GPa}$, $G_{12} = 3.5\,\text{GPa}$, and $\nu_{12} = 0.25$. By the use of a strain gauge, the normal strain along the fiber direction is determined to be $\varepsilon_{11} = 0.001$. Determine the internal pressure in the vessel.



(Figure 2.18 from Gibson 2012)

6. Classical Laminate Theory I

The Classical Theory of Plates

- The kinematics of a thin plate on the $x-y$ plane can be written as

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u(x,y) + z\theta_y(x,y) - y\theta_x(x,y) \\ v(x,y) - z\theta_x(x,y) + x\theta_z(x,y) \\ w(x,y) \end{bmatrix},$$

  where $u(x,y), v(x,y)$ are in-plane deformations, $w(x,y)$ is the out-of-plane deformation, $\theta_x(x,y), \theta_y(x,y)$ are out-of-plane (bending) rotations, and $\theta_z(x,y)$ is the in-plane (drilling) rotation.

  - Functional dependence for the quantities are only in terms of x and y, the in-plane coordinates, stemming from the assumption that lines along the thickness of the plate remain perpendicular to the mid-plane.
  - Higher order plate theories extend this by adding terms proportional to $z^2, z^{\vec{3}}, \dots,$ but this is not our concern for now.
  - Further, classical plate theory is only concerned with flexure (bending) so we leave out θ_z.

- The in-plane strains from the above displacement field are expressed as:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{bmatrix} - z \begin{bmatrix} -\theta_{y,x} \\ \theta_{x,y} \\ \theta_{x,x} - \theta_{y,y} \end{bmatrix}.$$

6. Classical Laminate Theory II

The Classical Theory of Plates

- By default, $\varepsilon_{zz} = 0$. The out-of-plane shears, on the other hand, can be written as

$$\gamma_{xz} = \theta_y + w_{,x}, \qquad \gamma_{yz} = -\theta_x + w_{,y}.$$

  - Requiring these to be zero amounts to the **Kirchhoff Assumptions** and allows relating the rotations to gradients of the flexural deformation:

$$\theta_x = w_{,y}, \quad \theta_y = -w_{,x}.$$

- Along with this, the strains become:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{bmatrix} - z \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$
$$\implies \boxed{\underset{\sim}{\varepsilon} = \underset{\sim}{u}' - z\underset{\sim}{w}''}.$$

- Assuming $\mathbb{C}$ to be the 3×3 constitutive (stiffness) matrix, the stresses can be expressed as:

$$\underset{\sim}{\sigma} = \mathbb{C}\underset{\sim}{u}' - z\mathbb{C}\underset{\sim}{w}''.$$

6. Classical Laminate Theory III

The Classical Theory of Plates

- The virtual work simplifies as:

$$\delta\mathcal{U} = \delta\underset{\sim}{\varepsilon}^T\underset{\sim}{\sigma} = \left(\delta\underset{\sim}{u}'^T - z\delta\underset{\sim}{w}''^T\right)\left(\mathbb{C}\underset{\sim}{u}' - z\mathbb{C}\underset{\sim}{w}''\right)$$

$$= \begin{bmatrix}\delta\underset{\sim}{u}'^T & \delta\underset{\sim}{w}''^T\end{bmatrix}\begin{bmatrix}\mathbb{C} & -z\mathbb{C}\\ -z\mathbb{C} & z^2\mathbb{C}\end{bmatrix}\begin{bmatrix}\underset{\sim}{u}'\\ \underset{\sim}{w}''\end{bmatrix}$$

$$\delta U = \int_{-\frac{t}{2}}^{\frac{t}{2}} \delta\underline{U}dz = \begin{bmatrix}\delta\underset{\sim}{u}'^T & \delta\underset{\sim}{w}''^T\end{bmatrix}\begin{bmatrix}\int\mathbb{C}dz & -\int z\mathbb{C}dz\\ -\int z\mathbb{C}dz & \int z^2\mathbb{C}dz\end{bmatrix}\begin{bmatrix}\underset{\sim}{u}'\\ \underset{\sim}{w}''\end{bmatrix}$$

$$\implies \delta U = \begin{bmatrix}\delta\underset{\sim}{u}'^T & \delta\underset{\sim}{w}''^T\end{bmatrix}\begin{bmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{B} & \mathbb{D}\end{bmatrix}\begin{bmatrix}\underset{\sim}{u}'\\ \underset{\sim}{w}''\end{bmatrix},$$

which is the standard form of the constitutive relationship for a thin plate.

6. Classical Laminate Theory IV

The Classical Theory of Plates

- Written in terms of just the displacement components and stresses this can be expressed as,

$$\delta U = \begin{bmatrix} \delta \underset{\sim}{u}'^T & \delta \underset{\sim}{w}''^T \end{bmatrix} \begin{bmatrix} \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} \underset{\sim}{\sigma} dz \\ \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} z \underset{\sim}{\sigma} dz \end{bmatrix},$$

and this implies the following conjugate pairs

$$\left(\begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{bmatrix}, \begin{bmatrix} N_{xx} = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} dz \\ N_{yy} = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} dz \\ N_{xy} = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} dz \end{bmatrix} \right), \quad \left(\begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}, \begin{bmatrix} -M_{yx} = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} -z\sigma_{xx} dz \\ M_{xy} = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} -z\sigma_{yy} dz \\ M_{\times} = -M_{yy} = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} -z\sigma_{xy} dz \end{bmatrix} \right).$$

6. Classical Laminate Theory V

The Classical Theory of Plates

- So in summary we have,

$$\begin{bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{bmatrix} = \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B} & \mathbb{D} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}' \\ \underset{\sim}{w}'' \end{bmatrix}.$$

- This $\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B} & \mathbb{D} \end{bmatrix}$ matrix is known as the **Laminate Stiffness Matrix**.
- For an Isotropic plate we have,

$$\mathbb{C} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix},$$

which leads to

$$\mathbb{A} = \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad \mathbb{D} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

6. Classical Laminate Theory: Stiffness of Laminated Composites

- Suppose we had different laminate plies along the thickness, such that the constitutive matrix is $\mathbb{C}_i$ for $z \in (z_i, z_{i+1})$ and $-\frac{t}{2} = z_1 < \cdots < z_N = \frac{t}{2}$.
- Then the $A - B - D$ matrices are written as the sums,

$$\mathbb{A} = \sum_i (z_{i+1} - z_i)\mathbb{C}_i, \quad \mathbb{B} = \sum_i \frac{z_{i+1}^2 - z_i^2}{2}\mathbb{C}_i, \quad \mathbb{D} = \sum_i \frac{z_{i+1}^3 - z_i^3}{3}\mathbb{C}_i.$$

- Unlike isotropic plates, composite laminates can have non-zero $\mathbb{B}$ matrix (moment-planar coupling), bending-twisting coupling, etc.
- Taking another look at the general integrals

$$\mathbb{A} = \int \mathbb{C}(z)dz, \quad \mathbb{B} = \int -z\mathbb{C}(z)dz, \quad \mathbb{D} = \int z^2\mathbb{C}(z)dz,$$

  it is clear that $\mathbb{B}$ is the only matrix that can become zero (since $\mathbb{C}(z)$ can never be completely odd) - this happens when $\mathbb{C}(z)$ is an **even function of** z.
  - In laminated composite parlance, we call this a "symmetric laminate".
  - Physically, this means that laminae are stacked in such a manner so that $\mathbb{C}(z) = \mathbb{C}(-z)$.
- There is no way for $\mathbb{A}$ or $\mathbb{D}$ to completely vanish, but the stress-shear coupling can be made to vanish.

6.1. Generally Orthotropic Laminae

Classical Laminate Theory

We already derived the constitutive relations for a generally orthotropic ply as a function of the fiber setting angle θ.

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & 0\\ & C_{22} & 0\\ & & C_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix}, \quad \begin{bmatrix}\sigma_{x}\\ \sigma_{y}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}C'_{11} & C'_{12} & C'_{13}\\ & C'_{22} & C'_{23}\\ & & S'_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{x}\\ \varepsilon_{y}\\ \gamma_{xy}\end{bmatrix}$$

$$C'_{11} = C_{11}c^4 + C_{22}s^4 + (2C_{33} + C_{12})2c^2s^2$$

$$C'_{22} = C_{11}s^4 + C_{22}c^4 + (2C_{33} + C_{12})2c^2s^2$$

$$C'_{33} = (C_{11} + C_{22} - 2C_{33} - 2C_{12})c^2s^2 + C_{33}(c^4 + s^4)$$

$$C'_{12} = (C_{11} + C_{22} - 4C_{33})c^2s^2 + C_{12}(c^4 + s^4)$$

$$C'_{13} = (C_{11} - 2C_{33} - C_{12})c^3s - (C_{22} - 2C_{33} - C_{12})cs^3$$

$$C'_{23} = (C_{11} - 2C_{33} - C_{12})cs^3 - (C_{22} - 2C_{33} - C_{12})c^3s.$$

$$c = \cos\theta, \quad s = \sin\theta$$

From the above, it is clear that:

- $S'_{11}(\theta), S'_{12}(\theta), S'_{22}(\theta), S'_{33}(\theta)$ are all even functions in θ ($f(-\theta) = f(\theta)$), and
- $S'_{13}(\theta), S'_{23}(\theta)$ are odd functions in θ ($f(-\theta) = -f(\theta)$)m

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$$C'_{11} = C_{11}c^4 + C_{22}s^4 + (2C_{33} + C_{12})2c^2s^2$$

C'_{3}

C'_{13}

C'_{23}

So we can say

$$\underline{\underline{S}}'(\theta) + \underline{\underline{S}}'(-\theta) = \begin{bmatrix}\times & \times & 0\\ & \times & 0\\ \text{sym} & & \times\end{bmatrix},$$

i.e., the specially orthotropic structure can be recovered from this.

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$$C'_{11} = C_{11}c^4 + C_{22}s^4 + (2C_{33} + C_{12})2c^2s^2$$

Balanced Laminates

- For each θ oriented laminate there's a $-\theta$ oriented laminate somewhere.
- Here, the $\mathbb{A}$ matrix will show the specially orthotropic structure but nothing can be said about the $\mathbb{B}$ and $\mathbb{D}$ matrices.

$$C'_{23} = (C_{11} - 2C_{33} - C_{12})cs^3 - (C_{22} - 2C_{33} - C_{12})c^3s.$$

$$c = \cos\theta, \quad s = \sin\theta$$

From the above, it is clear that:

- $S'_{11}(\theta), S'_{12}(\theta), S'_{22}(\theta), S'_{33}(\theta)$ are all even functions in θ $(f(-\theta) = f(\theta))$, and
- $S'_{13}(\theta), S'_{23}(\theta)$ are odd functions in θ $(f(-\theta) = -f(\theta))$m

6.1. Generally Orthotropic Laminae: The Quasi-Isotropic Laminate

Classical Laminate Theory

- An isotropic laminate will have as it's $\mathbb{C}$ matrix:

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ & C_{11} & 0 \\ & & \frac{C_{11}-C_{12}}{2} \end{bmatrix},$$

  and the same structure will be carried across to the A-B-D matrices.
- It is possible to achieve this structure in the $\mathbb{A}$ matrix through some trickery. We start by observing that the following geometric series adds up to zero:

$$\sum_{k=0}^{N-1} e^{ik\frac{2m\pi}{N}} = \sum_{k=0}^{N-1} \left(e^{i\frac{2m\pi}{N}}\right)^k = \frac{1-\left(e^{i\frac{2m\pi}{N}}\right)^N}{1-e^{i\frac{2m\pi}{N}}} = 0.$$

- Splitting the exponent into real and imaginary portions yields the identities:

$$\sum_{k=0}^{N-1} \cos\left(k\frac{2m\pi}{N}\right) = 0, \qquad \sum_{k=0}^{N-1} \sin\left(k\frac{2m\pi}{N}\right) = 0.$$

6.1. Generally Orthotropic Laminae: The Quasi-Isotropic Laminate

Classical Laminate Theory

- Using trigonometric identities, the laminate stiffness coefficients can be decomposed as follows:

Stiffness	**Constant Term**	$\cos 2\theta$ **Coefficient**	$\cos 4\theta$ **Coefficient**
C'_{11}	$\frac{4C_{33}+3C_{22}+2C_{12}+3C_{11}}{8}$	$\frac{C_{11}-C_{22}}{2}$	$\frac{C_{11}+C_{22}-2C_{12}-4C_{33}}{8}$
C'_{22}	$\frac{4C_{33}+3C_{22}+2C_{12}+3C_{11}}{8}$	$-\frac{C_{11}-C_{22}}{2}$	$\frac{C_{11}+C_{22}-2C_{12}-4C_{33}}{8}$
C'_{33}	$\frac{C_{11}+C_{22}-2C_{12}+4C_{33}}{8}$	0	$-\frac{C_{11}+C_{22}-2C_{12}-4C_{33}}{8}$
C'_{12}	$\frac{C_{11}+C_{22}+6C_{12}-4C_{33}}{8}$	0	$-\frac{C_{11}+C_{22}-2C_{12}-4C_{33}}{8}$
Stiffness	**Constant Term**	$\sin 2\theta$ **Coefficient**	$\sin 4\theta$ **Coefficient**
C'_{13}	0	$\frac{C_{11}-C_{22}}{4}$	$\frac{C_{11}+C_{22}-2C_{12}-4C_{33}}{8}$
C'_{23}	0	$\frac{C_{11}-C_{22}}{4}$	$-\frac{C_{11}+C_{22}-2C_{12}-4C_{33}}{8}$

- Having N plies of equal thickness with the k^{th} ply having angle $\theta_k = k\frac{\pi}{N}$ will result in $\mathbb{A}$ having just the “Constant Terms” from the above due to the trigonometric identity established above. It is also readily observed that $C'_{33} = \frac{C'_{11}-C'_{12}}{2}$.

6.1. Generally Orthotropic Laminae: The Quasi-Isotropic Laminate

Classical Laminate Theory

- Using trigonometric identities, the laminate stiffness coefficients can be decomposed as follows:

Stiffness			**efficient**
C'_{11}			$C_{12}-4C_{33}$
C'_{22}			$C_{12}-4C_{33}$
C'_{33}			$2C_{12}-4C_{33}$
C'_{12}			$2C_{12}-4C_{33}$
Stiffness			**efficient**
C'_{13}			$C_{12}-4C_{33}$
C'_{23}	0	$\frac{C_{11}-C_{22}}{4}$	$-\frac{C_{11}+C_{22}-2C_{12}-4C_{33}}{8}$

This shows that we can achieve a quasi-isotropic laminate if we chose fibre angles as $\theta_k = \theta_0 + k\frac{\pi}{N}$ with $k = 0, \dots, N-1$. θ_0 can be any starting angle. For example:

$N = 2$ $[0°/90°]$, which happens to be specially orthotropic also.

$N = 3$ $[0°/60°/120°]$, $[-60°/0°/60°]$, etc.

$N = 4$ $[0°/45°/90°/135°]$, etc.

Note that 120° is the same as −60°, and 135° is the same as −45° since these are fiber setting angles.

- Having N plies of equal thickness with the k^{th} ply having angle $\theta_k = k\frac{\pi}{N}$ will result in $\mathbb{A}$ having just the "Constant Terms" from the above due to the trigonometric identity established above. It is also readily observed that $C'_{33} = \frac{C'_{11}-C'_{12}}{2}$.

6.2. The Laminate Orientation Code

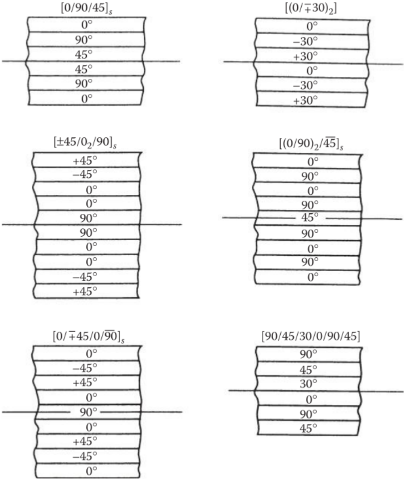
Classical Laminate Theory

- Ply angles separated by slashes, ordered from top to bottom
- Subscript “s” for symmetric laminates
- Numerical subscripts for repetitions
- Center ply with an overbar for odd laminates

(See sec. 7.1 in Gibson 2012)

Types

- Symmetric, Antisymmetric, Asymmetric, Balanced
- Angle-Ply ($\pm\theta$); Cross-Ply ($0^\circ, 90^\circ$); $\pi/4$ laminates



(Figure 7.1 from Gibson 2012)

6.2. The Laminate Orientation Code

Classical Laminate Theory

- Ply angles separate
  from top to bottom
- Subscript “s” for sy
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  laminates

(S

Typ

- Symmetric, Antisymm
  Balanced
- Angle-Ply (±θ); Cros
  laminates

Summary of Laminate Stiffnesses

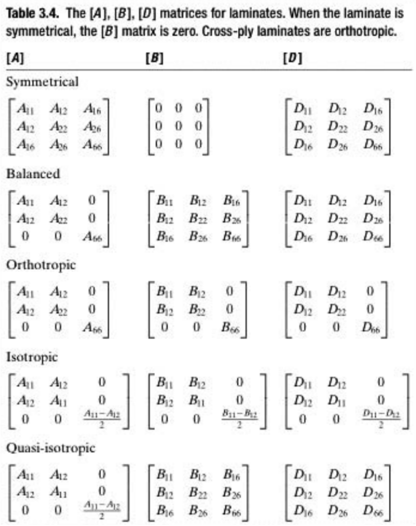
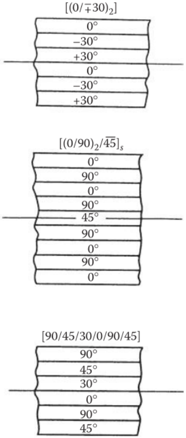


Table 3.4. The [*A*], [*B*], [*D*] matrices for laminates. When the laminate is symmetrical, the [*B*] matrix is zero. Cross-ply laminates are orthotropic.

	[*A*]	**[*B*]**	**[*D*]**
Symmetrical	$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$
Balanced	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$
Orthotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$
Isotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}-A_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & \frac{B_{11}-B_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & \frac{D_{11}-D_{12}}{2} \end{bmatrix}$
Quasi-isotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}-A_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$

(Table 3.4 from Kollár and Springer 2003)



Gibson 2012)

6.3. Laminated Beams

Classical Laminate Theory

- Consider a beam with a symmetric section on the $x-y$ plane. Invoking Kirchhoff kinematic assumptions we have: $\varepsilon_x = u' - yv''$.
- The stress distribution will depend on the section-coordinate. In general we will have: $\sigma_x = E_x(y)\varepsilon_x = E_x(y)\left(u' - yv''\right)$.
- We get the effective normal reaction N_x by integrating the stress over the section:

$$N_x = \int_{\mathcal{A}} \sigma_x = \left[\int_{\mathcal{A}} E_x(y)\right] u' + \left[\int_{\mathcal{A}} -yE_x(y)\right] v''.$$

- Similarly we get the bending moment M_z as the first moment of the stress,

$$M_z = \int_{\mathcal{A}} -y\sigma_x = \left[\int_{\mathcal{A}} -yE_x(y)\right] u' + \left[\int_{\mathcal{A}} y^2 E_x(y)\right] v''.$$

- In summary we have the beam-analog of the laminate stiffness matrix,

$$\begin{bmatrix} N_x \\ M_z \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} u' \\ v'' \end{bmatrix}.$$

Important note: We have assumed that no torsion/twist is present. See Kollár and Springer 2003 for the general form.

6.3. Laminated Beams

Classical Laminate Theory

- For a laminated composite with a rectangular section with width b, the integrals may be simplified as,

$$A = \int_{\mathcal{A}} E_x(y) = \sum_{i=1}^{N} E_{x,i} b(y_{i+1} - y_i), \quad B = \int_{\mathcal{A}} -yE_x(y) = -\sum_{i=1}^{N} E_{x,i} b \frac{y_{i+1}^2 - y_i^2}{2}$$

$$D = \int_{\mathcal{A}} y^2 E_x(y) = \sum_{i=1}^{N} E_{x,i} b \frac{y_{i+1}^3 - y_i^3}{3}.$$

- For plies of uniform thickness we can write

$$y_i = -\frac{h}{2} + (i-1)\frac{h}{N},$$

which leads to:

$$A = \frac{h}{N} \sum_{i=1}^{N} E_{x,i}, \; B = \frac{h^2}{2N^2} \sum_{i=1}^{N} E_{x,i}(2i - N - 1),$$

$$D = \frac{h^3}{12N^3} \sum_{i=1}^{N} E_{x,i}(12i^2 - 12Ni + 12N^2 + 3N^2 + 6N + 4)$$

6.4. Numerical Example

Classical Laminate Theory

Determine the ABD matrix for the following composite beams where the ply thickness is 1 mm and beam width is 10 mm:

- $[0/90]_s$, and
- $[0/90/0/90]$.

Assume the following properties for each lamina: $E_1 = 140\,\text{GPa}$, $E_2 = 10\,\text{GPa}$, $G_{12} = 7\,\text{GPa}$, $\nu_{12} = 0.3$, $\nu_{23} = 0.2$.

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