



# AS2070: Aerospace Structural Mechanics

## Module 2: Composite Material Mechanics (V5)

**Instructor: Nidish Narayanaa Balaji**

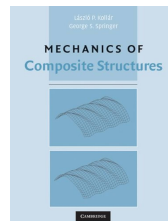
Dept. of Aerospace Engg., IIT Madras, Chennai

March 16, 2026

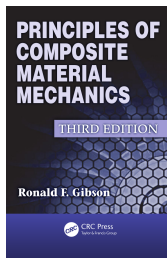
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(Also see Daniel and Ishai [2006](#))

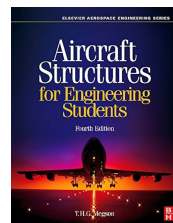
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Chapters 1-3, 11  
in Kollár and Springer  
([2003](#)).



Chapters 1-3  
in Gibson ([2012](#)).

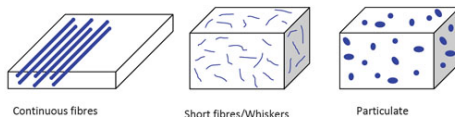


Chapter 25 in Megson  
([2013](#))

# 1.1. What are Composites?

## Introduction

- Structural material consisting of multiple non-soluble macro-constituents.
- Main motivation: material properties tailored to applications.
- Both stiffness and strength comes from the fibers/particles, and the matrix holds everything together.



*Types of composite materials (Figure from NPTEL Online-IIT KANPUR (2025))*

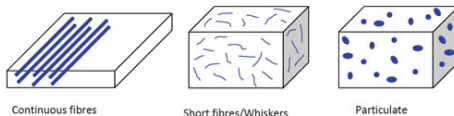
### Examples

- Reinforced concrete
- Wood (lignin matrix reinforced by cellulose fibers)
- Carbon-Fiber Reinforced Plastics (CFRP)

# 1.1. What are Composites?

## Introduction

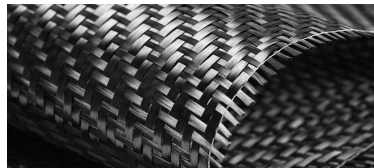
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## Examples

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- Carbon-Fiber Reinforced Plastics (CFRP)

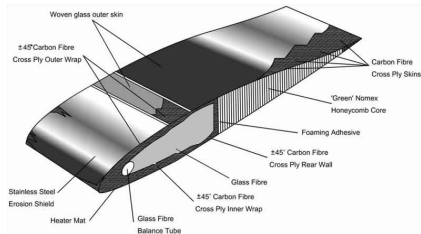


# 1.1. What are Composites?

## Introduction

- Structural material consisting of multiple non-soluble macro-constituents.

### CFRP Helicopter Blades



(Figures from Carbon Fiber Top Helicopter Blades 2025)

Exa

- Wood (lignin matrix reinforced by cellulose fibers)
- Carbon-Fiber Reinforced Plastics (CFRP)

- High fatigue resistance. But quite brittle.
- Main- and tail-planes, fuselages, etc. Helicopter blades.

AA.

# 1.1. What are Composites?

## Introduction

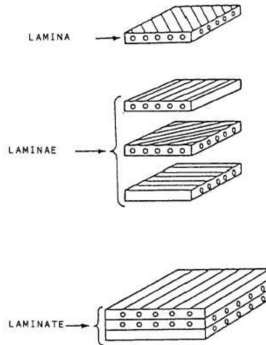
### Structural materials



### Examples

- Wood (lignin matrix, cellulose fibers)
- Carbon-Fiber Reinforced Polymer (CFRP)

## Laminated Composites



(Figure from Kalkan 2017)



quite brittle.

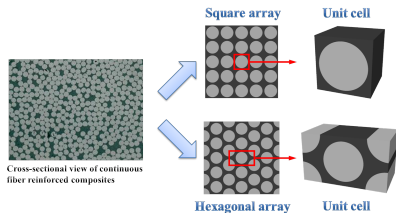
ges, etc. Helicopter

## 1.2. Modeling Composite Material

### Introduction

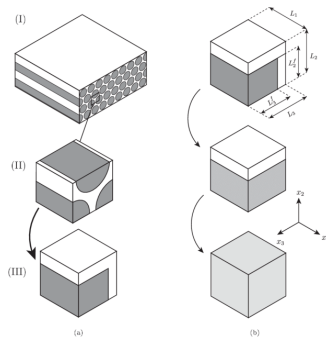
Two main approaches:

#### Micro-Mechanics



(Figure from "Micro-Mechanics of Failure" 2024)

#### Macro-Mechanics

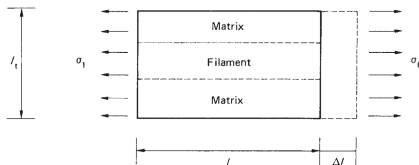


Homogenization of micro-structure (Figure from Skovsgaard and Heide-Jørgensen 2021)

# 1.3. Constitutive Modeling for Composites

## Introduction

### Axial Elongation



- Strain is fixed, but stress experienced by media differ.

$$\sigma_l = E_l \varepsilon_l$$

- Stress-strain relationship simplifies as,

$$\sigma_m = E_m \varepsilon_l, \quad \sigma_f = E_f \varepsilon_l$$

$$\sigma_l A = \sigma_m A_m + \sigma_f A_f$$

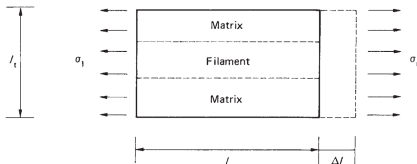
$$\Rightarrow E_l = \frac{A_f}{A} E_f + \frac{A_m}{A} E_m$$

(Figures from Megson 2013)

# 1.3. Constitutive Modeling for Composites

## Introduction

### Axial Elongation



- Strain is fixed, but stress experienced by media differ.

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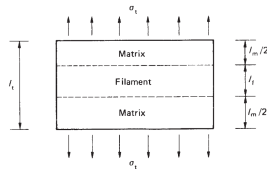
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$$\sigma_l A = \sigma_m A_m + \sigma_f A_f$$

$$\Rightarrow E_l = \frac{A_f}{A} E_f + \frac{A_m}{A} E_m$$

### Transverse Elongation



- Stress is fixed, strains differ:

$$\varepsilon_t l_t = \varepsilon_m l_m + \varepsilon_f l_f$$

$$\Rightarrow \frac{\sigma_t}{E_t} l_t = \frac{\sigma_t}{E_m} l_m + \frac{\sigma_t}{E_f} l_f$$

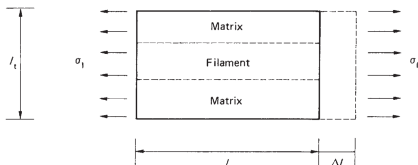
$$\Rightarrow \frac{1}{E_t} = \frac{1}{E_m} \frac{l_m}{l_t} + \frac{1}{E_f} \frac{l_f}{l_t}$$

(Figures from Megson 2013)

# 1.3. Constitutive Modeling for Composites

## Introduction: Poisson Effects

### Axial-Transverse Coupling



- Transverse displacement written as

$$\Delta_t = \nu_m \varepsilon_l l_m + \nu_f \varepsilon_l l_f := \nu_{lt} \varepsilon_l l_t$$

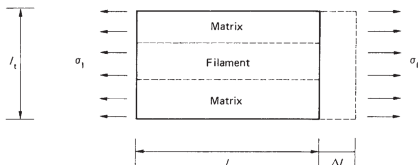
$$\Rightarrow \nu_{lt} = \frac{l_m}{l_t} \nu_m + \frac{l_f}{l_t} \nu_f$$

(Figures from Megson 2013)

# 1.3. Constitutive Modeling for Composites

## Introduction: Poisson Effects

### Axial-Transverse Coupling

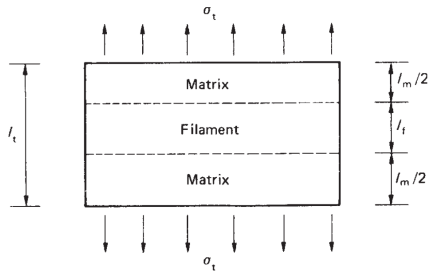


- Transverse displacement written as

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$$\Rightarrow \nu_{lt} = \frac{l_m}{l_t} \varepsilon_l + \frac{l_f}{l_t} \varepsilon_f$$

### Transverse-Axial Coupling



- Axial displacement written as

$$\nu_m \frac{\sigma_t}{E_m} = \nu_f \frac{\sigma_t}{E_f} := \nu_{tl} \frac{\sigma_t}{E_t}$$

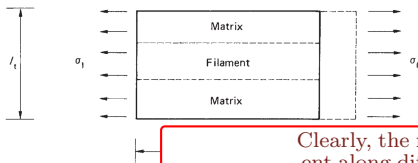
$$\Rightarrow \nu_{tl} = \frac{E_t}{E_l} \nu_{lt}$$

(Figures from Megson 2013)

# 1.3. Constitutive Modeling for Composites

## Introduction: Poisson Effects

### Axial-Transverse Coupling



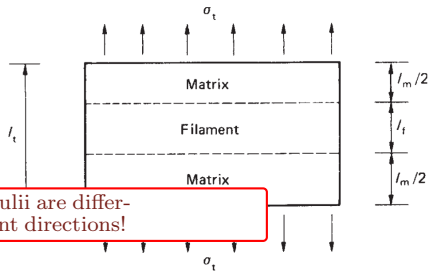
Clearly, the moduli are different along different directions!

- Transverse displacement written as

$$\Delta_t = \nu_m \varepsilon_l l_m + \nu_f \varepsilon_l l_f := \nu_{lt} \varepsilon_l l_t$$

$$\Rightarrow \nu_{lt} = \frac{l_m}{l_t} \varepsilon_l + \frac{l_f}{l_t} \varepsilon_f$$

### Transverse-Axial Coupling



- Axial displacement written as

$$\nu_m \frac{\sigma_t}{E_m} = \nu_f \frac{\sigma_t}{E_f} := \nu_{tl} \frac{\sigma_t}{E_t}$$

$$\Rightarrow \nu_{tl} = \frac{E_t}{E_l} \nu_{lt}$$

(Figures from Megson 2013)

# 1.3. Constitutive Modeling for Composites

## Introduction: Anisotropy

### General Anisotropy (aka “Triclinic”)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

# 1.3. Constitutive Modeling for Composites

## Introduction: Anisotropy

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### Monoclinic: Single Plane of Symmetry

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

# 1.3. Constitutive Modeling for Composites

Introduction: Anisotropy

## Orthotropic: Three Orthogonal Planes of Symmetry

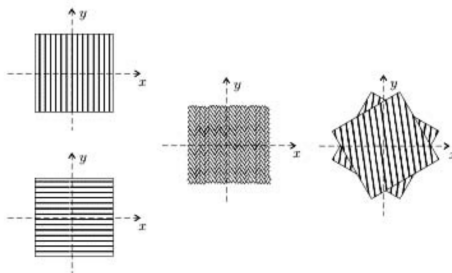
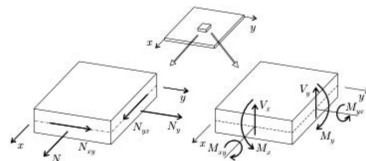
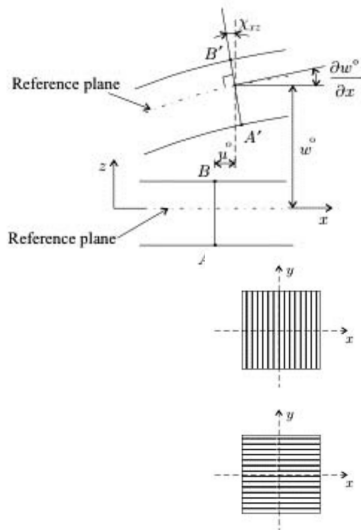
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

## Transversely Isotropic

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

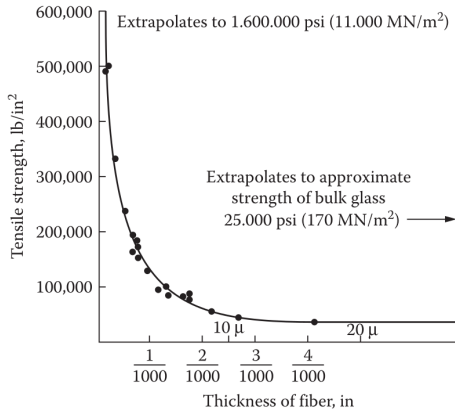
# 1.4. Classical Laminate Theory

## Introduction



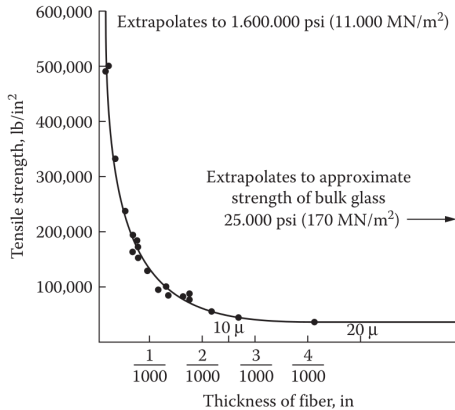
Figures from Kollár and Springer 2003

## 2. Composite Materials

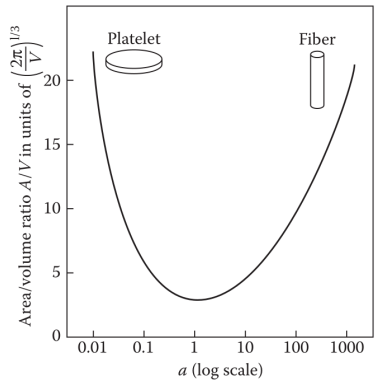


Griffith's experiments with glass fibres (1920) (Figure from Gibson 2012)

## 2. Composite Materials



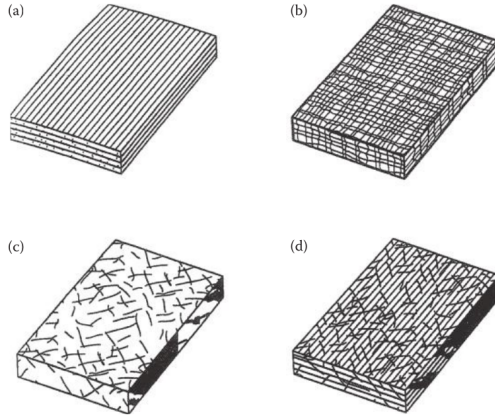
Griffith's experiments with glass fibres (1920) (Figure from Gibson 2012)



(Figure from Gibson 2012)

## 2.1. Types of Composite Materials

### Composite Materials



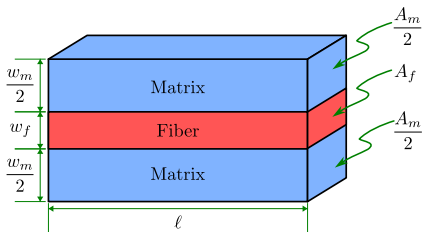
**FIGURE 1.4**

Types of fiber-reinforced composites. (a) Continuous fiber composite, (b) woven composite, (c) chopped fiber composite, and (d) hybrid composite.

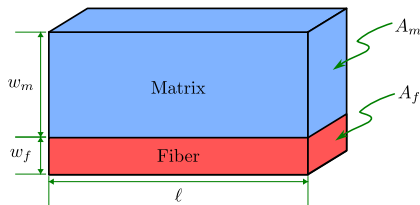
*(Figure from Gibson 2012)*

### 3. Micro-Mechanics Descriptions

- We shall now seek a “homogenized” understanding of the mechanics of a composite material.
- By homogenization, we shall abstract away all the spatial information to such an extent that we shall claim that the following two abstractions of a composite material behave identically.



(a) Abstraction 1 of a composite material



(b) Abstraction 2 of a composite material

- We shall analyze the mechanics of abstraction 2 to derive an “equivalent” elastic description of the material as a whole. (We will restrict ourselves to the 2D plane for this)

## 3.1. The Rule of Mixtures

### Micro-Mechanics Descriptions

The rule of mixtures is a very simple framework for developing expressions for homogenized mechanical properties.

### Basic Definitions

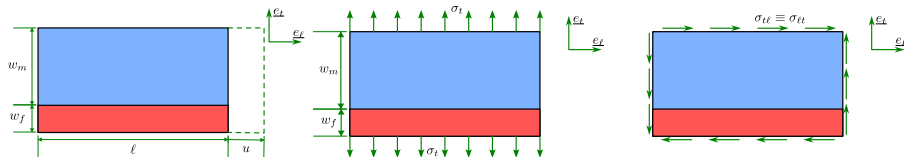
Subscripts  $(\cdot)_f$ ,  $(\cdot)_m$ ,  $(\cdot)_v$ , and  $(\cdot)_c$  denote quantities corresponding to the fiber, matrix, void, and composite (as a whole).

**Volume Fraction**  $v_f = \frac{V_f}{V_c}$ ,  $v_m = \frac{V_m}{V_c}$ ,  $v_v = \frac{V_v}{V_c}$  such that  $v_f + v_m + v_v = 1$ .

Note that composite density  $\rho_c = \rho_f v_f + \rho_m v_m$ .

**Weight Fraction**  $w_f = \frac{\rho_f}{\rho_c} v_f$

- We shall consider the behavior under the following 3 fundamental cases.



- (a) The abstract material under longitudinal extension (iso-strain)    (b) The abstract material under transverse extension (iso-stress)    (c) The abstract material under in-plane shear (iso-stress)

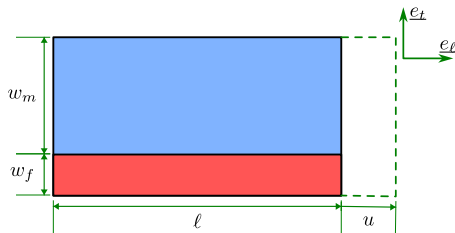
# 3.1. The Rule of Mixtures: Case 1: Longitudinal Iso-Strain Extension

## Micro-Mechanics Descriptions

- Under longitudinal extension, the matrix as well as the fiber are under the same strain, i.e.,  $\varepsilon_\ell = \frac{u}{\ell}$ . The stresses, respectively, are

$$\sigma_m = E_m \varepsilon_\ell = E_m \frac{u}{\ell}; \quad \sigma_f = E_f \varepsilon_\ell = E_f \frac{u}{\ell}.$$

- Subscripts  $m$  and  $f$  denote matrix and fiber properties.



- Converting the above stresses to forces (by multiplying with  $A_m$  and  $A_f$  respectively) and summing them up leads to the overall reaction force:

$$F_{tot} = (A_m E_m + A_f E_f) \frac{u}{\ell} := A_{tot} E_\ell \frac{u}{\ell}.$$

- Here we introduce  $E_\ell$  as the effective longitudinal modulus, so that we have:

$$E_\ell = \frac{A_m}{A_{tot}} E_m + \frac{A_f}{A_{tot}} E_f \implies \boxed{E_\ell = v_m E_m + v_f E_f}.$$

- This is closely related to the springs-in-parallel formula that you may already be familiar with. The jargon name for this is *Voigt Model*.

# 3.1. The Rule of Mixtures: Case 1: Longitudinal Iso-Strain Extension

## Micro-Mechanics Descriptions

- Under longitudinal extension, the matrix as well as the fiber are under the same

### Poisson's Effect

- The transverse displacements due to Poisson's effect can be written as

$$u_{t,m} = -\nu_m \frac{u}{\ell} w_m, \quad u_{t,f} = -\nu_f \frac{u}{\ell} w_f.$$

- The overall transverse displacement is:

$$u_t = -(\nu_m w_m + \nu_f w_f) \frac{u}{\ell} := -\nu_{\ell t} \frac{u}{\ell} w_{tot},$$

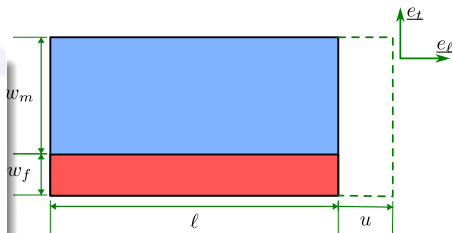
where we have substituted the homogenized formula in the end.

- So the effective Poisson's ratio is:

$$\nu_{\ell t} = v_m \nu_m + v_f \nu_f$$

- Here we have used  $\frac{w_m}{w_{tot}} = v_m$  and  $\frac{w_f}{w_{tot}} = v_f$ .

- This is closely related to the springs-in-parallel formula that you may already be familiar with. The jargon name for this is *Voigt Model*.



multiplying with  $A_m$  and  $A_f$  respectively)  
reaction force:

### Notation for Poisson's ratio

**subscripts:**  $\nu_{\langle \text{cause} \rangle \langle \text{effect} \rangle}$

For example:  $\nu_{\ell t}$  encodes the transverse strain caused by axial strains such that:

$$\epsilon_t = -\nu_{\ell t} \epsilon_\ell$$



udin

$f E_f$

multi

$w_f$

$w_m$

$\epsilon_t$

$\epsilon_\ell$

$u$

$\ell$

$w_f$

$w_m$

$u_{t,m}$

$u_{t,f}$

$u_t$

$\epsilon_\ell$

$\epsilon_t$

$\nu_{\ell t}$

$\nu_m$

$\nu_f$

$v_m$

$v_f$

$\frac{w_m}{w_{tot}}$

$\frac{w_f}{w_{tot}}$

$\frac{u}{\ell}$

$\frac{u_t}{\ell}$

$\frac{u_{t,m}}{\ell}$

$\frac{u_{t,f}}{\ell}$

$\frac{u_t}{\ell}$

$\frac{u}{\ell}$

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### 3.1. The Rule of Mixtures: Case 2: Transverse Iso-Stress Extension

#### Micro-Mechanics Descriptions

- Under transverse loading, the iso-stress condition is very naturally considered.
- The elongations in the transverse directions are written as:

$$u_{t,m} = \frac{\sigma_t}{E_m} w_m, \quad u_{t,f} = \frac{\sigma_t}{E_f} w_f.$$

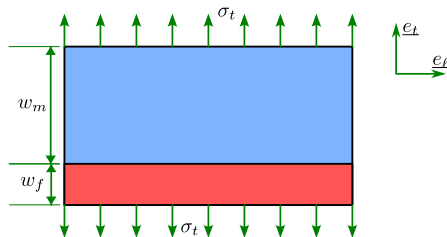
- The overall elongation can be expressed as

$$u_t = \left( \frac{w_m}{E_m} + \frac{w_f}{E_f} \right) \sigma_t := \frac{w_{tot}}{E_t} \sigma_t.$$

- By equating the coefficients, we get the effective transverse modulus as

$$E_t = \left( \frac{v_m}{E_m} + \frac{v_f}{E_f} \right)^{-1},$$

which closely resembles the springs-in-series formula. The jargon name for this is *Reuß model*.



# 3.1. The Rule of Mixtures: Case 2: Transverse Iso-Stress Extension

## Micro-Mechanics Descriptions

- Under transverse loading, the iso-stress

### Poisson's Effect

- The **longitudinal displacements** due to Poisson's effect can be written as

$$u_{\ell,m} = -\frac{\nu_m}{E_m} \sigma_t \ell, \quad u_{\ell,f} = -\frac{\nu_f}{E_f} \sigma_t \ell.$$

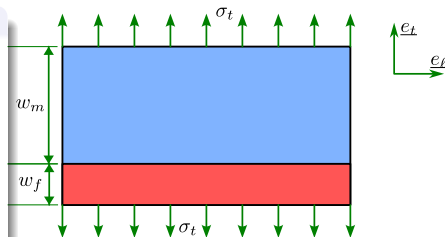
- Since the fiber and matrix get displaced differently, we write down the overall averaged (effective homogenized) longitudinal displacement as:

$$u_{\ell} = v_m u_{\ell,m} + v_f u_{\ell,f} \\ := -\frac{\nu_{t\ell}}{E_t} \sigma_t = -\left( \nu_m \frac{v_m}{E_m} + \nu_f \frac{v_f}{E_f} \right) \sigma_t \ell$$

- So the effective Poisson's ratio is:

$$\nu_{t\ell} = \frac{E_f v_m}{E_f v_m + E_m v_f} \nu_m + \frac{E_m v_f}{E_f v_m + E_m v_f} \nu_f$$

model.



### Notation for Poisson's ratio

**subscripts:**  $\nu_{\langle \text{cause} \rangle \langle \text{effect} \rangle}$

For example:  $\nu_{\ell t}$  encodes the transverse strain caused by axial strains such that:

$$\varepsilon_t = -\nu_{\ell t} \varepsilon_{\ell}$$

Reuß

### 3.1. The Rule of Mixtures: Case 3: In-plane Iso-Stress Shearing

#### Micro-Mechanics Descriptions

- Under shear loading, the iso-stress condition is very naturally considered.
- The shear strains (angles) are:

$$\gamma_{tl,m} = \frac{\sigma_{tl}}{G_m}, \quad \gamma_{tl,f} = \frac{\sigma_{tl}}{G_f}.$$

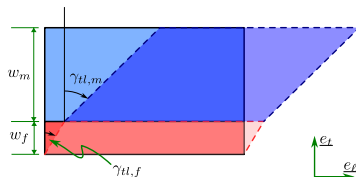
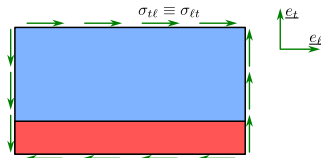
- The overall shear strain (deflection angle) is:

$$\begin{aligned} \gamma_{tl} &= \gamma_{tl,m} \frac{w_m}{w_{tot}} + \gamma_{tl,f} \frac{w_f}{w_{tot}} \\ &:= \frac{\sigma_{tl}}{G_{tl}} = \left( \frac{v_m}{G_m} + \frac{v_f}{G_f} \right) \sigma_{tl}. \end{aligned}$$

- From this, we write down the overall shear modulus as

$$G_{tl} = \left( \frac{v_m}{G_m} + \frac{v_f}{G_f} \right)^{-1}.$$

- This is also a Reuß model (springs-in-series model).

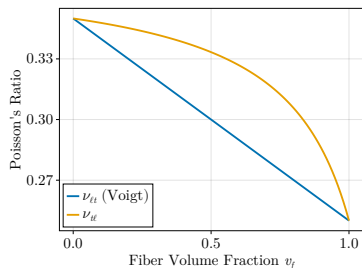
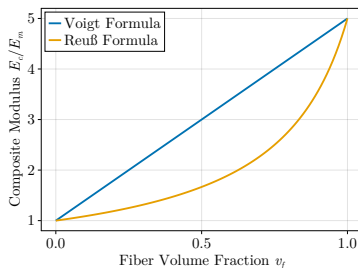


# 3.1. The Rule of Mixtures: Case 3: In-plane Iso-Stress Shearing

## Micro-Mechanics Descriptions

- Under shear loading, the iso-stress condition is very naturally considered.

### The rule of mixtures, visualized.



modulus as

$$G_{tl} = \left( \frac{v_m}{G_m} + \frac{v_f}{G_f} \right)^{-1}.$$



- This is also a Reuß model (springs-in-series model).

# 3.1. The Rule of Mixtures: Case 3: In-plane Iso-Stress Shearing

## Micro-Mechanics Descriptions

RoM is not always satisfactory!

- Under very narrow conditions, the Rule of Mixtures (ROM) can be used to estimate the transverse Young's modulus ( $E_t$ ) and shear modulus ( $G_{tl}$ ) of a composite. **NOTE: The Reuβ formula has been observed to consistently under-predict the moduli that are experimentally measured. This affects the  $E_t$ ,  $G_{tl}$ , and the  $\nu_{tl}$  estimates shown above.**

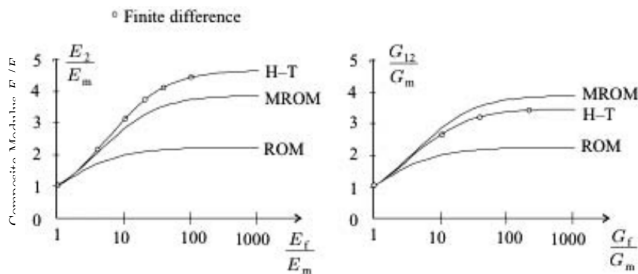


Figure 11.8: The transverse Young and shear moduli calculated by the rule of mixtures (ROM), the modified rule of mixtures (MROM), the Halpin-Tsai (H-T) equations, and the finite difference solutions (circles) of Adams and Doner ( $\nu_f = 0.55$ ).

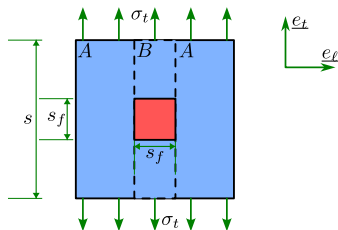
(Figure 11.8 from Kollár and Springer 2003)

- This is also a Reuβ model (springs-in-series model).

### 3.1. The Rule of Mixtures

#### Micro-Mechanics Descriptions

- The mismatch is related to the fact that our idealized picture was a poor representation of reality to begin with. Let us see how far we get with added geometrical detailing in our abstraction.



- Since the Voigt model is so successful, we look at the cross-section and divide it into regions A and B, which are acting as springs in parallel.
- For region B, the transverse modulus is written using the Reuß formula as:

$$E_{Bt} = \left( \frac{s_f/s}{E_f} + \frac{1 - s_f/s}{E_m} \right)^{-1}$$

$$= \left( \frac{\sqrt{v_f}}{E_f} + \frac{1 - \sqrt{v_f}}{E_m} \right)^{-1}.$$

- Now we apply the Voigt formula to obtain the overall transverse modulus:

$$E_t = \left( 1 - \frac{s_f}{s} \right) E_m + \frac{s_f}{s} E_{Bt}$$

$$= E_m \left[ \left( 1 - \sqrt{v_f} \right) + \frac{\sqrt{v_f}}{1 - \sqrt{v_f} \left( 1 - \frac{E_m}{E_f} \right)} \right]$$

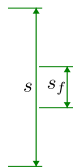
*An abstraction with slightly more geometrical details: a square fiber embedded in a square region of the matrix*

$$V_f = s_f^2, \quad V_{tot} = s^2, \quad v_f = \frac{V_f}{V_{tot}} = \frac{s_f^2}{s^2}$$

## 3.1. The Rule of Mixtures

### Micro-Mechanics Descriptions

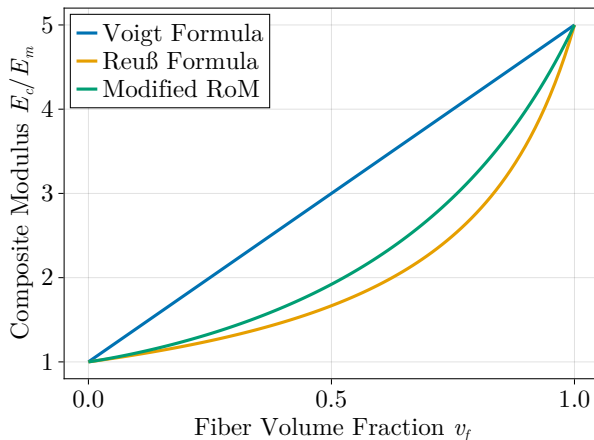
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An abstraction of  
fiber embedded in

$$V_f = s_f^2$$

This "Modified" Rule of Mixtures (mRoM) presents just one modification to RoM, but it can be seen to pro-



The same formula may also be used for the shear modulus.

representation  
tailing in

successful, we  
divide it into  
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modulus is  
a as:

$$\left( \frac{E_m}{E_f} \right)^{-1}$$

$$\left( \frac{E_m}{E_f} \right)^{-1}$$

la to obtain  
:

$$\left[ \frac{\sqrt{v_f}}{\left( 1 - \frac{E_m}{E_f} \right)} \right]$$

## 3.1. The Rule of Mixtures

### Micro-Mechanics Descriptions

- Even the mRoM is found insufficient in a lot of cases, so a heuristic formula known as the **Halpin-Tsai Equation** is quite popular.

(Recommended reading: Sec. 3.2.3 in Daniel and Ishai [2006](#))

### The Halpin-Tsai Equation

$$E_t = E_m \frac{1 + \xi \eta v_f}{1 - \eta v_f}, \quad \eta = \frac{E_f - E_m}{E_f + \xi E_m}$$

$$= E_m \frac{E_f + \xi E_m + \xi v_f (E_f - E_m)}{E_f + \xi E_m - v_f (E_f - E_m)}$$

**Note:**  $\xi = 2$  for circular section fibers.  $\xi = \frac{2a}{b}$  for rectangular fibers ( $b$  being loaded side).

- This is parameterized by  $\xi$ , which allows us to recover both the Voigt and Reuß models.

#### Case 1: $\xi \rightarrow 0$

$$E_t = \left( \frac{v_f}{E_f} + \frac{1 - v_f}{E_m} \right)^{-1}$$

Series, *Reuss* model.

#### Case 2: $\xi \rightarrow \infty$

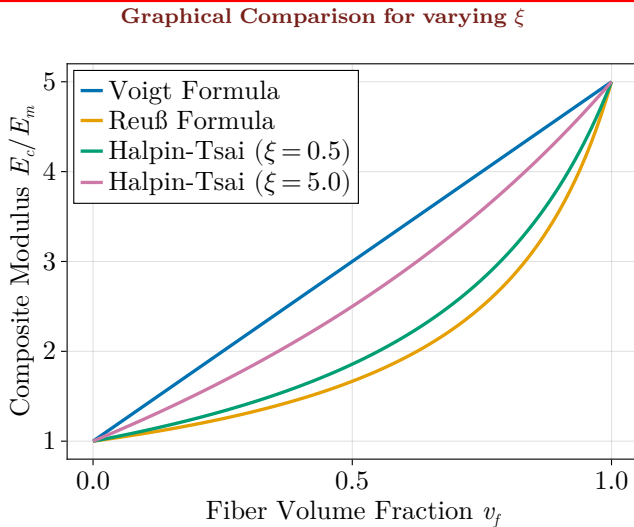
$$E_2 = E_f v_f + E_m (1 - v_f)$$

Parallel, *Voigt* model.

### 3.1. The Rule of Mixtures

#### Micro-Mechanics Descriptions

- Even the



Note:  $\xi$

- This

own as

(Hashai 2006)

models.

## 3.2. Numerical Example

### Micro-Mechanics Descriptions

(from Kollár and Springer 2003)

Consider a Graphite/Epoxy unidirectional ply. Matrix properties are given with subscript  $m$  in the table below. Nominal properties with fiber volume fraction  $v_f = 60\%$  are also given. Assume that the fibers show anisotropy ( $E_{f1} \neq E_{f2}$ ). Use the modified Rule of Mixtures.

|       | $E_1$ | $E_2$ | $G_{12}$ | $\nu_{12}$ | $E_m$ | $G_m$ | $\nu_m$ |
|-------|-------|-------|----------|------------|-------|-------|---------|
| Value | 148   | 9.65  | 4.55     | 0.3        | 4.1   | 1.5   | 0.35    |

*All moduli in GPa.*

Estimate the following:

- Fiber modulus properties
- Composite material moduli for volume fraction  $v_f = 0.55$ .

*Note:* It is common to denote the fiber-longitudinal direction as 1 and the transverse direction as 2.

## 3.2. Numerical Example

### Micro-Mechanics Descriptions

(from Kollár and Springer 2003)

Consider a Graphite/Epoxy unidirectional ply. Matrix properties are given with subscript  $m$  in the table below. Nominal properties with fiber volume fraction  $v_f = 60\%$  are also given. Assume that the fibers show anisotropy ( $E_{f1} \neq E_{f2}$ ). Use the modified Rule of Mixtures.

|       | $E_1$ | $E_2$ | $G_{12}$ | $\nu_{12}$ | $E_m$ | $G_m$ | $\nu_m$ |
|-------|-------|-------|----------|------------|-------|-------|---------|
| Value | 148   | 9.65  | 4.55     | 0.3        | 4.1   | 1.5   | 0.35    |

*All moduli in GPa.*

Estimate the following:

- Fiber modulus properties
- Composite

*Note:* It is c

### Answers:

- $E_{f1} \approx 240$  GPa,  $\nu_{f12} = 0.27$ ,  $E_{f2} \approx 23$  GPa,  $G \approx 23$  GPa.
- For  $v_f = 0.55$ , we have  $E_1 \approx 130$  GPa,  $\nu_{12} = 0.31$ ,  $E_2 \approx 8.8$  GPa,  $G_{12} \approx 4$  GPa.

ection as 2.

## 4.1. Macro-Mechanics Descriptions

### Material Symmetry and Anisotropy

#### Material Symmetry

The study of material symmetry is concerned with finding answers to the question:  
If the strain field on a deformable object is changed, how does the stress field change?

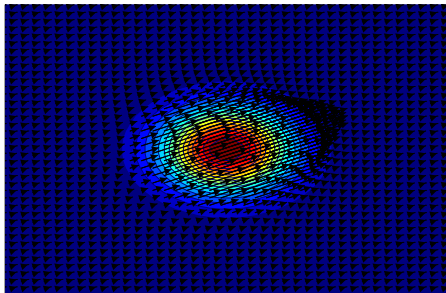
## 4.1. Macro-Mechanics Descriptions

### Material Symmetry and Anisotropy

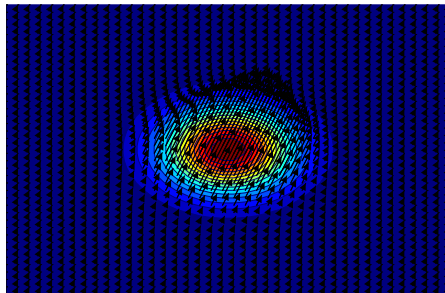
#### Material Symmetry

The study of material symmetry is concerned with finding answers to the question:  
If the strain field on a deformable object is changed, how does the stress field change?

Consider the following Deformation Fields



*Deformation Case 1*



*Deformation Case 2 (Case 1 Rotated)*

## 4.1. Macro-Mechanics Descriptions

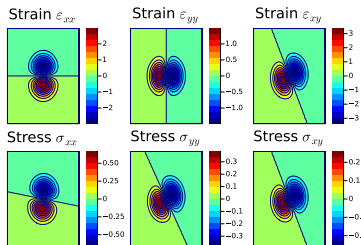
### Material Symmetry and Anisotropy

#### Material Symmetry

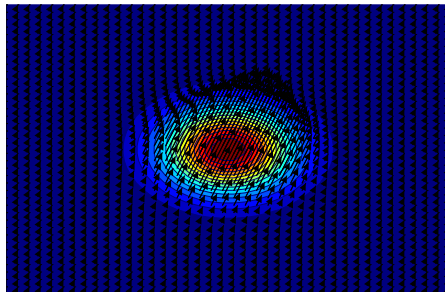
The study of material symmetry is concerned with finding answers to the question:  
If the strain field on a deformable object is changed, how does the stress field change?

Consider the following Deformation Fields

#### Stress and Strain Field



*Isotropic Stress-Strain Relationship*



*Deformation Case 2 (Case 1 Rotated)*

## 4.1. Macro-Mechanics Descriptions

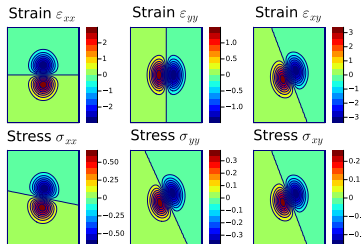
### Material Symmetry and Anisotropy

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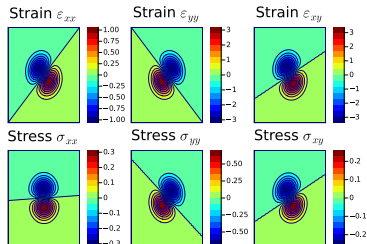
Consider the following Deformation Fields

#### Stress and Strain Field



*Isotropic Stress-Strain Relationship*

#### Stress and Strain Field



*Isotropic Stress-Strain Relationship*

## 4.1. Macro-Mechanics Descriptions

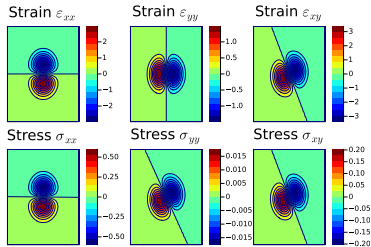
### Material Symmetry and Anisotropy

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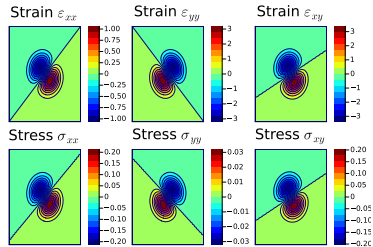
Consider the following Deformation Fields

#### Stress and Strain Field



*Anisotropic Case*

#### Stress and Strain Field



*Anisotropic Case*

## 4.1. Macro-Mechanics Descriptions

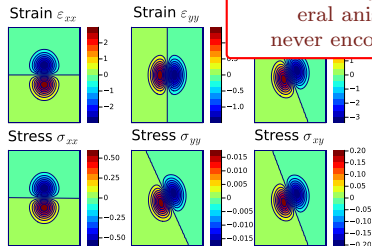
### Material Symmetry and Anisotropy

#### Material Symmetry

The study of material symmetry is concerned with finding answers to the question:  
If the strain field on a deformable object is changed, how does the stress field change?

Consider the following Deformation Fields

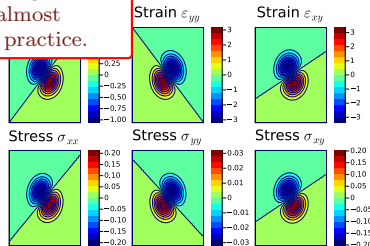
#### Stress and Strain Field



*Anisotropic Case*

Most materials exhibit some sort of symmetry and general anisotropy is almost never encountered in practice.

#### Stress and Strain Field



*Anisotropic Case*

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

How do stresses and strains transform under coordinate change?

- Suppose  $\underline{x} \in \mathbb{R}^3$  are the coordinates of a point in 3D space.
- Let  $\underline{x}' \in \mathbb{R}^3$  be the coordinates under transformation.
- We will write:  $\underline{x}' = \underline{Q} \underline{x}$ , with  $\underline{Q}^{-1} = \underline{Q}^T$ .

#### Strains

- $\underline{\underline{\varepsilon}} = \frac{1}{2} (\nabla_{\underline{x}} \underline{u} + \nabla_{\underline{x}} \underline{u}^T)$
- $\nabla_{\underline{x}'} \underline{u}' = \underline{Q} \nabla_{\underline{x}} \underline{u} \underline{Q}^{-1} \implies \underline{\underline{\varepsilon}}' = \underline{Q} \underline{\underline{\varepsilon}} \underline{Q}^T$ .

#### Stresses

- Cauchy Stress Definition:  $\underline{t} = \underline{\underline{\sigma}} \underline{n}$
- $\underline{Q} \underline{t} = \underline{t}' = \underline{\underline{\sigma}}' \underline{n}' = \underline{\underline{\sigma}}' \underline{Q} \underline{n} = \underline{Q} \underline{\underline{\sigma}} \underline{n}$   
 $\implies \underline{\underline{\sigma}}' = \underline{Q} \underline{\underline{\sigma}} \underline{Q}^T$

#### Reflections

Note that reflections may be expressed as a coordinate change with  $\underline{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$  (reflection about the  $xy$  plane).

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

- Under reflection about the  $xy$  plane, the strain transforms as,

$$\begin{aligned} \begin{bmatrix} \varepsilon'_x & \frac{\gamma'_{xy}}{2} & \frac{\gamma'_{xz}}{2} \\ \text{sym} & \varepsilon'_y & \frac{\gamma'_{yz}}{2} \\ & & \varepsilon'_z \end{bmatrix} &= \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \text{sym} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ & & \varepsilon_z \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & -\frac{\gamma_{xz}}{2} \\ \text{sym} & \varepsilon_y & -\frac{\gamma_{yz}}{2} \\ & & \varepsilon_z \end{bmatrix} \end{aligned}$$

- So in Voigt notation we have,

$$\begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_z \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \quad \begin{bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{xz} \\ \tau'_{yz} \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

Similarly for Stress

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

- Under reflection about the  $xy$  plane, the strain transforms as,

$$\begin{bmatrix} \epsilon' & \gamma'_{xy} & \gamma'_{xz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon & \gamma_{xy} & \gamma_{xz} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

If a material were symmetric about the  $xy$  plane, then reflecting the strain field about the  $xy$  plane will result in a stress field that is reflected about the same  $xy$  plane.

#### Note

- Strain field reflection is a kinematic operation/configuration change.
- Change in the Stress field is the effect that the above kinematic change results in.
- If the material happens to be symmetric about the reflection plane, then this change will be a reflection.

$$\begin{bmatrix} \epsilon'_x \\ \epsilon'_y \\ \epsilon'_z \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \gamma_{yz} \end{bmatrix} \begin{bmatrix} \tau'_{yz} \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \tau_{yz} \end{bmatrix}$$

Similarly for Stress

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

- We have said the following :

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

(The  $C_{22}$  position in the original image is circled and labeled 'sym')

Recall that this symmetry follows from strain energy existence

$$\begin{bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{xz} \\ \tau'_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_z \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{bmatrix}$$

(The  $C_{22}$  position in the original image is circled and labeled 'sym')

(The  $C$  matrix has to be the same in both the original and the reflected coordinate systems, if the material shows reflection symmetry)

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

- We have said the following :

This leads to

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ & & C_{33} & C_{34} & -C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & -C_{46} \\ \text{sym} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ \text{sym} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

Recall that this symmetry follows from strain energy existence

$$\begin{bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{yz} \\ \tau'_{zx} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ \text{sym} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_z \\ \gamma'_{xy} \\ \gamma'_{yz} \\ \gamma'_{zx} \end{bmatrix}$$

(The C matrix has to be the same in both the original and the reflected coordinate systems, if the material shows reflection symmetry)

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

- We have said the following :

This leads to

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ & & C_{33} & C_{34} & -C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & -C_{46} \\ \text{sym} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ \text{sym} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

Recall that this symmetry follows from strain energy existence

$$\begin{bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{yz} \\ \tau'_{zx} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_z \\ \gamma'_{xy} \\ \gamma'_{yz} \\ \gamma'_{zx} \end{bmatrix}$$

(The  $\underline{C}$  matrix has to be the same in both the original and the reflected coordinate systems, if the material shows reflection symmetry)

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

- We have said the following :

This leads to

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ & & C_{33} & C_{34} & -C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & -C_{46} \\ \text{sym} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ \text{sym} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

Recall that this symmetry follows from strain energy existence

Finally we see that material symmetry about the  $xz$  plane implies the following simplification to the constitutive relationship.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ \text{sym} & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

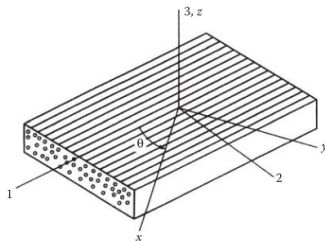
This is known as a **Monoclinic Material** (13 constants). This is also quite rare to encounter in practice.

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

Suppose all the three fundamental planes are planes of symmetry, we have an **Orthotropic Material** (9 constants).

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & \text{sym} & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix}$$



(Figure 2.5 from Gibson 2012)

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

Suppose all the three fundamental planes are planes of symmetry, we have an **Orthotropic Material** (9 constants).

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & \text{sym} & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix}$$

3, z

Notice that  $(\sigma_{(1,2,3)}, \varepsilon_{(1,2,3)})$  and  $(\tau_{(12,13,23)}, \gamma_{(12,13,23)})$  are naturally decoupled as a consequence of symmetry in this coordinate system.

Also note,

- Specially orthotropic
- Generally orthotropic

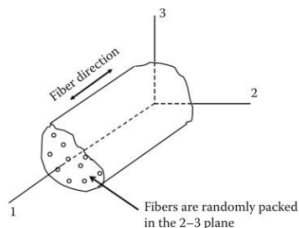
x

(Figure 2.5 from Gibson 2012)

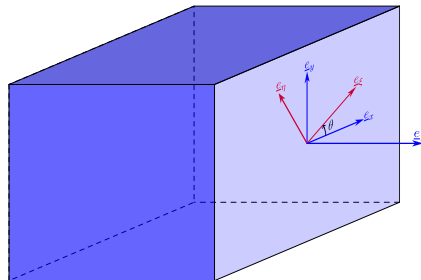
## 4.1. Material Symmetry and Anisotropy: Transverse Isotropy

### Macro-Mechanics Descriptions

- In continuous fiber reinforced composites, it is often the case that the fibers are randomly distributed on a plane. This leads to planar isotropy in the plane perpendicular to the fiber stacking direction.
- How do the stresses and strains transform on the plane?



(Figure 2.6 from Gibson 2012)



$$\begin{aligned}
 (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}) &\rightarrow (\sigma_\xi, \sigma_\eta, \sigma_z, \tau_{\xi\eta}, \tau_{\xi z}, \tau_{\eta z}) \\
 (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) &\rightarrow (\varepsilon_\xi, \varepsilon_\eta, \varepsilon_z, \gamma_{\xi\eta}, \gamma_{\xi z}, \gamma_{\eta z})
 \end{aligned}$$

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

• Here,  $\underline{Q} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $\underline{\sigma}' = \underline{Q} \underline{\sigma} \underline{Q}^T$ . So we get

$$\begin{bmatrix} \sigma_{\xi\xi} & \sigma_{\xi\eta} & \sigma_{\xi z} \\ \sigma_{\xi\eta} & \sigma_{\eta\eta} & \sigma_{\eta z} \\ \sigma_{\xi z} & \sigma_{\eta z} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta & \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - \sigma_{xy} \sin 2\theta & \sigma_{xz} \cos \theta + \sigma_{yz} \sin \theta \\ \sigma_{xy} \cos 2\theta - \frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta & \sigma_{yz} \cos \theta - \sigma_{xz} \sin \theta & \sigma_{zz} \\ \sigma_{xz} \cos \theta + \sigma_{yz} \sin \theta & \sigma_{yz} \cos \theta - \sigma_{xz} \sin \theta & \sigma_{zz} \end{bmatrix}$$

In Voigt notation, we can write the same as

$$\begin{bmatrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \sigma_{zz} \\ \sigma_{\xi\eta} \\ \sigma_{\xi z} \\ \sigma_{\eta z} \end{bmatrix} = \begin{bmatrix} \frac{1+\cos 2\theta}{2} & \frac{1-\cos 2\theta}{2} & 0 & \sin 2\theta & 0 & 0 \\ \frac{1-\cos 2\theta}{2} & \frac{1+\cos 2\theta}{2} & 0 & -\sin 2\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\sin 2\theta}{2} & \frac{\sin 2\theta}{2} & 0 & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

# 4.1. Material Symmetry and Anisotropy

## Macro-Mechanics Descriptions

- $$\begin{bmatrix} \cos \theta & \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{bmatrix} + \begin{bmatrix} -\sin \theta & \cos \theta & 0 \\ \cos \theta & \sin \theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{bmatrix}$$

### Strain Transformation in Voigt Notation

$$= \begin{bmatrix} \sigma \\ - \\ \end{bmatrix} \begin{bmatrix} \epsilon_{\xi\xi} \\ \epsilon_{\eta\eta} \\ \epsilon_{zz} \\ \gamma_{\xi\eta} \\ \gamma_{\xi z} \\ \gamma_{\eta z} \end{bmatrix} = \begin{bmatrix} \frac{1+\cos 2\theta}{2} & \frac{1-\cos 2\theta}{2} & 0 & \frac{\sin 2\theta}{2} & 0 & 0 \\ \frac{1-\cos 2\theta}{2} & \frac{1+\cos 2\theta}{2} & 0 & -\frac{\sin 2\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\sin 2\theta & \sin 2\theta & 0 & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

Note that we have used  $\gamma_{xy} = 2\epsilon_{xy}$  for the shear strains so some terms will be different from the stress transformation.

$$\begin{bmatrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \sigma_{zz} \\ \sigma_{\xi\eta} \\ \sigma_{\xi z} \\ \sigma_{\eta z} \end{bmatrix} = \begin{bmatrix} \frac{1+\cos 2\theta}{2} & \frac{1-\cos 2\theta}{2} & 0 & \sin 2\theta & 0 & 0 \\ \frac{1-\cos 2\theta}{2} & \frac{1+\cos 2\theta}{2} & 0 & -\sin 2\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\sin 2\theta}{2} & \frac{\sin 2\theta}{2} & 0 & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

- The stresses and strains transform as follows on the plane:

$$\sigma_{\xi} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{\eta} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$(\sigma_z = \sigma_z)$$

$$\tau_{\xi\eta} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\tau_{\xi z} = \tau_{xz} \cos \theta + \tau_{yz} \sin \theta$$

$$\tau_{\eta z} = -\tau_{xz} \sin \theta + \tau_{yz} \cos \theta$$

$$\varepsilon_{\xi} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\varepsilon_{\eta} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$(\varepsilon_z = \varepsilon_z)$$

$$\gamma_{\xi\eta} = -(\varepsilon_x - \varepsilon_y) \sin 2\theta + \gamma_{xy} \cos 2\theta$$

$$\gamma_{\xi z} = \gamma_{xz} \cos \theta + \gamma_{yz} \sin \theta$$

$$\gamma_{\eta z} = -\gamma_{xz} \sin \theta + \gamma_{yz} \cos \theta$$

- For an orthotropic material, the straight stresses/strains and shear stresses/strains are fully decoupled.
- So we will consider different cases of kinematic deformation fields to see if more can be said.

# 4.1. Material Symmetry and Anisotropy

## Macro-Mechanics Descriptions

### 1. Pure Out-Of-Plane Shear ( $\gamma_{xz} \neq 0$ )

- The stresses and strains are,

$$\sigma_\xi = 0$$

$$\sigma_\eta = 0$$

$$(\sigma_z = 0)$$

$$\tau_{\xi\eta} = 0$$

$$\begin{bmatrix} \tau_{\xi z} \\ \tau_{\eta z} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

$$= \begin{bmatrix} C_{55}\gamma_{xz} \cos \theta \\ -C_{55}\gamma_{xz} \sin \theta \end{bmatrix} := \begin{bmatrix} C_{55}\gamma_{\xi\eta} \\ C_{66}\gamma_{\eta z} \end{bmatrix}.$$

$$\varepsilon_\xi = 0$$

$$\varepsilon_\eta = 0$$

$$(\varepsilon_z = 0)$$

$$\gamma_{\xi\eta} = 0$$

$$\begin{bmatrix} \gamma_{\xi z} \\ \gamma_{\eta z} \end{bmatrix} = \begin{bmatrix} \gamma_{xz} \cos \theta \\ -\gamma_{xz} \sin \theta \end{bmatrix}$$

- Under symmetry,  $(\tau_{\xi z}, \tau_{\eta z})$  is related to  $(\gamma_{\xi z}, \gamma_{\eta z})$  in the same way that  $(\tau_{xz}, \tau_{yz})$  is related to  $(\gamma_{xz}, \gamma_{yz})$ .

- So we have,

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

• The

$\sigma_\xi =$

$\sigma_\eta =$

$(\sigma_z = 0)$

$\tau_{\xi\eta} =$

$\tau_{\xi z} = \tau_{\eta z} =$

$\tau_{\eta z} =$

• For fully

• So v said

$\frac{y}{r} \sin 2\theta$

$\frac{y}{r} \sin 2\theta$

ins are

can be

## 4.1. Material Symmetry and Anisotropy

Macro-Mech

### 2. Pure Out-Of-Plane Stretch ( $\varepsilon_z \neq 0$ )

- We have straight stresses  $\sigma_x = C_{13}\varepsilon_z, \sigma_y = C_{23}\varepsilon_z$ .
- Upon transformation we have,

$$\sigma_\xi = \left( \frac{C_{13} + C_{23}}{2} + \frac{C_{13} - C_{23}}{2} \cos 2\theta \right) \varepsilon_z$$

$$\sigma_\eta = \left( \frac{C_{13} + C_{23}}{2} - \frac{C_{13} - C_{23}}{2} \cos 2\theta \right) \varepsilon_z$$

$$\sigma_z = \sigma_z$$

$$\tau_{\xi\eta} = -\frac{C_{13} - C_{23}}{2} \sin 2\theta$$

$$\tau_{\xi z} = \tau_{\eta z} = 0$$

$$\varepsilon_\xi = 0$$

$$\varepsilon_\eta = 0$$

$$\varepsilon_z = \varepsilon_z$$

$$\gamma_{\xi\eta} = 0$$

$$\gamma_{\xi z} = \gamma_{\eta z} = 0$$

- For planar isotropy, the relationship between  $(\sigma_\xi, \sigma_\eta)$  and  $\sigma_z$  must be independent of  $\theta$ . This is only possible for  $C_{13} = C_{23}$ .
- So we have,

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ \text{sym} & & & & C_{55} & 0 \\ & & & & & C_{55} \end{bmatrix}$$

## 4.1. Material Symmetry and Anisotropy

### Macro-Mechanics Descriptions

#### 3. Pure In-Plane Stretch ( $\varepsilon_x \neq 0, \varepsilon_y = 0$ )

- From the constitutive properties we have  $\sigma_x = C_{11}\varepsilon_x$  and  $\sigma_y = C_{12}\varepsilon_x$ .

- Using this all the other components can be written as

$$\begin{aligned} \sigma_\xi &= \left( \frac{C_{11} + C_{12}}{2} + \frac{C_{11} - C_{12}}{2} \cos 2\theta \right) \varepsilon_x & \varepsilon_\xi &= \frac{1 + \cos 2\theta}{2} \varepsilon_x \\ \sigma_\eta &= \left( \frac{C_{11} + C_{12}}{2} + \frac{C_{11} - C_{12}}{2} \cos 2\theta \right) \varepsilon_x & \varepsilon_\eta &= \frac{1 - \cos 2\theta}{2} \varepsilon_x \\ (\sigma_z &= C_{12}\varepsilon_x + C_{22}\varepsilon_y \\ \tau_{\xi\eta} &= \sigma_z = 0 & \varepsilon_z &= 0 \\ \tau_{\xi z} &= \tau_{\xi\eta} = 0 & \gamma_{\xi\eta} &= 0 \\ \tau_{\eta z} &= \tau_{\xi z} = \tau_{\eta z} = 0. & \gamma_{\xi z} &= \gamma_{\eta z} = 0. \end{aligned}$$

- For the  $\sigma_\eta$  equality to hold, we need  $C_{22} = C_{11}$ . So we have

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & \cancel{C_{22}} & \begin{matrix} C_{11} \\ C_{13} \\ C_{33} \end{matrix} & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & \text{sym} & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{55} \end{bmatrix}$$

# 4.1. Material Symmetry and Anisotropy

## Macro-Mechanics Descriptions

### 4. Pure In-Plane Shear ( $\gamma_{xy} \neq 0$ )

- From the constitutive properties we have  $\tau_{xy} = C_{44}\gamma_{xy}$ .
- Using this all the other components can be written as

$$\sigma_\xi = C_{44}\gamma_{xy} \sin 2\theta = C_{11}\varepsilon_\xi + C_{12}\varepsilon_\eta \quad \varepsilon_\xi = \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\sigma_\eta = -C_{44}\gamma_{xy} \sin 2\theta = C_{12}\varepsilon_\xi + C_{11}\varepsilon_\eta \quad \varepsilon_\eta = -\frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\sigma_z = 0 \quad \varepsilon_z = 0$$

$$\tau_{\xi\eta} = C_{44}\gamma_{xy} \cos 2\theta \quad \gamma_{\xi\eta} = \gamma_{xy} \cos 2\theta$$

$$\tau_{\xi z} = \tau_{\eta z} = 0. \quad \gamma_{\xi z} = \gamma_{\eta z} = 0.$$

- So we have  $C_{44}\gamma_{xy} \sin 2\theta = \frac{C_{11}-C_{12}}{2}\gamma_{xy} \sin 2\theta$ . Therefore,

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}$$

sym

$C_{44}$  (green arrow pointing to  $\frac{C_{11}-C_{12}}{2}$ )

$$\frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\frac{\gamma_{xy}}{2} \sin 2\theta$$

$\theta$

raints are

re can be

## 4.1. Material Symmetry and Anisotropy

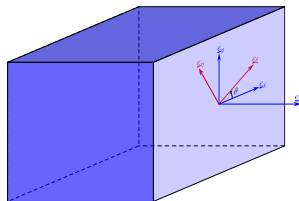
### Macro-Mechanics Descriptions

To Summarize,

a **Transversely Isotropic Material** constitution can be expressed as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ \textcolor{violet}{C}_{11} & \textcolor{blue}{C}_{13} & C_{33} & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ \text{sym} & & & & C_{55} & 0 \\ & & & & & \textcolor{red}{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

The material is fully characterized by five engineering constants.



## 4.1. Material Symmetry and Anisotropy: Engineering Constants

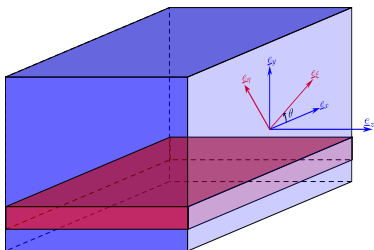
### Macro-Mechanics Descriptions

- In engineering practice, the constants are usually written easier in terms of compliance.
- For a specially orthotropic material the strain-stress relationship are usually expressed as,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & -\frac{\nu_{31}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_{11}} & -\frac{\nu_{23}}{E_{22}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\ & & & \frac{1}{G_{12}} & 0 & 0 \\ & \text{sym} & & & \frac{1}{G_{13}} & 0 \\ & & & & & \frac{1}{G_{23}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix}$$

## 5. Analysis of Planar Laminates

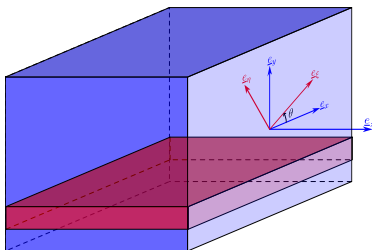
- Let us just consider one thin layer of a transversely isotropic material (continuously reinforced composite along a single direction).



$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ & \text{sym} & & & C_{55} & 0 \\ & & & & & C_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

## 5. Analysis of Planar Laminates

- Let us just consider one thin layer of a transversely isotropic material (continuously reinforced composite along a single direction).



$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ & \text{sym} & & & C_{55} & 0 \\ & & & & & C_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

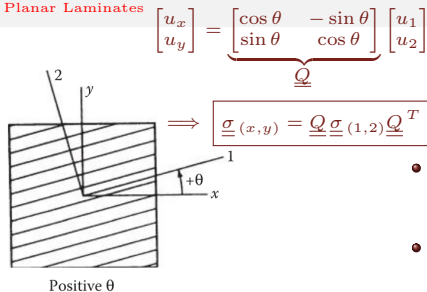
- We invoke plane stress assumptions, setting  $\sigma_y = 0$ . Let us also assume small shears,  $\tau_{xy} = 0, \tau_{yz} = 0$ .  
(Note:  $\varepsilon_z$  is not zero, and is implicitly defined)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} \quad (4 \text{ constants})$$

(Note change in notation in  $C_{ij}$ )

# 5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates



(Figure 2.11 from Gibson 2012)

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underline{\underline{\sigma}}(x, y) = \underline{\underline{Q}} \underline{\underline{\sigma}}_{(1,2)} \underline{\underline{Q}}^T$$

- If the coordinate system is not aligned with the fiber axes, **the stress and strain transformations need to be invoked.**
- In the constitutive relationship we have,

$$\underline{\underline{\sigma}}_{(1,2)} = \underline{\underline{C}} \underline{\underline{\varepsilon}}_{(1,2)}$$

$$\underline{\underline{T}}_{\sigma}^{-1} \underline{\underline{\sigma}}(x, y) = \underline{\underline{\sigma}}_{(1,2)} = \underline{\underline{C}} \underline{\underline{\varepsilon}}_{(1,2)} = \underline{\underline{C}} \underline{\underline{T}}_{\varepsilon}^{-1} \underline{\underline{\varepsilon}}(x, y)$$

$$\Rightarrow \underline{\underline{\sigma}}(x, y) = \underbrace{\underline{\underline{T}}_{\sigma} \underline{\underline{C}} \underline{\underline{T}}_{\varepsilon}^{-1}}_{\underline{\underline{C'}}} \underline{\underline{\varepsilon}}(x, y)$$

$$\text{where } \underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}.$$

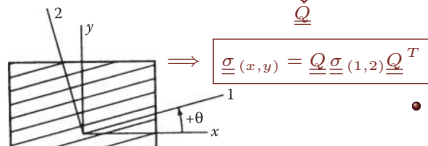
$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2 \theta & \sin^2 \theta & -2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & 2 \cos \theta \sin \theta \\ \cos \theta \sin \theta & -\cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}}_{\underline{\underline{T}}_{\sigma}} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\underline{\underline{T}}_{\sigma}^{-1} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}$$

# 5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



- If the coordinate system is not aligned with the fiber axes, the stress and strain transformations need to be invoked.

Recall that Strain Transformation looks slightly different because of our definition of shear strain  $\gamma_{xy} = 2\varepsilon_{xy}$ .

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2 \theta & \sin^2 \theta & -\cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & \cos \theta \sin \theta \\ 2 \cos \theta \sin \theta & -2 \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}}_{\underline{\underline{T}}_\varepsilon} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_y \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \sin \theta & \cos \theta & 2 \cos \theta \sin \theta \\ \cos \theta \sin \theta & -\cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}}_{\underline{\underline{T}}_\sigma} \begin{bmatrix} \sigma_2 \\ \tau_{12} \end{bmatrix}$$

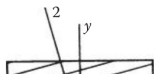
where  $\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}$ .

$$\underline{\underline{T}}_\sigma^{-1} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}$$

## 5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\underline{Q}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\underline{\sigma}(x, y) = \underline{Q} \underline{\sigma}_{(1,2)} \underline{Q}^T$$

**Transformed  $\underline{C}$  Matrix ( $\underline{\sigma} = \underline{C} \underline{\varepsilon}$ )**

$$\underline{C}' = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} \\ C'_{12} & C'_{22} & C'_{23} \\ C'_{13} & C'_{23} & C'_{33} \end{bmatrix}$$

$$C'_{11} = C_{11}c^4 + C_{22}s^4 + (2C_{33} + C_{12})2c^2s^2$$

$$C'_{22} = C_{11}s^4 + C_{22}c^4 + (2C_{33} + C_{12})2c^2s^2$$

$$C'_{33} = (C_{11} + C_{22} - 2C_{33} - 2C_{12})c^2s^2 + C_{33}(c^4 + s^4)$$

$$C'_{12} = (C_{11} + C_{22} - 4C_{33})c^2s^2 + C_{12}(c^4 + s^4)$$

$$C'_{13} = (C_{11} - 2C_{33} - C_{12})c^3s - (C_{22} - 2C_{33} - C_{12})cs^3$$

$$C'_{23} = (C_{11} - 2C_{33} - C_{12})cs^3 - (C_{22} - 2C_{33} - C_{12})c^3s.$$

ordinate system is not aligned with fiber axes, the stress and strain components need to be invoked.

utive relationship we have,

$$\underline{\varepsilon}_{(1,2)} = \underline{C} \underline{\varepsilon}_{(1,2)}$$

$$\underline{\varepsilon}_{(1,2)} = \underline{C} \underline{\varepsilon}_{(1,2)} = \underline{C} \underline{T}^{-1} \underline{\varepsilon}_{(x,y)}$$

$$\underline{\sigma}(x, y) = \underbrace{\underline{T} \underline{\sigma} \underline{T}^{-1}}_{\underline{C}'} \underline{\varepsilon}_{(x,y)}$$

$$\text{where } \underline{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}.$$

$$\underline{T}^{-1} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}$$

## 5.1. Generally Orthotropic Laminates

### Analysis of Planar Laminates

- Compliance is often more convenient:

$$\underline{\varepsilon}(x, y) = \underline{T} \underline{\varepsilon} \underline{T}^{-1} \underline{\sigma}(x, y)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{13} \\ & S'_{22} & S'_{23} \\ & & S'_{33} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$S'_{11} = S_{11}c^4 + S_{22}s^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{22} = S_{11}s^4 + S_{22}c^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{33} = (2S_{11} + 2S_{22} - S_{33} - 4S_{12})2c^2s^2 + S_{33}(c^4 + s^4)$$

$$S'_{12} = (S_{11} + S_{22} - S_{33})c^2s^2 + S_{12}(c^4 + s^4)$$

$$S'_{13} = (2S_{11} - S_{33} - 2S_{12})c^3s - (2S_{22} - S_{33} - 2S_{12})cs^3 \quad \nu_{yx} = E_y \left[ \frac{\nu_{21}}{E_2} (c^4 + s^4) \right.$$

$$\left. - \left( \frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} \right) c^2s^2 \right]$$

$$S'_{23} = (2S_{11} - S_{33} - 2S_{12})cs^3 - (2S_{22} - S_{33} - 2S_{12})c^3s.$$

- Based on this we can write,

$$E_x = \left[ \frac{c^4}{E_1} + \frac{s^4}{E_2} + \left( \frac{1}{G_{12}} - \frac{2\nu_{21}}{E_2} \right) c^2s^2 \right]^{-1}$$

$$E_y = \left[ \frac{s^4}{E_1} + \frac{c^4}{E_2} + \left( \frac{1}{G_{12}} - \frac{2\nu_{21}}{E_2} \right) c^2s^2 \right]^{-1}$$

$$G_{xy} = \left[ \frac{c^4 + s^4}{G_{12}} + \left( \frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{2G_{12}} + 2\frac{\nu_{21}}{E_2} \right) 4c^2s^2 \right]^{-1}$$

- In the material principal directions we have,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Engineering Constants:  $E_1, E_2, G_{12}, \nu_{12}$

- It is customary to express the laminate constitutive relationship as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

## 5.1. Generally Orthotropic Laminates

### Analysis of Planar Laminates

- Compliance is often more convenient:

$$\underline{\underline{\varepsilon}}(x, y) = \underline{\underline{T}} \underline{\underline{\varepsilon}} \underline{\underline{T}}^{-1} \underline{\underline{\sigma}}(x, y)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{13} \\ & S'_{22} & S'_{23} \\ & & S'_{33} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$S'_{11} = S_{11}c^4 + S_{22}s^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{22} = S_{11}s^4 + S_{22}c^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{33} = (2S_{11} + 2S_{22} - S_{33} - 4S_{12})2c^2s^2 + S_{33}(c^4 + s^4)$$

$$S'_{12} = (S_{11} + S_{22} - S_{33})c^2s^2 + S_{12}(c^4 + s^4)$$

$$S'_{13} = (2S_{11} - S_{33} - 2S_{12})c^3s - (2S_{22} - S_{33} - 2S_{12})cs^3$$

$$S'_{23} = (2S_{11} - S_{33} - 2S_{12})cs^3 - (2S_{22} - S_{33} - 2S_{12})c^3s.$$

- In the material principal directions we have,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Engineering Constants:  $E_1, E_2, G_{12}, \nu_{12}$

- Based on this we can write,

The Shear Constants can be written as

$$\eta_{xy,x} = G_{xy} \left[ \left( \frac{2}{E_1} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2} \right) c^3 s - \left( \frac{2}{E_2} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2} \right) cs^3 \right]$$

$$\eta_{xy,y} = G_{xy} \left[ \left( \frac{2}{E_1} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2} \right) cs^3 - \left( \frac{2}{E_2} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2} \right) c^3 s \right]$$

$$- \left( \frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} \right) c^2 s^2]$$

- It is customary to express the laminate constitutive relationship as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

# 5.1. Generally Orthotropic Laminates

## Analysis of Planar Laminates

- Compliance is often

$$\underline{\underline{\varepsilon}}(x, y) = \underline{\underline{T}} \underline{\underline{\varepsilon}} \underline{\underline{T}}^{-1} \underline{\underline{\sigma}}(x, y)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} \\ & S'_{22} \\ S'_{13} & S'_{23} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$S'_{11} = S_{11}c^4 + S_{22}s^4$$

$$S'_{22} = S_{11}s^4 + S_{22}c^4$$

$$S'_{33} = (2S_{11} + 2S_{22} - S_{33})s^2c^2$$

$$S'_{12} = (S_{11} + S_{22} - S_{33})s^2c^2$$

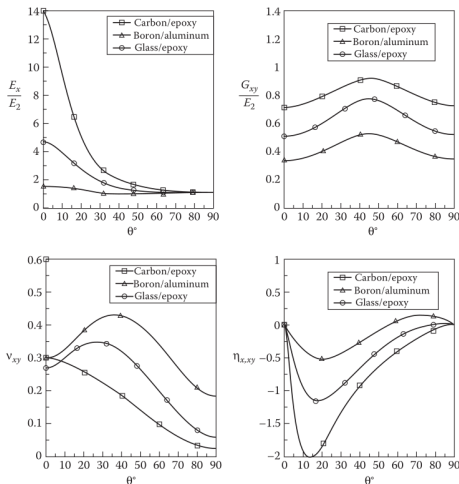
$$S'_{13} = (2S_{11} - S_{33} - 2\nu_{12})s^3c$$

$$S'_{23} = (2S_{11} - S_{33} - 2\nu_{12})sc^3$$

- In the material principal

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \nu_{12} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

## Off-Axis Modulii



(Figure 2.14 from Gibson 2012)

[ $\gamma_{xy}$ ]

write,

can be written as

$$\left( \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2} \right) c^3 s$$

$$\left( \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2} \right) cs^3$$

$$\left( \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2} \right) cs^3$$

$$\left( \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2} \right) c^3 s$$

$$\left( \frac{1}{G_{12}} \right) c^2 s^2$$

$$\left( \frac{1}{G_{12}} \right) c^2 s^2$$

express the laminate

as

$$\begin{bmatrix} \frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}} \\ \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}} \\ \frac{\eta_{yx,xy}}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Engineering Constants:  $E_1, E_2, G_{12}, \nu_{12}$

## 5.2. Numerical Examples: 1

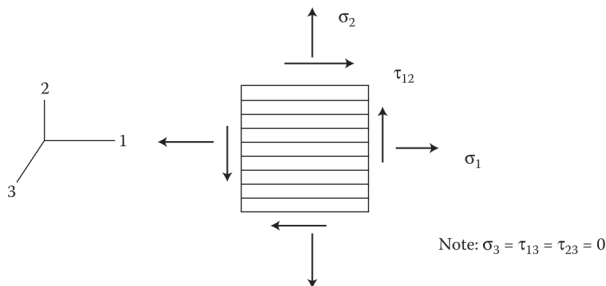
Analysis of Planar Laminates (Example 2.2 from Gibson 2012)

Consider an orthotropic laminate with the properties

$$E_1 = 140 \text{ GPa}, E_2 = 10 \text{ GPa}, G_{12} = 7 \text{ GPa}, \nu_{12} = 0.3, \nu_{23} = 0.2.$$

Compute the strains if it is subjected to the following state of stress in the principal coordinates:

$$\sigma_1 = 70 \text{ MPa}, \sigma_2 = 140 \text{ MPa}, \tau_{12} = 35 \text{ MPa}, \sigma_3 = \tau_{12} = \tau_{23} = 0.$$

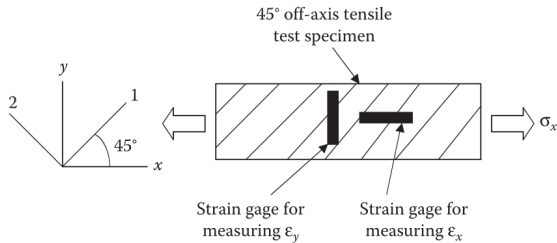


(Figure 2.10 from Gibson 2012)

## 5.2. Numerical Examples: 2

Analysis of Planar Laminates(Example 2.3 from Gibson 2012)

A  $45^\circ$  off-axis tensile test is conducted on a generally orthotropic test specimen by applying a normal stress  $\sigma_x$ . The specimen has strain gauges attached to measure axial and transverse strains ( $\epsilon_x, \epsilon_y$ ). How many engineering parameters can be estimated from measurements of  $\sigma_x, \epsilon_x, \epsilon_y$  ?

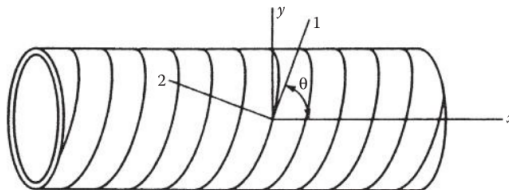


(Figure 2.15 from Gibson 2012)

## 5.2. Numerical Examples: 3

Analysis of Planar Laminates(Example 2.4 from Gibson 2012)

A filament-wound cylindrical pressure vessel of mean diameter  $d = 1$  m and wall thickness  $t = 20$  mm is subjected to an internal pressure,  $p$ . The filament- winding angle  $\theta = 53.1^\circ$  from the longitudinal axis of the pressure vessel, and the glass/epoxy material has the following properties:  $E_{11} = 40$  GPa,  $E_{22} = 10$  GPa,  $G_{12} = 3.5$  GPa, and  $\nu_{12} = 0.25$ . By the use of a strain gauge, the normal strain along the fiber direction is determined to be  $\varepsilon_{11} = 0.001$ . Determine the internal pressure in the vessel.



(Figure 2.18 from Gibson 2012)

## 6. Classical Laminate Theory

- In the Kirchhoff-Love Plate Theory we had,

$$\begin{bmatrix} \underline{N} \\ \underline{M} \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{B} & \underline{D} \end{bmatrix} \begin{bmatrix} \underline{u}' \\ \underline{w}'' \end{bmatrix}$$

where

$$\underline{\underline{A}} = \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad \underline{\underline{D}} = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad \underline{\underline{B}} = \underline{\underline{0}}.$$

- This can also be written in terms of thickness moments of the constitutive matrix

$$\underline{\underline{C}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \text{ as}$$

$$\underline{\underline{A}} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \underline{\underline{C}} dz, \quad \underline{\underline{B}} = \int_{-\frac{t}{2}}^{\frac{t}{2}} z \underline{\underline{C}} dz, \quad \underline{\underline{D}} = \int_{-\frac{t}{2}}^{\frac{t}{2}} z^2 \underline{\underline{C}} dz.$$

## 6. Classical Laminate Theory

- Suppose we had different laminate plies along the thickness, such that the constitutive matrix is  $\underline{\underline{C}}_i$  for  $z \in (z_i, z_{i+1})$  and  $-\frac{t}{2} = z_1 < \dots < z_N = \frac{t}{2}$ .
- Then the  $A - B - D$  matrices are written as the sums,

$$\underline{\underline{A}} = \sum_i (z_{i+1} - z_i) \underline{\underline{C}}_i, \quad \underline{\underline{B}} = \sum_i \frac{z_{i+1}^2 - z_i^2}{2} \underline{\underline{C}}_i, \quad \underline{\underline{D}} = \sum_i \frac{z_{i+1}^3 - z_i^3}{3} \underline{\underline{C}}_i.$$

- Unlike isotropic plates, composite laminates can have non-zero  $\underline{\underline{B}}$  matrix (moment-planar coupling), bending-twisting coupling, etc.
- This  $\begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}} & \underline{\underline{D}} \end{bmatrix}$  matrix is known as the **Laminate Stiffness Matrix**.

## 6.1. The Laminate Orientation Code

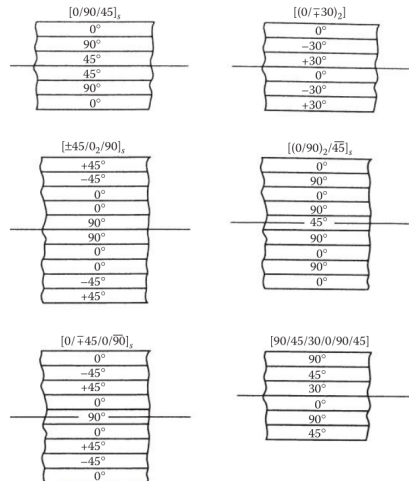
### Classical Laminate Theory

- Ply angles separated by slashes, ordered from top to bottom
- Subscript “s” for symmetric laminates
- Numerical subscripts for repetitions
- Center ply with an overbar for odd laminates

(See sec. 7.1 in Gibson 2012)

### Types

- Symmetric, Antisymmetric, Asymmetric
- Angle-Ply, Cross-Ply, Balanced,  $\pi/4$  laminates



(Figure 7.1 from Gibson 2012)

# 6.1. The Laminate Orientation Code

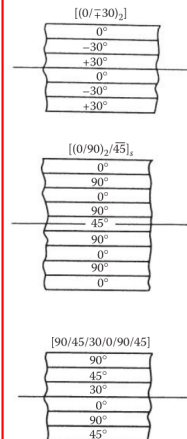
## Classical Laminate Theory

### Summary of Laminate Stiffnesses

**Table 3.4.** The  $[A]$ ,  $[B]$ ,  $[D]$  matrices for laminates. When the laminate is symmetrical, the  $[B]$  matrix is zero. Cross-ply laminates are orthotropic.

| $[A]$                                                                                                            | $[B]$                                                                                                            | $[D]$                                                                                                            |
|------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------|
| Symmetrical                                                                                                      |                                                                                                                  |                                                                                                                  |
| $\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$                                              | $\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$ |
| Balanced                                                                                                         |                                                                                                                  |                                                                                                                  |
| $\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$                     | $\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$ | $\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$ |
| Orthotropic                                                                                                      |                                                                                                                  |                                                                                                                  |
| $\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$                     | $\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}$                     | $\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$                     |
| Isotropic                                                                                                        |                                                                                                                  |                                                                                                                  |
| $\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}-A_{12}}{2} \end{bmatrix}$    | $\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & \frac{B_{11}-B_{12}}{2} \end{bmatrix}$    | $\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & \frac{D_{11}-D_{12}}{2} \end{bmatrix}$    |
| Quasi-isotropic                                                                                                  |                                                                                                                  |                                                                                                                  |
| $\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}-A_{12}}{2} \end{bmatrix}$    | $\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$ | $\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$ |

(Table 3.4 from Kollár and Springer 2003)



Gibson 2012)

- Ply angles separated from top to bottom
- Subscript “s” for symmetrical
- Numerical subscript for number of layers
- Center ply with an angle for odd-numbered laminates

### Types of Laminates

- Symmetric, Antisymmetric
- Angle-Ply, Cross-Ply, Quasi-isotropic laminates

## 6.2. Laminated Beams

### Classical Laminate Theory

- Consider a beam with a symmetric section on the  $x - y$  plane. Invoking Kirchhoff kinematic assumptions we have:  $\varepsilon_x = u' - yv''$ .
- The stress distribution will depend on the section-coordinate. In general we will have:  $\sigma_x = E_x(y)\varepsilon_x = E_x(y)(u' - yv'')$ .
- We get the effective normal reaction  $N_x$  by integrating the stress over the section:

$$N_x = \int_{\mathcal{A}} \sigma_x = \left[ \int_{\mathcal{A}} E_x(y) \right] u' + \left[ \int_{\mathcal{A}} -yE_x(y) \right] v''.$$

- Similarly we get the bending moment  $M_z$  as the first moment of the stress,

$$M_z = \int_{\mathcal{A}} -y\sigma_x = \left[ \int_{\mathcal{A}} -yE_x(y) \right] u' + \left[ \int_{\mathcal{A}} y^2 E_x(y) \right] v''.$$

- In summary we have the beam-analog of the laminate stiffness matrix,

$$\begin{bmatrix} N_x \\ M_z \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} u' \\ v'' \end{bmatrix}.$$

**Important note:** We have assumed that no torsion/twist is present. See Kollár and Springer 2003 for the general form.

## 6.2. Laminated Beams

### Classical Laminate Theory

- For a laminated composite with a rectangular section with width  $b$ , the integrals may be simplified as,

$$A = \int_{\mathcal{A}} E_x(y) = \sum_{i=1}^N E_{x,i} b (y_{i+1} - y_i), \quad B = \int_{\mathcal{A}} -y E_x(y) = - \sum_{i=1}^N E_{x,i} b \frac{y_{i+1}^2 - y_i^2}{2}$$

$$D = \int_{\mathcal{A}} y^2 E_x(y) = \sum_{i=1}^N E_{x,i} b \frac{y_{i+1}^3 - y_i^3}{3}.$$

- For plies of uniform thickness we can write

$$y_i = -\frac{h}{2} + (i-1) \frac{h}{N},$$

which leads to:

$$A = \frac{h}{N} \sum_{i=1}^N E_{x,i}, \quad B = \frac{h^2}{2N^2} \sum_{i=1}^N E_{x,i} (2i - N - 1),$$

$$D = \frac{h^3}{12N^3} \sum_{i=1}^N E_{x,i} (12i^2 - 12Ni + 12N^2 + 3N^2 + 6N + 4)$$

## 6.3. Numerical Example

### Classical Laminate Theory

Determine the ABD matrix for the following composite beams where the ply thickness is 1 mm and beam width is 10 mm:

- $[0/90]_s$ , and
- $[0/90/0/90]$ .

Assume the following properties for each lamina:  $E_1 = 140$  GPa,  $E_2 = 10$  GPa,  $G_{12} = 7$  GPa,  $\nu_{12} = 0.3$ ,  $\nu_{23} = 0.2$ .

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