AS2070: Aerospace Structural Mechanics

Module 2: Composite Material Mechanics (V1)

Instructor: Nidish Narayanaa Balaji

Dept. of Aerospace Engg., IIT Madras, Chennai

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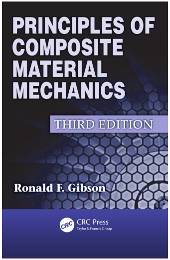
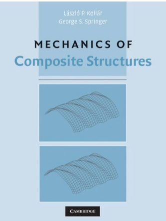
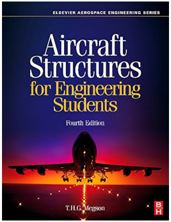
(Also see Daniel and Ishai 2006)



Chapters 1-3 in Gibson (2012).



Chapters 1-3, 11 in Kollár and Springer (2003).

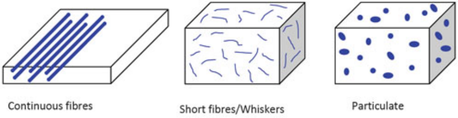


Chapter 25 in Megson (2013)

1.1. What are Composites?

Introduction

- Structural material consisting of multiple non-soluble macro-constituents.
- Main motivation: material properties tailored to applications.
- Both stiffness and strength comes from the fibers/particles, and the matrix holdes everything together.



Types of composite materials (Figure from NPTEL Online-IIT KANPUR (2025))

Examples

- Reinforced concrete
- Wood (lignin matrix reinforced by cellulose fibers)
- Carbon-Fiber Reinforced Plastics (CFRP)

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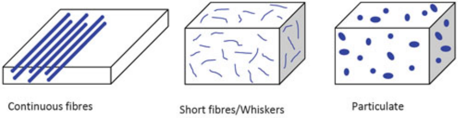

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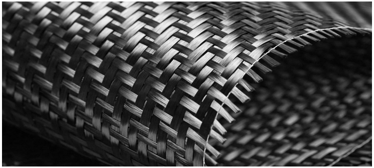
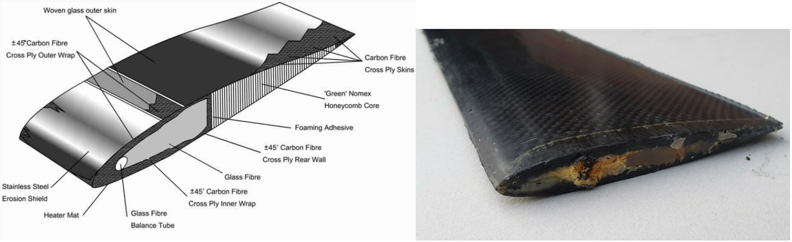


1.1. What are Composites?

Introduction

- Structural material consisting of multiple non-soluble macro-constituents.

CFRP Helicopter Blades



(Figures from Carbon Fiber Top Helicopter Blades 2025)

Exa

- Wood (lignin matrix reinforced by cellulose fibers)
- Carbon-Fiber Reinforced Plastics (CFRP)

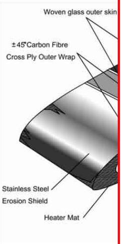
AA.

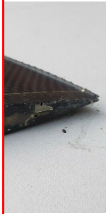
- High fatigue resistance. But quite brittle.
- Main- and tail-planes, fuselages, etc. Helicopter blades.

1.1. What are Composites?

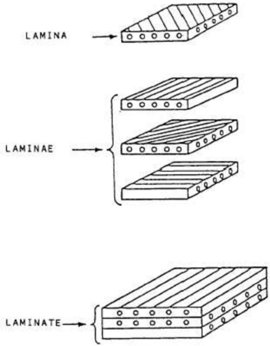
Introduction

- Structural ma ts.





Laminated Composites



(Figure from Kalkan 2017)

Exa

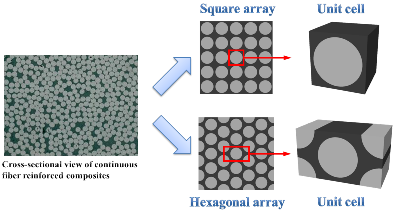
- AA.
- Wood (lignin ma quite brittle.
  fibers)
- Carbon-Fiber Re ges, etc. Helicopter

1.2. Modeling Composite Material

Introduction

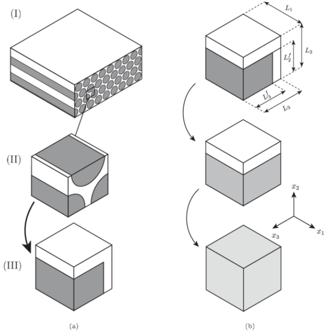
Two main approaches:

Micro-Mechanics



(Figure from "Micro-Mechanics of Failure" 2024)

Macro-Mechanics

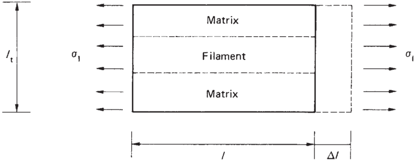


Homogenization of micro-structure (Figure from Skovsgaard and Heide-Jørgensen 2021)

1.3. Constitutive Modeling for Composites

Introduction

Axial Elongation



- Strain is fixed, but stress experienced by media differ.

$$\sigma_l = E_l \varepsilon_l$$

- Stress-strain relationship simplifies as,

$$\sigma_m = E_m \varepsilon_l, \quad \sigma_f = E_f \varepsilon_l$$

$$\sigma_l A = \sigma_m A_m + \sigma_f A_f$$

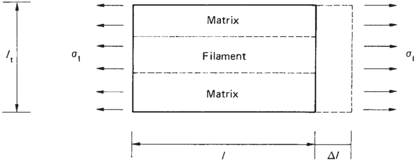
$$\implies \boxed{E_l = \frac{A_f}{A} E_f + \frac{A_m}{A} E_m}.$$

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction

Axial Elongation



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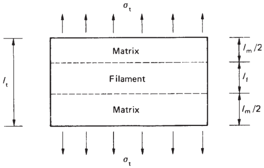
Transverse Elongation



- Stress is fixed, strains differ:

$$\varepsilon_t l_t = \varepsilon_m l_m + \varepsilon_f l_f$$

$$\implies \frac{\sigma_t}{E_t} l_t = \frac{\sigma_t}{E_m} l_m + \frac{\sigma_t}{E_f} l_f$$

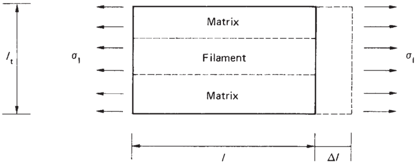
$$\implies \boxed{\frac{1}{E_t} = \frac{1}{E_m}\frac{l_m}{l_t} + \frac{1}{E_f}\frac{l_f}{l_t}}.$$

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction: Poisson Effects

Axial-Transverse Coupling



- Transverse displacement written as

$$\Delta_t = \nu_m \varepsilon_l l_m + \nu_f \varepsilon_l l_f := \nu_{lt} \varepsilon_l l_t$$

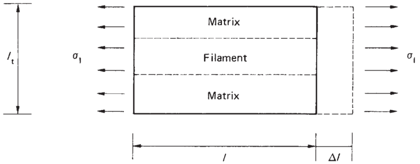
$$\implies \boxed{\nu_{lt} = \frac{l_m}{l_t}\varepsilon_l + \frac{l_f}{l_t}\varepsilon_f}.$$

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction: Poisson Effects

Axial-Transverse Coupling



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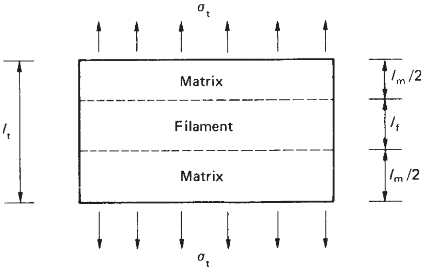
Transverse-Axial Coupling



- Axial displacement written as

$$\nu_m \frac{\sigma_t}{E_m} = \nu_f \frac{\sigma_t}{E_f} := \nu_{tl} \frac{\sigma_t}{E_t},$$

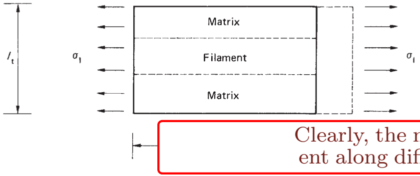
$$\implies \boxed{\nu_{tl} = \frac{E_t}{E_l}\nu_{lt}}.$$

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction: Poisson Effects

Axial-Transverse Coupling



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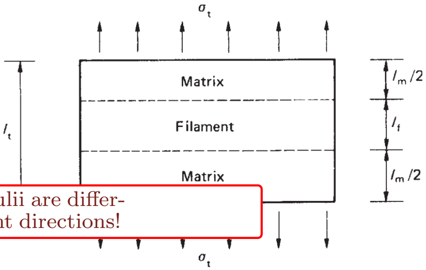
Transverse-Axial Coupling



- Axial displacement written as

$$\nu_m \frac{\sigma_t}{E_m} = \nu_f \frac{\sigma_t}{E_f} := \nu_{tl} \frac{\sigma_t}{E_t},$$

$$\implies \boxed{\nu_{tl} = \frac{E_t}{E_l} \nu_{lt}}.$$

Clearly, the modulii are different along different directions!

(Figures from Megson 2013)

1.3. Constitutive Modeling for Composites

Introduction: Anisotropy

General Anisotropy (aka "Triclinic")

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46}\\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56}\\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

1.3. Constitutive Modeling for Composites

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Monoclinic: Single Plane of Symmetry

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0\\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0\\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0\\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & C_{56}\\ 0 & 0 & 0 & 0 & C_{56} & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

1.3. Constitutive Modeling for Composites

Introduction: Anisotropy

Orthotropic: Three Orthogonal Planes of Symmetry

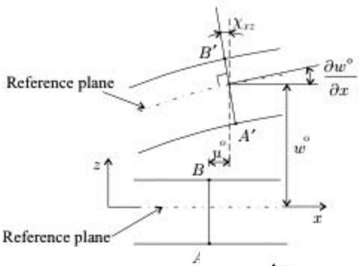
$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

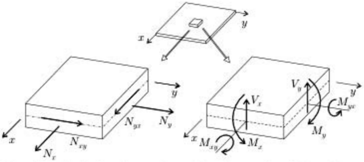
Transversely Isotropic

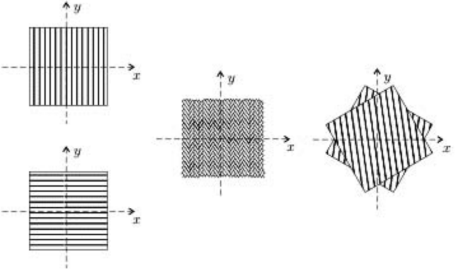
$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2}\end{bmatrix} \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

1.4. Classical Laminate Theory

Introduction

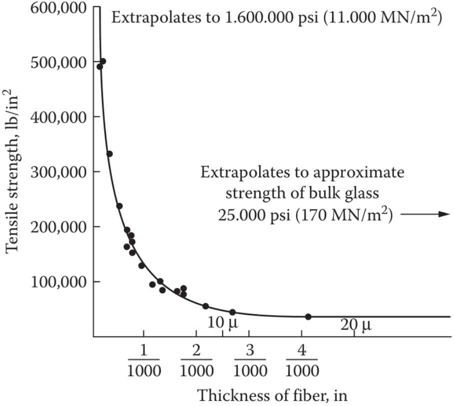






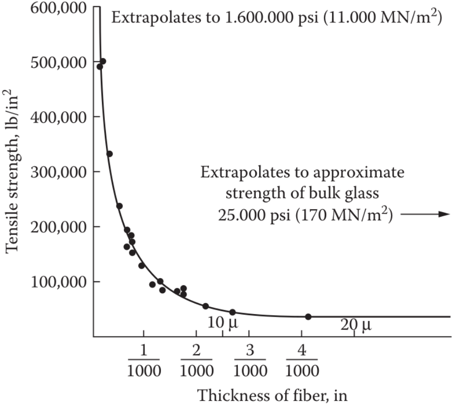
Figures from Kollár and Springer 2003

2. Composite Materials



Griffith's experiments with glass fibres (1920) (Figure from Gibson 2012)

2. Composite Materials



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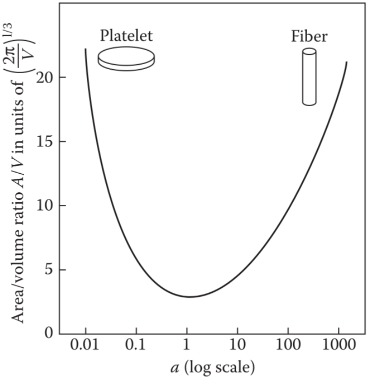


(Figure from Gibson 2012)

2.1. Types of Composite Materials

Composite Materials

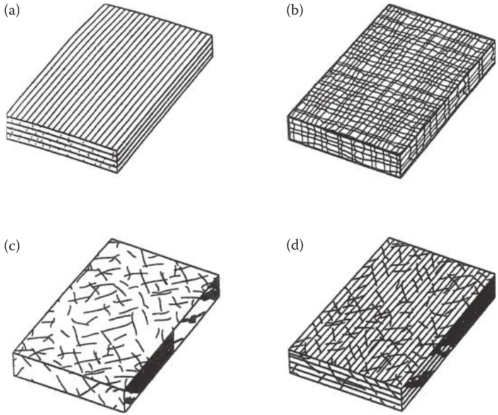


FIGURE 1.4
Types of fiber-reinforced composites. (a) Continuous fiber composite, (b) woven composite, (c) chopped fiber composite, and (d) hybrid composite.

(Figure from Gibson 2012)

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

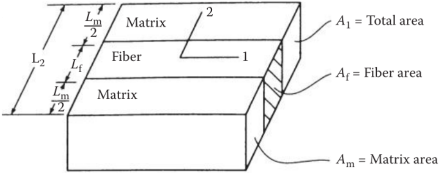
The rule of mixtures is introduced as a very simple framework for developing "overall"/representative mechanical properties.

Basic Definitions

Subscripts $(\cdot)_f$, $(\cdot)_m$, $(\cdot)_v$, and $(\cdot)_c$ denote quantities corresponding to the fiber, matrix, void, and composite (as a whole).

Volume Fraction $v_f = \frac{V_f}{V_c}, v_m = \frac{V_m}{V_c}, v_v = \frac{V_v}{V_c}$ such that $v_f + v_m + v_v = 1$.
Note that composite density $\rho_c = \rho_f v_f + \rho_m v_m$.

Weight Fraction $w_f = \frac{\rho_f}{\rho_c} v_f$



(Figure 3.5a from Gibson 2012)

$$E_1 = v_f E_f + v_m E_m$$

$$(\times) E_2 = \left(\frac{v_f}{E_f} + \frac{v_m}{E_m}\right)^{-1}$$

$$\nu_{12} = v_f \nu_f + v_m \nu_m$$

$$(\times) G_{12} = \left(\frac{v_f}{G_f} + \frac{v_m}{G_m}\right)^{-1}$$

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

The rule of mixtures is introduced as a very simple framework for developing “overall” /representative mechanical properties

RoM is not always satisfactory!

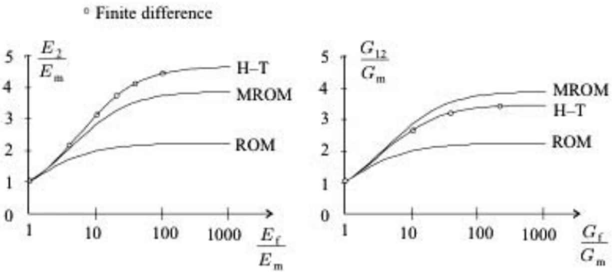


Figure 11.8: The transverse Young and shear moduli calculated by the rule of mixtures (ROM), the modified rule of mixtures (MROM), the Halpin–Tsai (H–T) equations, and the finite difference solutions (circles) of Adams and Doner ($v_f = 0.55$).

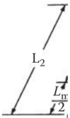
(Figure 11.8 from Kollár and Springer 2003)

Basic D

Subscrip ... rix, void, and composi

Volume

Weight



E_m

$)^{-1}$

$)^{-1}$

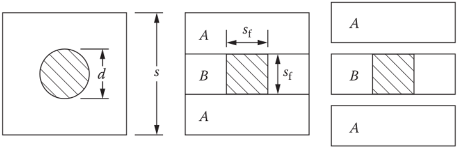
G_f G_m

(Figure 3.5a from Gibson 2012)

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

- The mismatch is related to the fact that our idealized picture was a poor representation of reality to begin with. More geometrical details of the fiber arrangement are necessary.



(Figure 3.8 from Gibson 2012)

$$s_f = \sqrt{\frac{\pi}{4}} d; s = \sqrt{\frac{\pi}{4v_f}} d.$$

$$\begin{aligned}
E_{B2} &= \left(\frac{\sqrt{v_f}}{E_f} + \frac{1-\sqrt{v_f}}{E_m} \right)^{-1} \\
&= \frac{E_m}{1 - \sqrt{v_f}(1 - \frac{E_m}{E_f})}
\end{aligned}$$

$$\begin{aligned}
E_2 &= E_{B2} \frac{s_f}{s} + E_m \frac{s - s_f}{s} \\
&= E_{B2}\sqrt{v_f} + E_m(1-\sqrt{v_f}) \\
&= E_m \left[(1-\sqrt{v_f}) + \frac{\sqrt{v_f}}{1-\sqrt{v_f}(1-\frac{E_m}{E_f})} \right]
\end{aligned}$$

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

(Recommended reading: Sec. 3.2.3 in Daniel and Ishai 2006)

The Halpin-Tsai Equation

$$\begin{aligned} E_2 &= E_m \frac{1+\xi\eta v_f}{1-\eta v_f}, \quad \eta = \frac{E_f - E_m}{E_f + \xi E_m} \\ &= E_m \frac{E_f + \xi E_m + \xi v_f(E_f - E_m)}{E_f + \xi E_m - v_f(E_f - E_m)} \end{aligned}$$

Note: $\xi = 2$ for circular section fibers. $\xi = \frac{2a}{b}$ for rectangular fibers (b being loaded side).

Case 1: $\xi \to 0$

$$E_2 = \left(\frac{v_f}{E_f} + \frac{1-v_f}{E_m}\right)^{-1}$$

Series, *Reuss* model.

Case 2: $\xi \to \infty$

$$E_2 = E_f v_f + E_m(1-v_f)$$

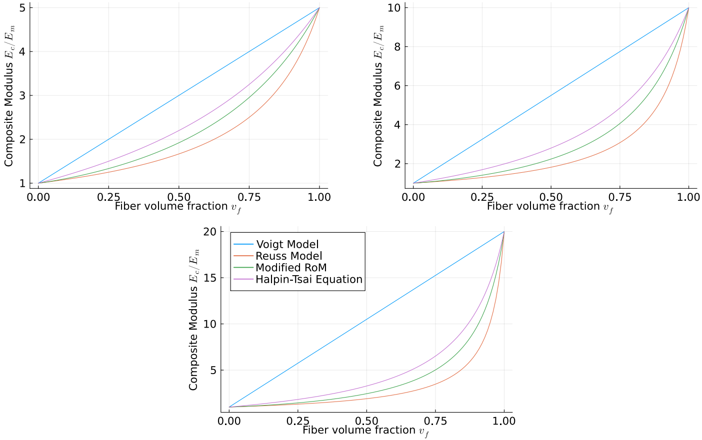
Parallel, *Voigt* model.

3.1. The Rule of Mixtures

Micro-Mechanics Descriptions

Graphical Comparison for varying $\frac{E_f}{E_m}$

Ishai 2006)



Note: ξ

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Series,

3.2. Numerical Example

Micro-Mechanics Descriptions

(from Kollár and Springer 2003)

Consider a Graphite/Epoxy unidirectional ply. Matrix properties are given with subscript m in the table below. Nominal properties with fiber volume fraction $v_f = 60\,\%$ are also given. Assume that the fibers show anisotropy ($E_{f1} \neq E_{f2}$).

	E_1	E_2	G_{12}	ν_{12}	E_m	G_m	ν_m
Value	148	9.65	4.55	0.3	4.1	1.5	0.35

All modulii in GPa.

Estimate the following:

- Fiber modulus properties
- Composite material modulii for volume fraction $v_f = 0.55$.

(Also discussed sensitivity analysis)

4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

Material Symmetry

The study of material symmetry is concerned with finding answers to the question:
If the strain field on a deformable object is changed, how does the stress field change?

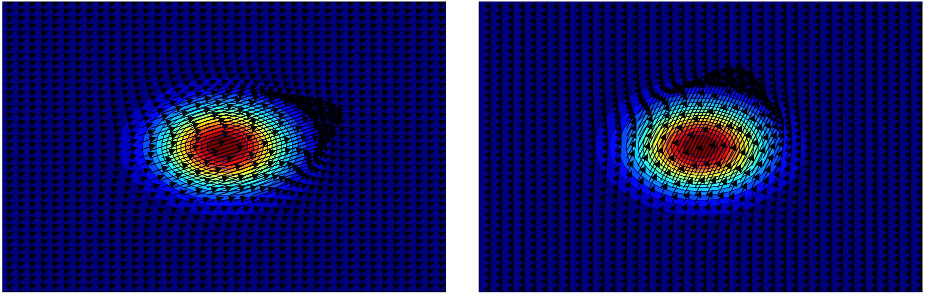
4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

Material Symmetry

The study of material symmetry is concerned with finding answers to the question:
If the strain field on a deformable object is changed, how does the stress field change?

Consider the following Deformation Fields



Deformation Case 1

Deformation Case 2 (Case 1 Rotated)

4.1. Macro-Mechanics Descriptions

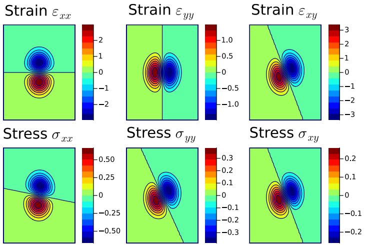
Material Symmetry and Anisotropy

Material Symmetry

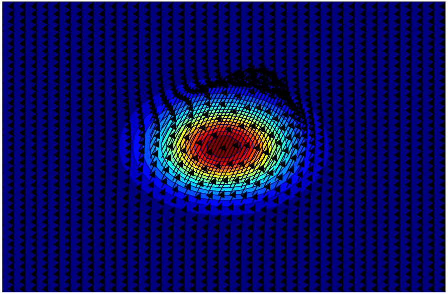
The study of material symmetry is concerned with finding answers to the question:
If the strain field on a deformable object is changed, how does the stress field change?

Consider the following Deformation Fields

Stress and Strain Field



Isotropic Stress-Strain Relationship



Deformation Case 2 (Case 1 Rotated)

4.1. Macro-Mechanics Descriptions

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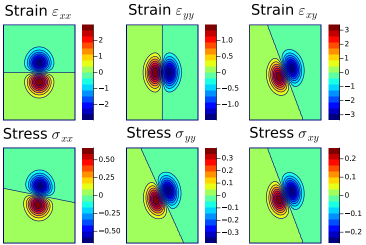

Isotropic Stress-Strain Relationship

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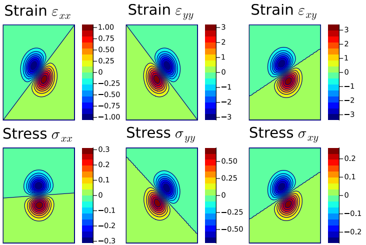

Isotropic Stress-Strain Relationship

4.1. Macro-Mechanics Descriptions

Material Symmetry and Anisotropy

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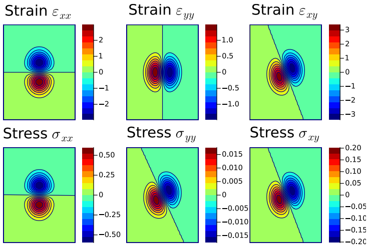
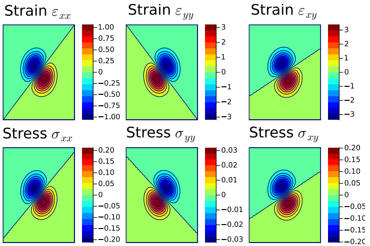


Anisotropic Case

Stress and Strain Field



Anisotropic Case

4.1. Macro-Mechanics Descriptions

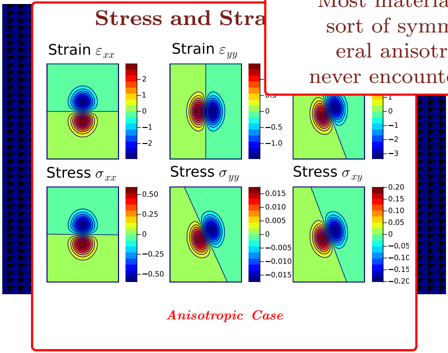
Material Symmetry and Anisotropy

Material Symmetry

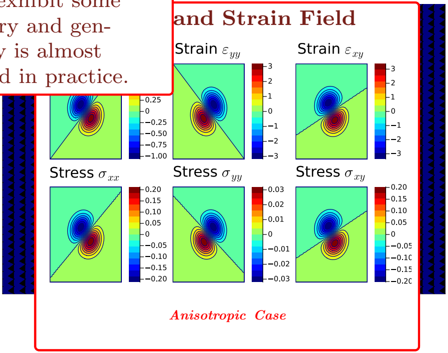
The study of material symmetry is concerned with finding answers to the question:
If the strain field on a deformable object is changed, how does the stress field change?

Consider the following Deformation Fields

Most materials exhibit some sort of symmetry and general anisotropy is almost never encountered in practice.



Anisotropic Case



Anisotropic Case

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

How do stresses and strains transform under coordinate change?

- Suppose $\underline{x} \in \mathbb{R}^3$ are the coordinates of a point in 3D space.
- Let $\underline{x}' \in \mathbb{R}^3$ be the coordinates under transformation.
- We will write: $\boxed{\underline{x}' = \underline{\underline{Q}}\,\underline{x}}$, with $\underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^{T}$.

Strains

- $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\nabla}_{\underline{x}}\underline{u} + \underline{\nabla}_{\underline{x}}\underline{u}^T\right)$
- $\underline{\nabla}_{\underline{x}'}\underline{u}' = \underline{\underline{Q}}\,\underline{\nabla}_{\underline{x}}\underline{u}\underline{\underline{Q}}^{-1} \implies \boxed{\underline{\underline{\varepsilon}}' = \underline{\underline{Q}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{Q}}^{T}}$.

Stresses

- Cauchy Stress Definition: $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$
- $\underline{\underline{Q}}\,\underline{t} = \underline{t}' = \underline{\underline{\sigma}}'\underline{n}' = \underline{\underline{\sigma}}'\underline{\underline{Q}}\,\underline{n} = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{n}$ $\implies \boxed{\underline{\underline{\sigma}}' = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{\underline{Q}}^{T}}$

Reflections

Note that reflections may be expressed as a coordinate change with $\underline{\underline{Q}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (reflection about the xy plane).

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- Under reflection about the xy plane, the strain transforms as,

$$\begin{bmatrix} \varepsilon_x' & \frac{\gamma_{xy}'}{2} & \frac{\gamma_{xz}'}{2} \\ & \varepsilon_y' & \frac{\gamma_{yz}'}{2} \\ \text{sym} & & \varepsilon_z' \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \text{sym} & & \varepsilon_z \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & -\frac{\gamma_{xz}}{2} \\ & \varepsilon_y & -\frac{\gamma_{yz}}{2} \\ \text{sym} & & \varepsilon_z \end{bmatrix}$$

- So in Voigt notation we have,

$$\begin{bmatrix} \varepsilon_x' \\ \varepsilon_y' \\ \varepsilon_y' \\ \gamma_{xy}' \\ \gamma_{xz}' \\ \gamma_{yz}' \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \qquad \begin{bmatrix} \sigma_x' \\ \sigma_y' \\ \sigma_y' \\ \tau_{xy}' \\ \tau_{xz}' \\ \tau_{yz}' \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

Similarly for Stress

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- Under reflection about the xy plane, the strain transforms as,

- So

If a material were symmetric about the xy plane, then reflecting the strain field about the xy plane will result in a stress field that is reflected about the same xy plane.

Note

- Strain field reflection is a kinematic operation/configuration change.
- Change in the Stress field is the effect that the above kinematic change results in.
- If the material happens to be symmetric about the reflection plane, then this change will be a reflection.

Similarly for Stress

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- We have said the following :

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & \text{sym} & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

Recall that this symmetry follows from strain energy existence

$$\begin{bmatrix}\sigma'_x\\ \sigma'_y\\ \sigma'_z\\ \tau'_{xy}\\ \tau'_{xz}\\ \tau'_{yz}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & \text{sym} & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon'_x\\ \varepsilon'_y\\ \varepsilon'_z\\ \gamma'_{xy}\\ \gamma'_{xz}\\ \gamma'_{yz}\end{bmatrix}$$

(The $\underline{\underline{C}}$ matrix is the same in both the original and the reflected coordinate systems)

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

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This leads to

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ & & C_{33} & C_{34} & -C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & -C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

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$$\boxed{\begin{bmatrix} \sigma_x' \\ \sigma_y' \\ \sigma_z' \\ \tau_{xy}' \\ \tau_{xz}' \\ \tau_{yz}' \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x' \\ \varepsilon_y' \\ \varepsilon_z' \\ \gamma_{xy}' \\ \gamma_{xz}' \\ \gamma_{yz}' \end{bmatrix}}$$

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$$\begin{bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{xz} \\ \tau'_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \varepsilon'_z \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{bmatrix}$$

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Recall that this symmetry follows from strain energy existence

Finally we see that material symmetry about the xz plane implies the following simplification to the constitutive relationship.

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

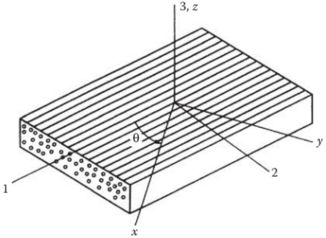
This is known as a **Monoclinic Material** (13 constants). This is also quite rare to encounter in practice.

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

Suppose all the three fundamental planes are planes of symmetry, we have an **Orthotropic Material** (9 constants).

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\tau_{12}\\\tau_{13}\\\tau_{23}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\gamma_{12}\\\gamma_{13}\\\gamma_{23}\end{bmatrix}$$



(Figure 2.5 from Gibson 2012)

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$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix}$$

Notice that $(\sigma_{(1,2,3)}, \varepsilon_{(1,2,3)})$ and $(\tau_{(12,13,23)}, \gamma_{(12,13,23)})$ are naturally decoupled as a consequence of symmetry in this coordinate system.

Also note,

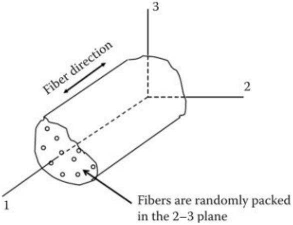
- Specially orthotropic
- Generally orthotropic

(Figure 2.5 from Gibson 2012)

4.1. Material Symmetry and Anisotropy: Transverse Isotropy

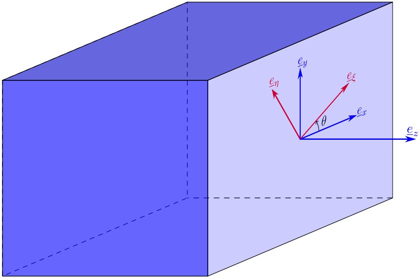
Macro-Mechanics Descriptions

- In continuous fiber reinforced composites, it is often the case that the fibers are randomly distributed on a plane. This leads to planar isotropy in the plane perpendicular to the fiber stacking direction.



(Figure 2.6 from Gibson 2012)

- How do the stresses and strains transform on the plane?



$$(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}) \rightarrow (\sigma_\xi, \sigma_\eta, \sigma_z, \tau_{\xi\eta}, \tau_{\xi z}, \tau_{\eta z})$$

$$(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \rightarrow (\varepsilon_\xi, \varepsilon_\eta, \varepsilon_z, \gamma_{\xi\eta}, \gamma_{\xi z}, \gamma_{\eta z})$$

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- The stresses and strains transform as follows on the plane:

$$\sigma_\xi = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\sigma_\eta = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$(\sigma_z = \sigma_z)$$
$$\tau_{\xi\eta} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$\tau_{\xi z} = \tau_{xz}\cos\theta + \tau_{yz}\sin\theta$$
$$\tau_{\eta z} = -\tau_{xz}\sin\theta + \tau_{yz}\cos\theta$$

$$\varepsilon_\xi = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$
$$\varepsilon_\eta = \frac{\varepsilon_x+\varepsilon_y}{2} - \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$
$$(\varepsilon_z = \varepsilon_z)$$
$$\gamma_{\xi\eta} = -(\varepsilon_x-\varepsilon_y)\sin 2\theta + \gamma_{xy}\cos 2\theta$$
$$\gamma_{\xi z} = \gamma_{xz}\cos\theta + \gamma_{yz}\sin\theta$$
$$\gamma_{\eta z} = -\gamma_{xz}\sin\theta + \gamma_{yz}\cos\theta$$

- For an orthotropic material, the straight stresses/strains and shear stresses/strains are fully decoupled.
- So we will consider different cases of kinematic deformation fields to see if more can be said.

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- The

$\sigma_\xi =$ … $\frac{y}{}$ $\sin 2\theta$

$\sigma_\eta =$ … $\frac{y}{}$ $\sin 2\theta$

$(\sigma_z = \sigma$

$\tau_{\xi\eta} = -$

$\tau_{\xi z} = \tau$

$\tau_{\eta z} = -$

- For … ins are fully
- So v … can be said

1. Pure Out-Of-Plane Shear ($\gamma_{xz} \neq 0$)

- The stresses and strains are,

$$\begin{aligned}
\sigma_\xi &= 0\\
\sigma_\eta &= 0\\
(\sigma_z &= 0)\\
\tau_{\xi\eta} &= 0\\
\begin{bmatrix}\tau_{\xi z}\\ \tau_{\eta z}\end{bmatrix} &= \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\tau_{xz}\\ \tau_{yz}\end{bmatrix}\\
&= \begin{bmatrix}C_{55}\gamma_{xz}\cos\theta\\ -C_{55}\gamma_{xz}\sin\theta\end{bmatrix} := \begin{bmatrix}C_{55}\gamma_{\xi\eta}\\ C_{66}\gamma_{\eta z}\end{bmatrix}.
\end{aligned}$$

$$\begin{aligned}
\varepsilon_\xi &= 0\\
\varepsilon_\eta &= 0\\
(\varepsilon_z &= 0)\\
\gamma_{\xi\eta} &= 0\\
\begin{bmatrix}\gamma_{\xi z}\\ \gamma_{\eta z}\end{bmatrix} &= \begin{bmatrix}\gamma_{xz}\cos\theta\\ -\gamma_{xz}\sin\theta\end{bmatrix}
\end{aligned}$$

- Under symmetry, $(\tau_{\xi z}, \tau_{\eta z})$ is related to $(\gamma_{\xi z}, \gamma_{\eta z})$ in the same way that (τ_{xz}, τ_{yz}) is related to $(\gamma_{xz}, \gamma_{yz})$.
- So we have,

$$\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\
 & C_{22} & C_{23} & 0 & 0 & 0\\
 & & C_{33} & 0 & 0 & 0\\
 & & & C_{44} & 0 & 0\\
 & \text{sym} & & & C_{55} & 0\\
 & & & & & \cancelto{C_{55}}{C_{66}}
\end{bmatrix}$$

4.1. Material Symmetry and Anisotropy

2. Pure Out-Of-Plane Stretch ($\varepsilon_z \neq 0$)

- We have straight stresses $\sigma_x = C_{13}\varepsilon_z, \sigma_y = C_{23}\varepsilon_z$.
- Upon transformation we have,

$$\begin{aligned}
\sigma_\xi &= \left(\frac{C_{13}+C_{23}}{2} + \frac{C_{13}-C_{23}}{2}\cos 2\theta\right)\varepsilon_z \\
\sigma_\eta &= \left(\frac{C_{13}+C_{23}}{2} - \frac{C_{13}-C_{23}}{2}\cos 2\theta\right)\varepsilon_z \\
\sigma_z &= \sigma_z \\
\tau_{\xi\eta} &= -\frac{C_{13}-C_{23}}{2}\sin 2\theta \\
\tau_{\xi z} &= \tau_{\eta z} = 0
\end{aligned}$$

$$\begin{aligned}
\varepsilon_\xi &= 0 \\
\varepsilon_\eta &= 0 \\
\varepsilon_z &= \varepsilon_z \\
\gamma_{\xi\eta} &= 0 \\
\gamma_{\xi z} &= \gamma_{\eta z} = 0
\end{aligned}$$

- For planar isotropy, the relationship between $(\sigma_\xi, \sigma_\eta)$ and σ_z must be independent of θ. This is only possible for $C_{13} = C_{23}$.
- So we have,

$$\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
 & C_{22} & \cancel{C_{23}} \nearrow C_{13} & 0 & 0 & 0 \\
 & & C_{33} & 0 & 0 & 0 \\
 & & & C_{44} & 0 & 0 \\
 & \text{sym} & & & C_{55} & 0 \\
 & & & & & C_{55}
\end{bmatrix}$$

- The

$\sigma_\xi = \frac{\sigma}{}$ … sin 2θ

$\sigma_\eta = \frac{\sigma}{}$ … sin 2θ

$(\sigma_z = \sigma$

$\tau_{\xi\eta} = -$

$\tau_{\xi z} = \tau$

$\tau_{\eta z} = -$

- For … ns are fully
- So w … can be said.

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

3. Pure In-Plane Stretch ($\varepsilon_x \neq 0, \varepsilon_y = 0$)

- From the constitutive properties we have $\sigma_x = C_{11}\varepsilon_x$ and $\sigma_y = C_{12}\varepsilon_x$.
- Using this all the other components can be written as

$$\begin{aligned}
\sigma_\xi &= \left(\frac{C_{11}+C_{12}}{2} + \frac{C_{11}-C_{12}}{2}\cos 2\theta\right)\varepsilon_x & \varepsilon_\xi &= \frac{1+\cos 2\theta}{2}\varepsilon_x\\
\sigma_\eta &= \left(\frac{C_{11}+C_{12}}{2} + \frac{C_{11}-C_{12}}{2}\cos 2\theta\right)\varepsilon_x & \varepsilon_\eta &= \frac{1-\cos 2\theta}{2}\varepsilon_x\\
&= C_{12}\varepsilon_x + C_{22}\varepsilon_y & &\\
\sigma_z &= 0 & \varepsilon_z &= 0\\
\tau_{\xi\eta} &= 0 & \gamma_{\xi\eta} &= 0\\
\tau_{\xi z} &= \tau_{\eta z} = 0. & \gamma_{\xi z} &= \gamma_{\eta z} = 0.
\end{aligned}$$

- For the σ_η equality to hold, we need $C_{22} = C_{11}$. So we have

$$\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\
 & \cancel{C_{22}} \nearrow C_{11} & C_{13} & 0 & 0 & 0\\
 & & C_{33} & 0 & 0 & 0\\
 & & & C_{44} & 0 & 0\\
 & \text{sym} & & & C_{55} & 0\\
 & & & & & C_{55}
\end{bmatrix}$$

- Th

$\sigma_\xi =$ … $\frac{xy}{2}\sin 2\theta$

$\sigma_\eta =$ … $\frac{xy}{2}\sin 2\theta$

$(\sigma_z =$

$\tau_{\xi\eta} =$

$\tau_{\xi z} =$

$\tau_{\eta z} =$

- For … ains are full
- So … can be said

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

4. Pure In-Plane Shear ($\gamma_{xy} \neq 0$)

- From the constitutive properties we have $\tau_{xy} = C_{44}\gamma_{xy}$.
- Using this all the other components can be written as

$$\begin{aligned}
\sigma_\xi &= C_{44}\gamma_{xy}\sin 2\theta = C_{11}\varepsilon_\xi + C_{12}\varepsilon_\eta \\
\sigma_\eta &= -C_{44}\gamma_{xy}\sin 2\theta = C_{12}\varepsilon_\xi + C_{11}\varepsilon_\eta \\
\sigma_z &= 0 \\
\tau_{\xi\eta} &= C_{44}\gamma_{xy}\cos 2\theta \\
\tau_{\xi z} &= \tau_{\eta z} = 0.
\end{aligned}
\qquad
\begin{aligned}
\varepsilon_\xi &= \frac{\gamma_{xy}}{2}\sin 2\theta \\
\varepsilon_\eta &= -\frac{\gamma_{xy}}{2}\sin 2\theta \\
\varepsilon_z &= 0 \\
\gamma_{\xi\eta} &= \gamma_{xy}\cos 2\theta \\
\gamma_{\xi z} &= \gamma_{\eta z} = 0.
\end{aligned}$$

- So we have $C_{44}\gamma_{xy}\sin 2\theta = \frac{C_{11}-C_{12}}{2}\gamma_{xy}\sin 2\theta$. Therefore,

$$\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
 & C_{11} & C_{13} & 0 & 0 & 0 \\
 & & C_{33} & 0 & 0 & 0 \\
 & & & \overset{\frac{C_{11}-C_{12}}{2}}{\cancel{C_{44}}} & 0 & 0 \\
 & \text{sym} & & & C_{55} & 0 \\
 & & & & & C_{55}
\end{bmatrix}$$

4.1. Material Symmetry and Anisotropy

Macro-Mechanics Descriptions

- The stresses and strains transform as follows on the plane:

σ_ξ … $\frac{\gamma_{xy}}{2}\sin 2\theta$

σ_η … $\frac{\gamma_{xy}}{2}\sin 2\theta$

$(\sigma_z$

$\tau_{\xi\eta}$ … 2θ

$\tau_{\xi z}$

$\tau_{\eta z}$

- … strains are

- … ore can be said.

To Summarize,
a **Transversely Isotropic Material**
constitution can be expressed as

$$\begin{bmatrix}\sigma_x\\\sigma_y\\\sigma_z\\\tau_{xy}\\\tau_{xz}\\\tau_{yz}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{11} & C_{13} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{55}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\varepsilon_z\\\gamma_{xy}\\\gamma_{xz}\\\gamma_{yz}\end{bmatrix}$$

The material is fully characterized by five engineering constants.

4.1. Material Symmetry and Anisotropy: Engineering Constants

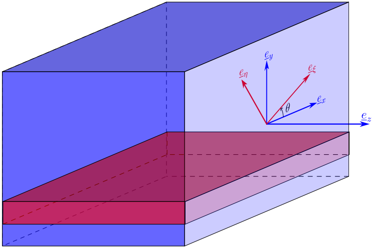
Macro-Mechanics Descriptions

- In engineering practice, the constants are usually written easier in terms of compliance.
- For a specially orthotropic material the strain-stress relationship are usually expressed as,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & -\frac{\nu_{31}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{32}}{E_{33}} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_{11}} & -\frac{\nu_{23}}{E_{22}} & \frac{1}{E_{33}} & 0 & 0 & 0 \\ & & & \frac{1}{G_{12}} & 0 & 0 \\ & \text{sym} & & & \frac{1}{G_{13}} & 0 \\ & & & & & \frac{1}{G_{23}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix}$$

5. Analysis of Planar Laminates

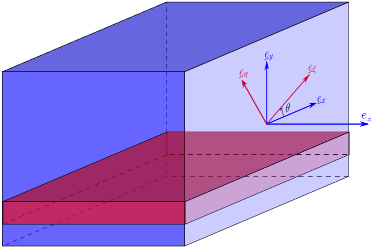
- Let us just consider one thin layer of a transversely isotropic material (continuously reinforced composite along a single direction).



$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{11} & C_{13} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{55}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

5. Analysis of Planar Laminates

- Let us just consider one thin layer of a transversely isotropic material (continuously reinforced composite along a single direction).



$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{11} & C_{13} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0\\ & \text{sym} & & & C_{55} & 0\\ & & & & & C_{55}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}$$

- We invoke plane stress assumptions, setting $\sigma_y = 0$. Let us also assume small shears, $\tau_{xy} = 0, \tau_{yz} = 0$.
  (Note: ε_z is not zero, and is implicitly defined)

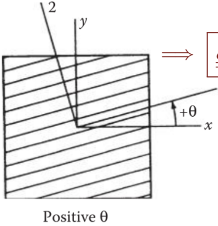
$$\boxed{\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{33}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix}} \text{(4 constants)}$$

(Note change in notation in C_{ij})

5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\Longrightarrow \boxed{\underline{\sigma}_{(x,y)} = \underline{\underline{Q}}\,\underline{\sigma}_{(1,2)}\underline{\underline{Q}}^T}$$

(Figure 2.11 from Gibson 2012)

- What if the coordinate system is not aligned with the fiber axes? The stress and strains transform
- In the constitutive relationship we have,

$$\underline{\sigma}_{(1,2)} = \underline{\underline{C}}\,\underline{\varepsilon}_{(1,2)}$$

$$\underline{\underline{T}}_{\sigma}^{-1}\underline{\sigma}_{(x,y)} = \underline{\sigma}_{(1,2)} = \underline{\underline{C}}\,\varepsilon_{(1,2)} = \underline{\underline{C}}\,\underline{\underline{T}}_{\varepsilon}^{-1}\underline{\varepsilon}_{(x,y)}$$

$$\Longrightarrow \underline{\sigma}_{(x,y)} = \underbrace{\underline{\underline{T}}_{\sigma}\underline{\underline{C}}\,\underline{\underline{T}}_{\varepsilon}^{-1}}_{\underline{\underline{C}}'}\varepsilon_{(x,y)}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underline{\underline{T}}_{\sigma}} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

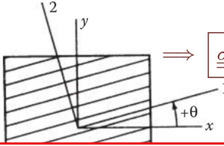
$$\text{where } \underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}.$$

$$\underline{\underline{T}}_{\sigma}^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\Longrightarrow \boxed{\underline{\underline{\sigma}}_{(x,y)} = \underline{\underline{Q}}\,\underline{\underline{\sigma}}_{(1,2)}\underline{\underline{Q}}^T}$$

Note that Strain Transformation looks slightly different because of our definition of shear strain $\gamma_{xy} = 2\varepsilon_{xy}$.

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2\theta & \sin^2\theta & -\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & \cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underline{\underline{T}}_\varepsilon} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_y \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \sin\theta & \cos\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underline{\underline{T}}_\sigma} \begin{bmatrix} \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\underline{\underline{T}}_\sigma^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

- What if the coordinate system is not aligned with the fiber axes? The stress ansform

utive relationship we have,

$$_{(1,2)} = \underline{\underline{C}}\,\underline{\varepsilon}_{(1,2)}$$

$$_{(1,2)} = \underline{\underline{C}}\,\varepsilon_{(1,2)} = \underline{\underline{C}}\,\underline{\underline{T}}_\varepsilon^{-1}\underline{\varepsilon}_{(x,y)}$$

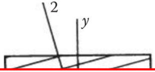
$$_{(x,y)} = \underbrace{\underline{\underline{T}}_\sigma\underline{\underline{C}}\,\underline{\underline{T}}_\varepsilon^{-1}}_{\underline{\underline{C}}'}\varepsilon_{(x,y)}$$

where $\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}$.

5.1. Generally Orthotropic Laminates: In-Plane Rotational Transformations

Analysis of Planar Laminates

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\underline{\underline{Q}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\Longrightarrow \boxed{\underline{\underline{\sigma}}_{(x,y)} = \underline{\underline{Q}}\,\underline{\underline{\sigma}}_{(1,2)}\underline{\underline{Q}}^T}$$

Transformed $\underline{\underline{C}}$ Matrix ($\underline{\sigma} = \underline{\underline{C}}\,\underline{\varepsilon}$)

$$\underline{\underline{C}}' = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} \\ C'_{12} & C'_{22} & C'_{23} \\ C'_{13} & C'_{23} & C'_{33} \end{bmatrix}$$

$$C'_{11} = C_{11}c^4 + C_{22}s^4 + (2C_{33} + C_{12})2c^2s^2$$

$$C'_{22} = C_{11}s^4 + C_{22}c^4 + (2C_{33} + C_{12})2c^2s^2$$

$$C'_{33} = (C_{11} + C_{22} - 2C_{33} - 2C_{12})c^2s^2 + C_{33}(c^4 + s^4)$$

$$C'_{12} = (C_{11} + C_{22} - 4C_{33})c^2s^2 + C_{12}(c^4 + s^4)$$

$$C'_{13} = (C_{11} - 2C_{33} - C_{12})c^3s - (C_{22} - 2C_{33} - C_{12})cs^3$$

$$C'_{23} = (C_{11} - 2C_{33} - C_{12})cs^3 - (C_{22} - 2C_{33} - C_{12})c^3s.$$

the coordinate system is not
with the fiber axes? The stress
ansform

utive relationship we have,

$$_{(1,2)} = \underline{\underline{C}}\,\varepsilon_{(1,2)}$$

$$_{(1,2)} = \underline{\underline{C}}\,\varepsilon_{(1,2)} = \underline{\underline{C}}\,\underline{\underline{T}}_\varepsilon^{-1}\varepsilon_{(x,y)}$$

$$_{(x,y)} = \underbrace{\underline{\underline{T}}_\sigma\underline{\underline{C}}\,\underline{\underline{T}}_\varepsilon^{-1}}_{\underline{\underline{C}}'}\varepsilon_{(x,y)}$$

$$\begin{bmatrix} \sigma_y \\ \tau_{xy} \end{bmatrix}$$

ere $\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}$.

$$\underline{\underline{T}}_\sigma^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

5.1. Generally Orthotropic Laminates

Analysis of Planar Laminates

- Compliance is often more convenient:

$$\underline{\varepsilon}_{(x,y)} = \underline{\underline{T}}_{\varepsilon}\underline{\underline{S}}\,\underline{\underline{T}}_{\sigma}^{-1}\underline{\sigma}_{(x,y)}$$

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}S'_{11} & S'_{12} & S'_{13}\\ & S'_{22} & S'_{23}\\ & & S'_{33}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

$$\begin{aligned}
S'_{11} &= S_{11}c^4 + S_{22}s^4 + (S_{33} + 2S_{12})c^2s^2\\
S'_{22} &= S_{11}s^4 + S_{22}c^4 + (S_{33} + 2S_{12})c^2s^2\\
S'_{33} &= (2S_{11} + 2S_{22} - S_{33} - 4S_{12})2c^2s^2 + S_{33}(c^4 + s^4)\\
S'_{12} &= (S_{11} + S_{22} - S_{33})c^2s^2 + S_{12}(c^4 + s^4)\\
S'_{13} &= (2S_{11} - S_{33} - 2S_{12})c^3s - (2S_{22} - S_{33} - 2S_{12})cs^3\\
S'_{23} &= (2S_{11} - S_{33} - 2S_{12})cs^3 - (2S_{22} - S_{33} - 2S_{12})c^3s.
\end{aligned}$$

- In the material principal directions we have,

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G_{12}}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}$$

Engineering Constants: $E_1, E_2, G_{12}, \nu_{12}$

- Based on this we can write,

$$E_x = \left[\frac{c^4}{E_1} + \frac{s^4}{E_2} + \left(\frac{1}{G_{12}} - \frac{2\nu_{21}}{E_2}\right)c^2s^2\right]^{-1}$$

$$E_y = \left[\frac{s^4}{E_1} + \frac{c^4}{E_2} + \left(\frac{1}{G_{12}} - \frac{2\nu_{21}}{E_2}\right)c^2s^2\right]^{-1}$$

$$\begin{aligned}
G_{xy} =&\Bigg[\frac{c^4 + s^4}{G_{12}} + \Bigg(\frac{1}{E_1} + \frac{1}{E_2}\\
&- \frac{1}{2G_{12}} + 2\frac{\nu_{21}}{E_2}\Bigg)4c^2s^2\Bigg]^{-1}
\end{aligned}$$

$$\begin{aligned}
\nu_{yx} =& E_y\Bigg[\frac{\nu_{21}}{E_2}(c^4 + s^4)\\
&- \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)c^2s^2\Bigg]
\end{aligned}$$

- It is customary to express the laminate constitutive relationship as

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}}\\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}}\\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

5.1. Generally Orthotropic Laminates

Analysis of Planar Laminates

- Compliance is often more convenient:

$$\underline{\varepsilon}_{(x,y)} = \underline{\underline{T}}_{\varepsilon}\underline{\underline{S}}\,\underline{\underline{T}}_{\sigma}^{-1}\underline{\sigma}_{(x,y)}$$

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}S'_{11} & S'_{12} & S'_{13}\\ & S'_{22} & S'_{23}\\ & & S'_{33}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

$$S'_{11} = S_{11}c^4 + S_{22}s^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{22} = S_{11}s^4 + S_{22}c^4 + (S_{33} + 2S_{12})c^2s^2$$

$$S'_{33} = (2S_{11} + 2S_{22} - S_{33} - 4S_{12})2c^2s^2 + S_{33}(c$$

$$S'_{12} = (S_{11} + S_{22} - S_{33})c^2s^2 + S_{12}(c^4 + s^4)$$

$$S'_{13} = (2S_{11} - S_{33} - 2S_{12})c^3s - (2S_{22} - S_{33} - 2S_{12})cs^3$$

$$S'_{23} = (2S_{11} - S_{33} - 2S_{12})cs^3 - (2S_{22} - S_{33} - 2S_{12})c^3s.$$

- In the material principal directions we have,

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G_{12}}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}$$

Engineering Constants: $E_1, E_2, G_{12}, \nu_{12}$

- Based on this we can write,

The Shear Constants can be written as

$$\begin{aligned}\eta_{xy,x} =& G_{xy}\left[\left(\frac{2}{E_1} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\right)c^3s\right.\\ &\left. - \left(\frac{2}{E_2} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\right)cs^3\right]\\ \eta_{xy,y} =& G_{xy}\left[\left(\frac{2}{E_1} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\right)cs^3\right.\\ &\left. - \left(\frac{2}{E_2} - \frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\right)c^3s\right]\end{aligned}$$

$$\nu_{yx} = E_y\Bigg[\ \frac{\ }{E_2}(c\ + s\ )$$

$$- \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)c^2s^2\Bigg]$$

- It is customary to express the laminate constitutive relationship as

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix}\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}}\\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}}\\ \frac{\eta_{x,xy}}{E_x} & \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

5.1. Generally Orthotropic Laminates

Analysis of Planar Laminates

- Compliance is often r

$$\underline{\varepsilon}_{(x,y)} = \underline{\underline{T}}_{\varepsilon}\underline{\underline{S}}\,\underline{\underline{T}}_{\sigma}^{-1}\underline{\sigma}_{(x,y}$$

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \Bigg[\begin{matrix}S'_{11} & S'_{12}\\ & S'_{22}\\ & \end{matrix}$$

$$S'_{11} = S_{11}c^4 + S_{22}s^4$$

$$S'_{22} = S_{11}s^4 + S_{22}c^4$$

$$S'_{33} = (2S_{11} + 2S_{22} -$$

$$S'_{12} = (S_{11} + S_{22} - S$$

$$S'_{13} = (2S_{11} - S_{33} - 2$$

$$S'_{23} = (2S_{11} - S_{33} - 2$$

- In the material princ

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix} = \Bigg[\begin{matrix}\frac{1}{E_1}\\ -\frac{\nu_1}{E_1}\\ 0\end{matrix}$$

write,

can be written as

$$\frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\Big) c^3 s$$

$$+ \frac{2\nu_{21}}{E_2}\Big) cs^3\Big]$$

$$\frac{1}{G_{12}} + \frac{2\nu_{21}}{E_2}\Big) cs^3$$

$$+ \frac{2\nu_{21}}{E_2}\Big) c^3 s\Big]$$

$$\frac{1}{G_{12}}\Big) c^2 s^2\Big]$$

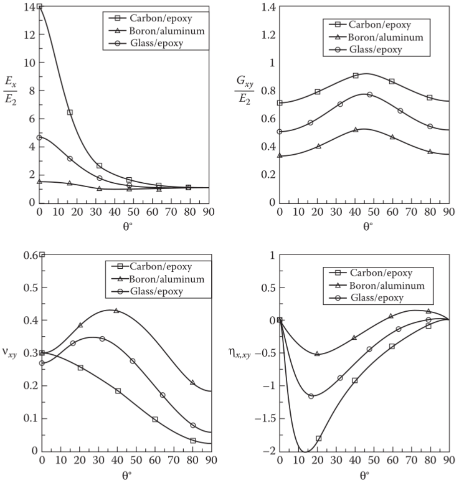
ress the laminate
ip as

$$\begin{matrix}\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy,x}}{G_{xy}}\\ \frac{1}{E_y} & \frac{\eta_{xy,y}}{G_{xy}}\\ \frac{\eta_{y,xy}}{E_y} & \frac{1}{G_{xy}}\end{matrix}\Bigg]\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

$$\gamma_{xy}\Big]$$

Engineering Constants: $E_1, E_2, G_{12}, \nu_{12}$

Off-Axis Modulii



(*Figure 2.14 from Gibson 2012*)

5.2. Numerical Examples: 1

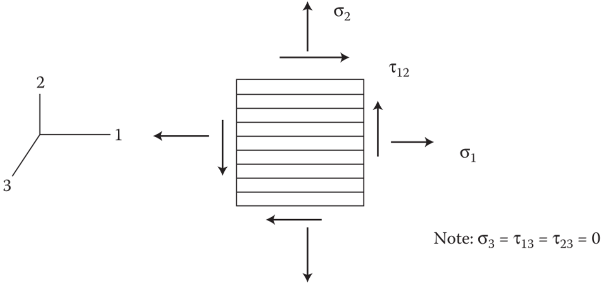
Analysis of Planar Laminates(Example 2.2 from Gibson 2012)

Consider an orthotropic laminate with the properties

$$E_1 = 140\,\text{GPa},\ E_2 = 10\,\text{GPa},\ G_{12} = 7\,\text{GPa},\ \nu_{12} = 0.3,\ \nu_{23} = 0.2.$$

Compute the strains if it is subjected to the following state of stress in the principal coordinates:

$$\sigma_1 = 70\,\text{MPa},\ \sigma_2 = 140\,\text{MPa},\ \tau_{12} = 35\,\text{MPa},\ \sigma_3 = \tau_{12} = \tau_{23} = 0.$$

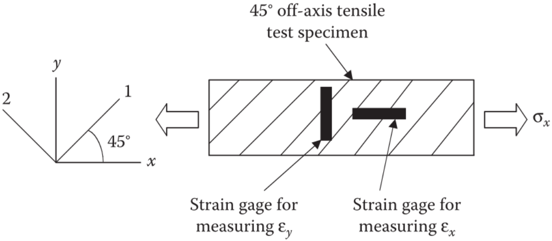


(Figure 2.10 from Gibson 2012)

5.2. Numerical Examples: 2

Analysis of Planar Laminates(Example 2.3 from Gibson 2012)

A 45° off-axis tensile test is conducted on a generally orthotropic test specimen by applying a normal stress σ_x. The specimen has strain gauges attached to measure axial and transverse strains $(\varepsilon_x, \varepsilon_y)$. How many engineering parameters can be estimated from measurements of $\sigma_x, \varepsilon_x, \varepsilon_y$?



(*Figure 2.15 from Gibson 2012*)

6. Classical Laminate Theory

- In the Kirchhoff-Love Plate Theory we had,

$$\begin{bmatrix}\underline{N}\\ \underline{M}\end{bmatrix} = \begin{bmatrix}\underline{\underline{A}} & \underline{\underline{B}}\\ \underline{\underline{B}} & \underline{\underline{D}}\end{bmatrix}\begin{bmatrix}\underline{u'}\\ \underline{w''}\end{bmatrix}$$

where

$$\underline{\underline{A}} = \frac{Et}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}, \quad \underline{\underline{D}} = \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}, \quad \underline{\underline{B}} = \underline{\underline{0}}.$$

- This can also be written in terms of thickness moments of the constitutive matrix $\underline{\underline{C}} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}$ as

$$\underline{\underline{A}} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \underline{\underline{C}}\, dz, \quad \underline{\underline{B}} = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\underline{\underline{C}}\, dz, \quad \underline{\underline{D}} = \int_{-\frac{t}{2}}^{\frac{t}{2}} z^2\underline{\underline{C}}\, dz.$$

6. Classical Laminate Theory

- Suppose we had different laminate plies along the thickness, such that the constitutive matrix is $\underline{\underline{C}}_i$ for $z \in (z_i, z_{i+1})$ and $-\frac{t}{2} = z_1 < \cdots < z_N = \frac{t}{2}$.
- Then the $A - B - D$ matrices are written as the sums,

$$\underline{\underline{A}} = \sum_i (z_{i+1} - z_i)\underline{\underline{C}}_i, \quad \underline{\underline{B}} = \sum_i \frac{z_{i+1}^2 - z_i^2}{2}\underline{\underline{C}}_i, \quad \underline{\underline{D}} = \sum_i \frac{z_{i+1}^3 - z_i^3}{3}\underline{\underline{C}}_i.$$

- Unlike isotropic plates, composite laminates can have non-zero $\underline{\underline{B}}$ matrix (moment-planar coupling), bending-twisting coupling, etc.
- This $\begin{bmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}} & \underline{\underline{D}} \end{bmatrix}$ matrix is known as the **Laminate Stiffness Matrix**.

6.1. The Laminate Orientation Code

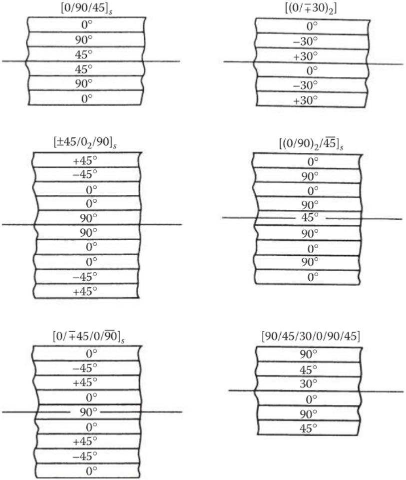
Classical Laminate Theory

- Ply angles separated by slashes, ordered from top to bottom
- Subscript “s” for symmetric laminates
- Numerical subscripts for repetitions
- Center ply with an overbar for odd laminates

(See sec. 7.1 in Gibson 2012)

Types

- Symmetric, Antisymmetric, Asymmetric
- Angle-Ply, Cross-Ply, Balanced, $\pi/4$ laminates



(Figure 7.1 from Gibson 2012)

6.1. The Laminate Orientation Code

Classical Laminate Theory

- Ply angles separate
  from top to bottom
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- Symmetric, Antisymi
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Summary of Laminate Stiffnesses

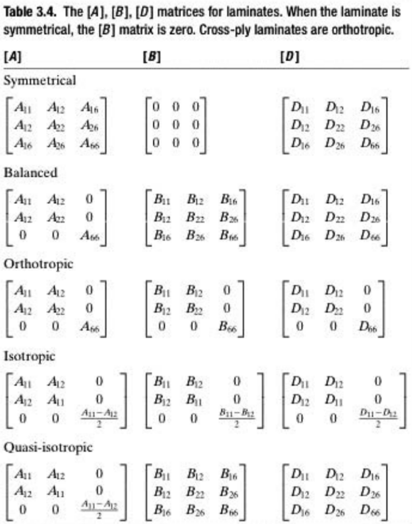
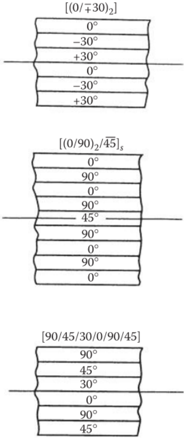


Table 3.4. The [*A*], [*B*], [*D*] matrices for laminates. When the laminate is symmetrical, the [*B*] matrix is zero. Cross-ply laminates are orthotropic.

	[*A*]	**[*B*]**	**[*D*]**
Symmetrical	$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$
Balanced	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$
Orthotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$
Isotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}-A_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & \frac{B_{11}-B_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & \frac{D_{11}-D_{12}}{2} \end{bmatrix}$
Quasi-isotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}-A_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$

(Table 3.4 from Kollár and Springer 2003)



Gibson 2012)

6.2. Laminated Beams

Classical Laminate Theory

- Consider a beam with a symmetric section on the $x-y$ plane. Invoking Kirchhoff kinematic assumptions we have: $\varepsilon_x = u' - yv''$.
- The stress distribution will depend on the section-coordinate. In general we will have: $\sigma_x = E_x(y)\varepsilon_x = E_x(y)\left(u' - yv''\right)$.
- We get the effective normal reaction N_x by integrating the stress over the section:

$$N_x = \int_{\mathcal{A}} \sigma_x = \left[\int_{\mathcal{A}} E_x(y)\right] u' + \left[\int_{\mathcal{A}} -yE_x(y)\right] v''.$$

- Similarly we get the bending moment M_z as the first moment of the stress,

$$M_z = \int_{\mathcal{A}} -y\sigma_x = \left[\int_{\mathcal{A}} -yE_x(y)\right] u' + \left[\int_{\mathcal{A}} y^2 E_x(y)\right] v''.$$

- In summary we have the beam-analog of the laminate stiffness matrix,

$$\begin{bmatrix} N_x \\ M_z \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} u' \\ v'' \end{bmatrix}.$$

Important note: We have assumed that no torsion/twist is present. See Kollár and Springer 2003 for the general form.

6.2. Laminated Beams

Classical Laminate Theory

- For a laminated composite with a rectangular section with width b, the integrals may be simplified as,

$$A = \int_{\mathcal{A}} E_x(y) = \sum_{i=1}^{N} E_{x,i} b(y_{i+1} - y_i), \quad B = \int_{\mathcal{A}} -yE_x(y) = -\sum_{i=1}^{N} E_{x,i} b \frac{y_{i+1}^2 - y_i^2}{2}$$

$$D = \int_{\mathcal{A}} y^2 E_x(y) = \sum_{i=1}^{N} E_{x,i} b \frac{y_{i+1}^3 - y_i^3}{3}.$$

- For plies of uniform thickness we can write

$$y_i = -\frac{h}{2} + (i-1)\frac{h}{N},$$

which leads to:

$$A = \frac{h}{N}\sum_{i=1}^{N} E_{x,i}, \; B = \frac{h^2}{2N^2}\sum_{i=1}^{N} E_{x,i}(2i - N - 1),$$

$$D = \frac{h^3}{12N^3}\sum_{i=1}^{N} E_{x,i}(12i^2 - 12Ni + 12N^2 + 3N^2 + 6N + 4)$$

6.3. Numerical Example

Classical Laminate Theory

Determine the ABD matrix for the following composite beams where the ply thickness is 1 mm and beam width is 10 mm:

- $[0/90]_s$, and
- $[0/90/0/90]$.

Assume the following properties for each lamina: $E_1 = 140\,\text{GPa}$, $E_2 = 10\,\text{GPa}$, $G_{12} = 7\,\text{GPa}$, $\nu_{12} = 0.3$, $\nu_{23} = 0.2$.

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