AS2070: Aerospace Structural Mechanics (V3)

Module 1: Elastic Stability

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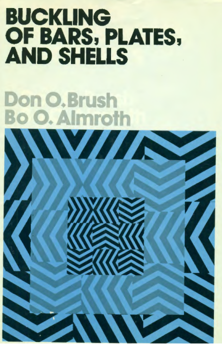
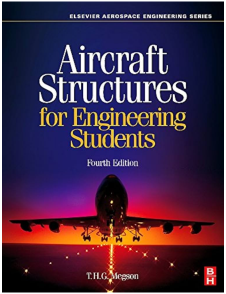


Chapters 1-3 in Brush and Almroth (1975).

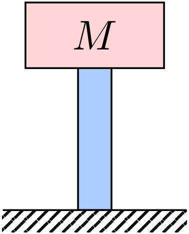


Chapters 7-9 in Megson (2013)

1. Introduction

Structural Stability: What?

- Consider supporting a mass M on the top of a rod.



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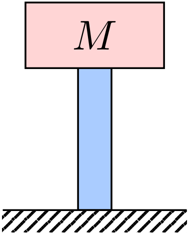
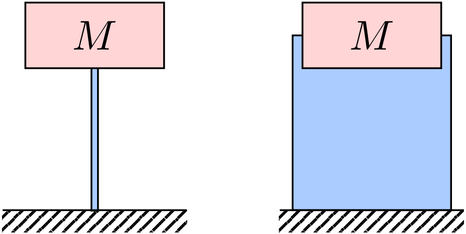


Two Extreme Cases:



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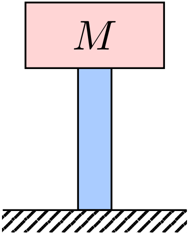

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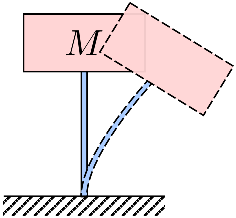
- Collapse is imminent on at least one!



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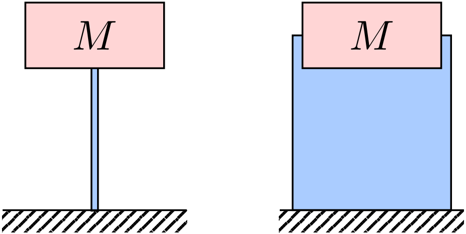

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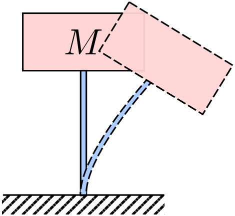
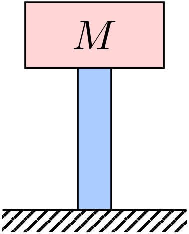


How can we mathematically describe this?



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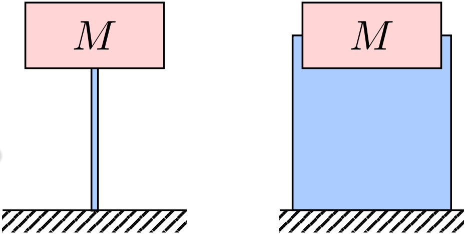


1. Introduction

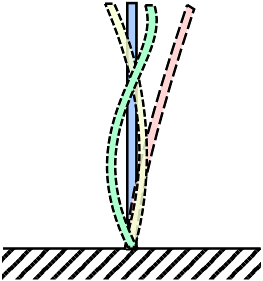
Structural Stability: Perturbation Behavior

Perturbation Behavior

Key insight we will invoke is behavior under **perturbation**:

How would the system respond if I slightly perturb it?

- Mathematically, by perturbation we mean *any change to the system's configuration.*
- In this case, this could be different deflection shapes.



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Structural Stability: Perturbation Behavior

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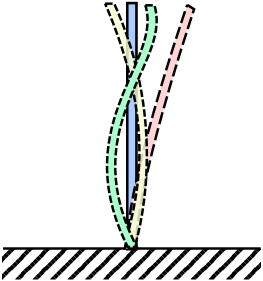
How would the system respond if I slightly perturb it?

- Mathematically, by perturbation we mean *any change to the system's configuration.*
- In this case, this could be different deflection shapes.

Question (Slightly more specific)

What will the system tend to do if an arbitrarily small magnitude of perturbation is introduced?

- Will it tend to **return to its original configuration**?
- Will it **blow up**?
- Will it do **something else entirely**?



1.1. Elastic Stability
Introduction

What do these words mean?

Elastic → Reversible → Conservative

Conservative System

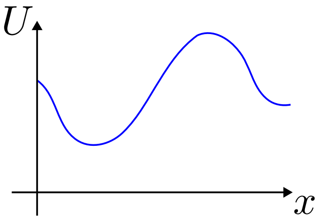
- The restoring force of a conservative system can be written using a gradient of a **potential function**:
$$\underline{F} = -\nabla U.$$

Equilibrium

- System achieves equilibrium when $\underline{F} = \underline{0}$, i.e.,
$$\nabla U = 0.$$

1D Example

Consider a system whose configuration is expressed by the scalar x and the potential is as shown.



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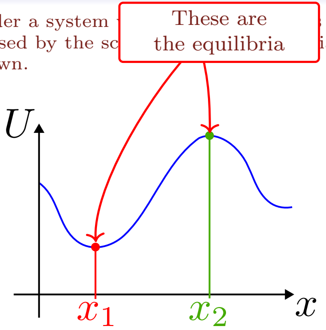

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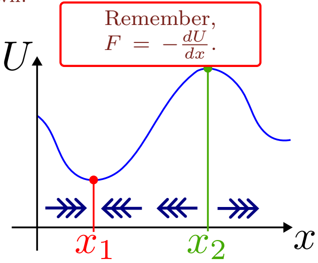

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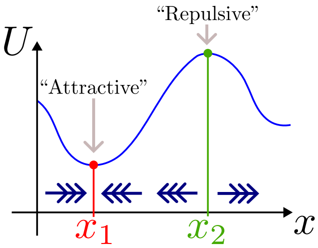

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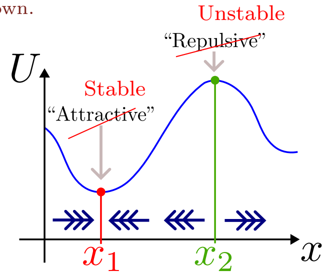
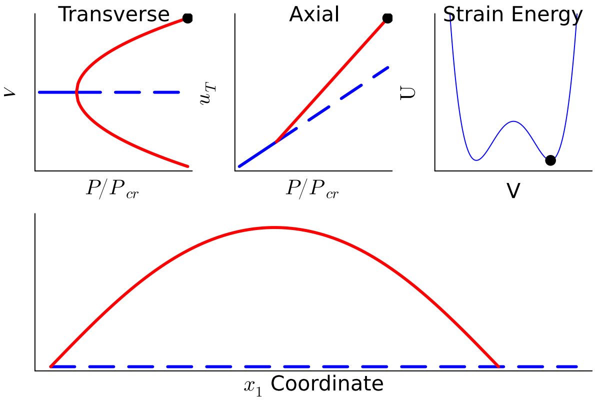


1.2. Bifurcation

Introduction

A system is said to have **undergone a bifurcation** if its state of stability has changed due to the variation of some parameter.

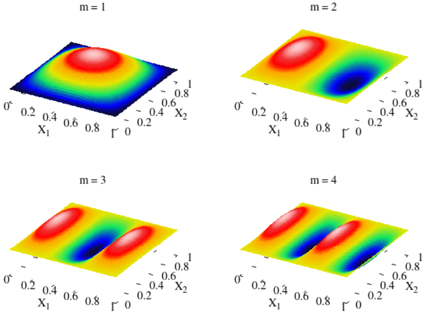


Example: *A pinned-pinned beam undergoing axial loading.*

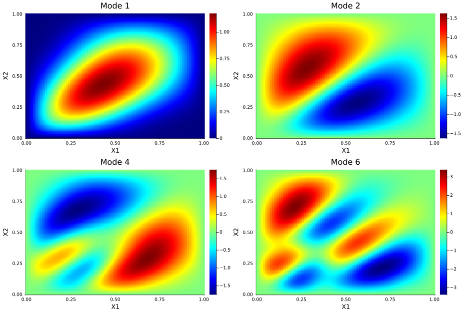
1.3. Modes of Stability Loss

Introduction

The **configuration** that a system can assume as it undergoes a bifurcation is the *mode* of the stability loss.



Example: *Thin plate (pinned) under axial loading*



Example: *Thin plate (pinned) under shear loading*

2. The Principle of Virtual Work

An Optimization Framework for Mechanics

- Let us consider a point supported by a linear spring acted upon by a force.

Kinematics

Assume displacement to be restricted to the $\underline{e_x}$ axis:

$$\underline{u} = u_x \underline{e_x}.$$

Equilibrium Condition

Net forces acting on the object is zero at equilibrium:

$$\sum F_x = F - F_{spring} = 0.$$

Constitutive Modeling

Spring reaction is linearly proportional to displacement.

$$F_{spring} = k u_x.$$

- While the above three already allow us to solve the problem of finding the equilibrium configuration of the system, we turn our attention to what happens in the vicinity of the equilibrium now.
- We define **Virtual Displacement** as an *infinitesimal displacement that is consistent with constraints*. (we read kinematic assumptions as constraints here) And we denote this by $\delta \underline{u}$.
- Since this is infinitesimal by definition, the associated work done, i.e., **Virtual Work** is written as

$$\delta W = \left(\sum F_x\right) \delta u_x = (F - k u_x) \delta u_x.$$

- Being a small quantity, regular calculus rules apply to the $\delta(\cdot)$ if considered as an infinitesimal operator just as they do for $d(\cdot)$, so the above simplifies as:

$$\delta W = \delta \left(F u_x - \frac{k}{2} u_x^2 \right).$$

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We have now defined a new quantity: **Work Potential**, with units of energy. This is written as $\boxed{W = \Pi - U}$.

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Load Contribution $\mathbf{\Pi = F u_x}$

Elastic Contribution $\mathbf{U = \frac{1}{2} k u_x^2}$

$$\delta W = \delta \left(F u_x - \frac{k}{2} u_x^2 \right).$$

2. The Principle of Virtual Work

An Optimization Framework for Mechanics

- The **Work Potential** is quite a helpful quantity for us because
  1. **the first derivative $\partial W/\partial u_x$ represents the overall force $\sum F_x$** acting on the system (static equilibrium is the same as stating $\partial W/\partial u_x = 0$), and
  2. **the second derivative $\partial^2 W/\partial u_x^2$ at equilibrium represents the surplus force as we move away** from the equilibrium (because this represents $\partial(\sum F_x)/\partial u_x$).
- An equilibrium can be sought by finding u_x for $\partial W/\varepsilon u_x = 0$.
- For classifying the equilibria that have been found, we use the following principles (based on arguments about what is happening to the surplus force as we move away):
  - If $\partial^2 W/\partial u_x^2 < 0$, the equilibrium is stable.
  - If $\partial^2 W/\partial u_x^2 > 0$, the equilibrium is unstable.
  - If $\partial^2 W/\partial u_x^2 = 0$, the equilibrium is neutrally stable (up to second order).
- The above is mathematically identical to an optimization problem posed as

$$\underset{u_x \in \mathbb{R}}{\text{extremize}} \quad W,$$

and the **maxima of this optimization problem are stable equilibria**, the **minima are unstable**, and **the saddles are neutral**.

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For the Spring Example,

$$W = Fu_x - \frac{k}{2}u_x^2, \quad \frac{\partial W}{\partial u_x} = F - ku_x, \quad \frac{\partial^2 W}{\partial u_x^2} = -k.$$

- There exists exactly only equilibrium point for fixed F, k: $u_x^* = \frac{F}{k}$, and
- this equilibrium is unconditionally stable (always stable).

- The above is mathematically identical to an optimization problem posed as

$$\underset{u_x \in \mathbb{R}}{\text{extremize}} \quad W,$$

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2.1. A Rigid Column Under Axial Load

The Principle of Virtual Work

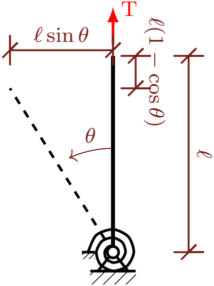
- Now let us consider a rigid column supported by a pin as shown.
- It is also supported by a torsional spring which is governed by a linear constitutive law ($M_{spring} = k\theta$).
- Let's count the work done by each member (only once per member)

Elastic Contributions

- Rotational angle is the work-conjugate of moment.

$$M_{spring}\delta\theta = k\theta\delta\theta = \delta\left(\frac{1}{2}k\theta^2\right),$$

$$\implies \boxed{U = \frac{k}{2}\theta^2}.$$



Load Contributions

- The load at the tip is $T\underline{e_y}$.
- The displacement of the tip is $\boxed{-\ell\sin\theta\underline{e_x} - \ell(1-\cos\theta)\underline{e_y}}$. So we have:

$$\Pi = T\ell(\cos\theta - 1).$$

- We **do not** count moment contributions separately.

- So the work potential is written as: $\Pi - U = \boxed{W = T\ell(\cos\theta - 1) - \frac{k}{2}\theta^2}$.

2.1. A Rigid Column Under Axial Load

The Principle of Virtual Work

Work Potential

$$W = T\ell(\cos\theta - 1) - \frac{k}{2}\theta^2, \quad \frac{\partial W}{\partial \theta} = -(T\ell\sin\theta + k\theta), \quad \frac{\partial^2 W}{\partial\theta^2} = -(T\ell\cos\theta + k).$$

- Under leading term small angle assumptions ($\sin\theta \approx \theta, \cos\theta \approx 1$) we have,

$$\theta_{eq} = 0 \quad \text{(always)}, \left.\frac{\partial^2 W}{\partial\theta^2}\right|_{\theta_{eq}} = -(T\ell + k).$$

  The equilibrium is stable as long as $T > -\frac{k}{\ell}$.

  - Our prediction for $T < -\frac{k}{\ell}$ is just that there exists a **trivial equilibrium** **($\theta_{eq} = 0$) that is unstable**.

- Under two-term small angle assumptions ($\sin\theta \approx \theta - \frac{\theta^3}{6}$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$) we have,

$$\theta_{eq} = 0, \pm\sqrt{6\left(1 + \frac{k}{T\ell}\right)}, \quad \text{with} \quad \frac{\partial^2 W}{\partial\theta^2} = -(T\ell + k), 2(T\ell + k),$$

  where we now predict two additional solutions.

  - Here we have a more precise prediction for $T < -\frac{k}{\ell}$: the non-trivial θ solutions are stable.

2.1. A Rigid Column Under Axial Load

The Principle of Virtual Work

W = … $+ k)$.

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- Ou… $(\theta_{eq} = 0)$ **tha…**
- Under t… …e have,

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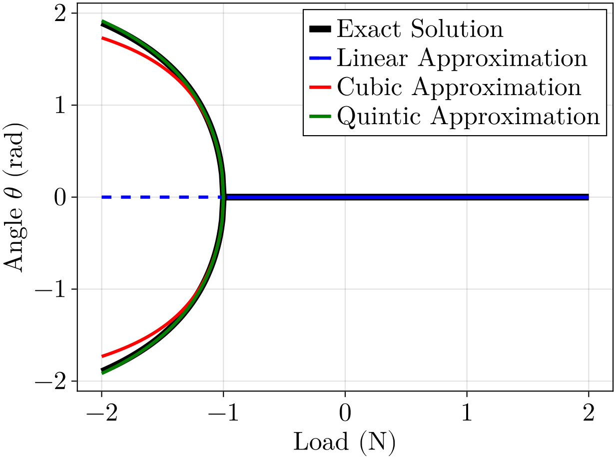
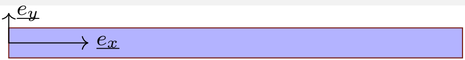
where w…

- Her… …ons are stable.

The "Forced Response Curve" $(k = 1\,\mathrm{N/m}, \ell = 1\,\mathrm{m})$



3. Euler Bernoulli Beam Theory



Kinematic Assumptions

1. Small displacements and rotations
2. Plane sections remain planar and do not change shape
3. Neutral axis remains perpendicular to the section
4. Small strain
5. Von Karman strain assumption

- Assumptions 1 and 2 imply that the deformation field can be written as

$$u_x = u(x) - y\theta(x)$$
$$u_y = v(x)$$

- Assumption 3 implies that $\theta = \frac{dv}{dx} = v'$ so we have

$$u_x = u - yv'$$
$$u_y = v$$

- Assumption 4 implies that the axial strain can be written as

$$\varepsilon_x = u_x' + \frac{1}{2}\left({u_x'}^2 + {u_y'}^2\right)$$

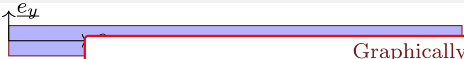
- With assumption 5 we drop the ${u_x'}^2$ term (as it will certainly be smaller than u_x', which is small to begin with as of assumption 4). So we have

$$\boxed{\varepsilon_x = u_x' + \frac{1}{2}{u_y'}^2}.$$

Constitutive Assumption

We shall assume that we're looking at a slender beam so plane stress assumptions are applicable and $\sigma_x = E_y \varepsilon_x$ with E_y being the Young's modulus.

3. Euler Bernoulli Beam Theory



- Assumption 2 implies that $\theta = \frac{dv}{x} = v'$ so

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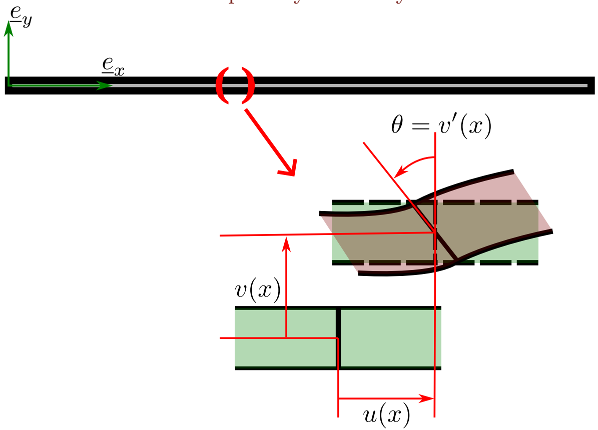
Graphically Summary



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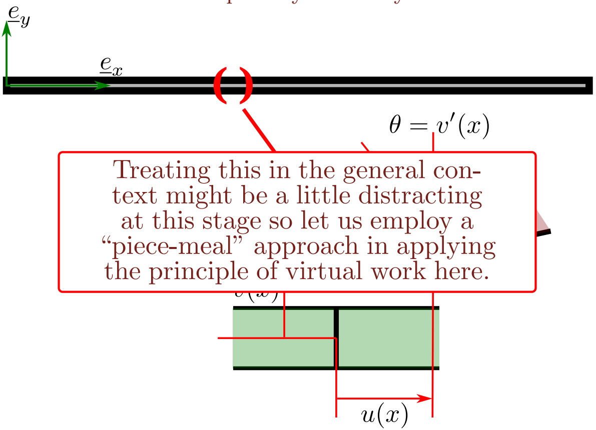

Treating this in the general context might be a little distracting at this stage so let us employ a "piece-meal" approach in applying the principle of virtual work here.

Constitutive Assumption

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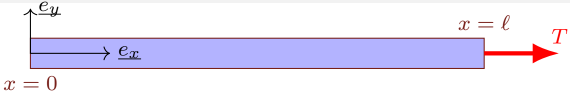
3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory



- Let us first suppose that only axial deformations are present, so $u_x = u(x)$ and $u_y = 0$. So the axial strain is $\boxed{\varepsilon_x = u'}$.
- Let us also suppose that the following boundary conditions are provided:
  - The left face does not deform axially ($u_x = 0$ for $x = 0$), and
  - A load of T is uniformly applied at the right face.

A Note on Virtual Displacements Here

- Unlike the previous examples where the kinematic deformation was just a single scalar, here deformation $u(x)$ is a function of the spatial coordinate x, i.e., a **field**.
- So the virtual displacement is also a **field**, $\delta u(x)$.

Load Virtual Work

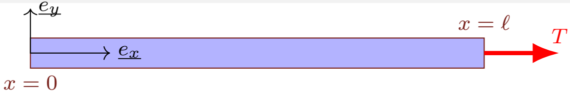
$$\delta \Pi = T \delta u(x)\Big|_{x=\ell}$$

Elastic Virtual Work

$$\delta U = \int_0^\ell \left[\int_{\mathcal{A}} \left(\sigma_x(x) \delta \varepsilon_x(x) \right) dA \right] dx$$

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory



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A Note on Virtual

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- So the virtual displacement is also a **field**, δ...

This is the **elastic virtual energy density** $\sigma_x \delta\varepsilon_x$. It has to be integrated over the complete 3-dimensional beam to get the energy.

Load Virtual Work

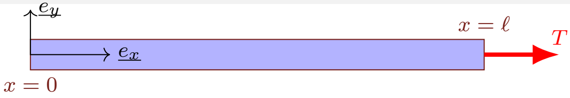
$$\delta\Pi = T\delta u(x)\Big|_{x=\ell}$$

Elastic Virtual Work

$$\delta U = \int_0^\ell \left[\int_{\mathcal{A}} \sigma_x(x)\delta\varepsilon_x(x)\, dA\right] dx$$

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory



- Let us first suppose that only axial deformations are present, so $u_x = u(x)$ and $u_y = 0$. So the axial strain is $\boxed{\varepsilon_x = u'}$.
- Let us also suppose that the following boundary conditions are provided:
  - The left face does not deform axially ($u_x = 0$ for $x = 0$), and
  - A load of T is uniformly applied at the right face.

A Note on Virtual Displacements Here

- Unlike the previous examples where the kin ... r, here deformation $u(x)$ is a function of the spatia ...
- So the virtual displacement is also a **field**, $\delta u(x)$.

This is the integral over the cross section.

Load Virtual Work

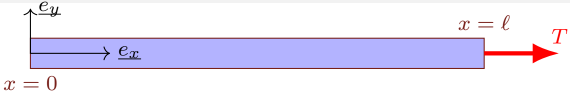
$$\delta \Pi = T \delta u(x)\Big|_{x=\ell}$$

Elastic Virtual Work

$$\delta U = \int_0^\ell \left[\int_{\mathcal{A}} (\sigma_x(x) \delta \varepsilon_x(x))\, dA \right] dx$$

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory



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  - The left face does not deform axially ($u_x = 0$ for $x = 0$), and
  - A load of T is uniformly applied at the right face.

A Note on Virtual Displacements Here

- Unlike the previous examples where the kin ar, here deformation $u(x)$ is a function of the spatia
- So the virtual displacement is also a **field**, $\delta u(x)$.

And this is the integral over the span of the beam.

Load Virtual Work

$$\delta \Pi = T \delta u(x)\Big|_{x=\ell}$$

Elastic Virtual Work

$$\delta U = \int_0^\ell \left[\int_{\mathcal{A}} (\sigma_x(x) \delta \varepsilon_x(x)) \, dA \right] dx$$

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory

Load Virtual Work

$$\delta\Pi = T\delta u(x)\Big|_{x=\ell}$$

Elastic Virtual Work

$$\delta U = \int_0^\ell \left[\int_{\mathcal{A}} \left(\sigma_x(x)\delta\varepsilon_x(x)\right) dA\right] dx$$

- From kinematics we have $\boxed{\varepsilon_x = u'}$ and from constitution we have $\boxed{\sigma_x = E_y \varepsilon_x}$, so together we have $\boxed{\sigma_x = E_y u'}$.
- The "inner" integral of the elastic virtual work leads to:

$$\int_{\mathcal{A}} E_y u'(x)\delta u'(x) dA = E_y A u' \delta u',$$

where A is the area of the cross section. (We can do this because $u(x)$ is only a function of x).

- The "outer" integral gets simplified (using chain rule) as

$$\int_0^\ell E_y A u'(x)\delta u'(x) dx = -\int_0^\ell \left[E_y A u'(x)\right]' \delta u(x) dx + \left[E_y A u'(x)\right]\delta u(x)\Big|_{x=0}^{\ell}$$

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory

Load Virtual Work

$$\delta\Pi = T\delta u(x)\Big|_{x=\ell}$$

Elastic Virtual Work

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$$\int_{\mathcal{A}} E_y u'(x)\delta u'(x) dA = E_y A u'\delta u',$$

where A is the area of ... use $u(x)$ is only a function of x).

This is an integral over the domain $x \in (0, \ell)$

- The "outer" integral gets simplified (using chain rule) as

$$\int_0^\ell E_y A u'(x)\delta u'(x) dx = -\int_0^\ell [E_y A u'(x)]'\delta u(x) dx + [E_y A u'(x)]\delta u(x)\Big|_{x=0}^{\ell}$$

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory

Load Virtual Work

$$\delta\Pi = T\delta u(x)\Big|_{x=\ell}$$

Elastic Virtual Work

$$\delta U = \int_0^\ell \left[\int_{\mathcal{A}} (\sigma_x(x)\delta\varepsilon_x(x))\, dA\right] dx$$

- From kinematics we have $\boxed{\varepsilon_x = u'}$ and from constitution we have $\boxed{\sigma_x = E_y\varepsilon_x}$, so together we have $\boxed{\sigma_x = E_y u'}$.
- The "inner" integral of the elastic virtual work leads to:

$$\int_{\mathcal{A}} E_y u'(x)\delta u'(x) dA = E_y A u'\delta u',$$

where A is the area of the cross section. (We can [illegible] n of x).

- The "outer" integral gets simplified (using chain rule) as

$$\int_0^\ell E_y A u'(x)\delta u'(x) dx = -\int_0^\ell [E_y A u'(x)]'\delta u(x) dx + [E_y A u'(x)]\delta u(x)\Big|_{x=0}^{\ell}$$

These are (2) boundary terms

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory

Load Virtual Work

Elastic Virtual Work

So What IS this Virtual Displacement ?

- We defined it as an **infinitesimal** deformation field that **obeys kinematic constraints exactly**.
- The kinematic constraint in this case is that $u(x) = 0$ for $x = 0$, that's all.
- So δu can be ANY FUNCTION $g(x)$ with $g(x = 0) = 0$.

  (More rigorously, we require square integrability, but we can gloss over this for now)
- Distinguish this with $u(x)$, which describes the physical **deformation** of the beam. As they stand, $\delta u(x)$ and $u(x)$ are two completely different functions.

We restate **The Principle of Virtual Work**:

The virtual work must be zero for ANY CHOICE of virtual displacement $\delta u(x)$ if physical displacement $u(x)$ corresponds to equilibrium.

- Fro … so

  toge
- The

  whe … function

  of x
- The

$$\int_0^\ell E_y A u'(x) \delta u'(x) dx = -\int_0^\ell [E_y A u'(x)]' \delta u(x) dx + [E_y A u'(x)] \delta u(x) \Big|_{x=0}^{\ell}$$

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory

- Combining all that we have, we can write

$$\delta W := \delta\Pi - \delta U = \int_0^\ell \left[E_y A u'(x)\right]' \delta u(x) dx + \left[T - E_y A u'(x)\right] \delta u(x)\Big|_{x=\ell} + \left[E_y A u'(x)\right] \delta u(x)\Big|_{x=0}$$

- δW has to be zero **FOR ALL CHOICES OF $\delta u(x)$**, as per the principle of virtual work. So we can write

$$\begin{aligned} \left[E_y A u'(x)\right]' &= 0, & x &\in (0, \ell), \\ E_y A u'(x) &= 0, & x &\in \{\ell\}, \\ u(x) &= 0, & x &\in \{0\}. \end{aligned}$$

- Since $u(x) = 0$ AND $\delta u(x) = 0$, we trivially satisfy $E_y A u'(x) \delta u(x)\Big|_{x=0} = 0$.
- For the $x = \ell$ boundary condition, $\delta u(x) = 0$ is too restrictive and not valid - so we set the coefficient of $\delta u(x)$ to zero.
- For the integral term we set the integrand to zero at all points in the open domain.

3.1. The Axial Deformation Problem

Euler Bernoulli Beam Theory

Formal Solution

We have

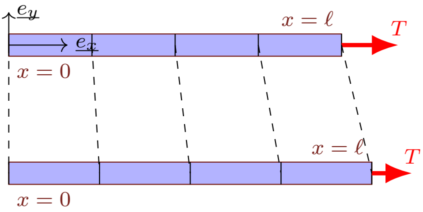
$$\begin{aligned} \left[E_y A u'(x)\right]' &= 0, \qquad x \in (0, \ell), \\ E_y A u'(x) &= 0, \qquad x \in \{\ell\}, \\ u(x) &= 0, \qquad x \in \{0\}, \end{aligned}$$

which can be solved by

$$E_y A u(x) = Tx, \quad \forall \quad x \in [0, \ell].$$

(Notice that we now have closed brackets above)

- The stress at all points within the domain is $\sigma_x = E_y u' = \frac{T}{A}$.
- The above is a fact we may also have observed through drawing the free-body diagram, so this is not really world shattering news to us.



A Visualization of the Solution

3.2. The Euler Buckling Problem

Euler Bernoulli Beam Theory

- Let us now consider the prospect of small transverse deformation **in addition** to the existing axial field $u(x) = \frac{T}{E_y A} x$.
- We shall assume that the transverse deformations are small enough so as not to affect the existing axial field.
- So the strain and stress fields are now written as:

Strain Field

$$\varepsilon_x = \cancel{u'}^{\frac{T}{E_y A}} - y v'' + \frac{v'^2}{2}$$
$$= \frac{T}{E_y A} - y v'' + \frac{v'^2}{2}$$

Stress Field

$$\sigma_x = \cancel{E_y u'}^{\frac{T}{A}} - E_y y v'' + \frac{E_y}{2} v'^2$$
$$= \frac{T}{A} - E_y y v'' + \frac{E_y}{2} v'^2$$

- When we take variations on the strain field (we need this for δU), we will keep T fixed - i.e., the axially applied load is constant. So the **variational strain**, i.e., virtual strain is:

$$\delta\varepsilon_x = -y\delta v'' + v'\delta v'.$$

- The virtual elastic work now gets expressed as

$$\delta U = \int_0^\ell \left(\int_{\mathcal{A}} \sigma_x \delta\varepsilon_x dA \right) dx.$$

3.2. The Euler Buckling Problem

Euler Bernoulli Beam Theory

Stress

$$\sigma_x = \frac{T}{A} - E_y y v'' + \frac{E_y}{2} v'^2$$

Virtual Strain

$$\delta\varepsilon_x = -y\delta v'' + v'\delta v'$$

Virtual Elastic Work

$$\delta U = \int_0^\ell \left(\int_{\mathcal{A}} \sigma_x \delta\varepsilon_x dA \right) dx.$$

- The section integral simplifies as:

$$\int_{\mathcal{A}} \sigma_x \delta\varepsilon_x dA = Tv'\delta v' + EIv''\delta v'' + \mathcal{O}(v^2),$$

  where we have not pursued terms that are quadratic or higher order in v or its derivatives. (I show this in the appendix for post buckling)
- Using the above, the span integral can be simplified through the application of chain rule as

$$\delta U = \int_0^\ell Tv'\delta v' + EIv''\delta v'' dx = \int_0^\ell -Tv''\delta v - EIv'''\delta v' dx + Tv'\delta v\Big|_{x=0}^{\ell} + EIv''\delta v'\Big|_{x=0}^{\ell}$$

$$= \int_0^\ell \left(EIv'''' - Tv''\right)\delta v dx + EIv''\delta v'\Big|_{x=0}^{\ell} + \left(Tv' - EIv'''\right)\delta v\Big|_{x=0}^{\ell}$$

3.2. The Euler Buckling Problem

Euler Bernoulli Beam Theory

Stress

$$\sigma_x = \frac{T}{A} - E_y y v'' + \frac{E_y}{2} v'^2$$

Virtual Strain

$$\delta\varepsilon_x = -y\delta v'' + v'\delta v'$$

Virtual Elastic Work

$$\delta U = \int_0^\ell \left(\int_{\mathcal{A}} \sigma_x \delta\varepsilon_x dA \right) dx.$$

- The section integral simplifies as:

$$\int_{\mathcal{A}} \sigma_x \delta\varepsilon_x dA = Tv'\delta v' + EIv''\delta v'' + \mathcal{O}(v^2),$$

where we have not pursued terms that are quadratic or higher order in v or its derivatives. (I show this in the appendix for post buckling)

- U... be simplified through the application of chain r...

This term leads to

$$EIv'''' - Tv'' = 0, \quad x \in (0, \ell),$$

the general differential equation governing Euler Buckling.

$$\ldots v''\delta v - EIv'''\delta v' dx + Tv'\delta v\Big|_{x=0}^{\ell} + EIv''\delta v'\Big|_{x=0}^{\ell}$$

$$= \int_0^\ell \left(EIv'''' - Tv''\right)\delta v dx + EIv''\delta v'\Big|_{x=0}^{\ell} + \left(Tv' - EIv'''\right)\delta v\Big|_{x=0}^{\ell}$$

3.2. The Euler Buckling Problem

Euler Bernoulli Beam Theory

Stress

$$\sigma_x = \frac{T}{A} - E_y y v'' + \frac{E_y}{2} v'^2$$

Virtual Strain

$$\delta\varepsilon_x = -y\delta v'' + v'\delta v'$$

Virtual Elastic Work

$$\delta U = \int_0^\ell \left(\int_{\mathcal{A}} \sigma_x \delta\varepsilon_x dA \right) dx.$$

- The section integral simplifies as:

$$\int \sigma_x \delta\varepsilon_x dA = Tv'\delta v' + EIv''\delta v'' + \mathcal{O}(v^2),$$

This term leads to

$$EIv'' = 0 \,(\text{or})\, v' = 0, \quad x \in \{0, \ell\},$$

i.e., either have a moment-free boundary (like a pinned hinge or a free edge), or restrict the rotation (like a clamped edge).

where we have no ... higher order in v or its derivatives. (I sho ... ng)

- Using the above, ... ugh the application of chain rule as

$$\delta U = \int_0^\ell Tv'\delta v' \ldots \int_0 \ldots dx + Tv'\delta v\Big|_{x=0}^{\ell} + EIv''\delta v'\Big|_{x=0}^{\ell}$$

$$= \int_0^\ell \left(EIv'''' - Tv''\right)\delta v dx + EIv''\delta v'\Big|_{x=0}^{\ell} + \left(Tv' - EIv'''\right)\delta v\Big|_{x=0}^{\ell}$$

3.2. The Euler Buckling Problem

Euler Bernoulli Beam Theory

Stress

$$\sigma_x = \frac{T}{A} - E_y y v'' + \frac{E_y}{2} v'^2$$

Virtual Strain

$$\delta\varepsilon_x = -y\delta v'' + v'\delta v'$$

Virtual Elastic Work

$$\delta U = \int_0^\ell \left(\int_{\mathcal{A}} \sigma_x \delta\varepsilon_x dA \right) dx.$$

- The section integral simplifies as:

$$\int_{\mathcal{A}} \sigma_x \delta\varepsilon_x dA = Tv'\delta v' + EIv''\delta v'' + \mathcal{O}(v^2),$$

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- Using the above, the span integral c… n of chain rule as

$$\delta U = \int_0^\ell Tv'\delta v' + EIv''\delta v'' dx = \int_0^\ell \cdots \Big|_{x=0}^{\ell} + EIv''\delta v' \Big|_{x=0}^{\ell}$$

$$= \int_0^\ell \left(EIv'''' - Tv''\right) \delta v dx + EIv''\delta v' \Big|_{x=0}^{\ell} + \left(Tv' - EIv'''\right) \delta v \Big|_{x=0}^{\ell}$$

This term leads to

$$Tv' - EIv''' = 0 \text{ (or) } v = 0, \quad x \in \{0, \ell\},$$

i.e., either have the shear force equal $-Tv'$ or restrict the transverse deformation at the ends.

3.2. The Euler Buckling Problem

Euler Bernoulli Beam Theory

Stress

$\sigma_x = \frac{T}{A}$ ... $\frac{E y}{}$ ...

Virtual Strain

Virtual Elastic Work

δU ... $\int^{\ell} \left(\int \ldots \delta \ldots dA \right) dx.$

- The se...

where ... deriva...

- Using ... f chain rule a...

The Final Set of Equilibrium Equations

- Applying the principle of virtual work we finally have

$$EIv'''' + Pv'' = 0, \quad x \in (0, \ell)$$
$$EIv'' = 0 \,(\text{or})\, v' = 0, \quad x \in \{0, \ell\}$$
$$EIv''' + Tv' = 0 \,(\text{or})\, v = 0, \quad x \in \{0, \ell\}$$

where we have used $P = -T$, the **compressive load**.

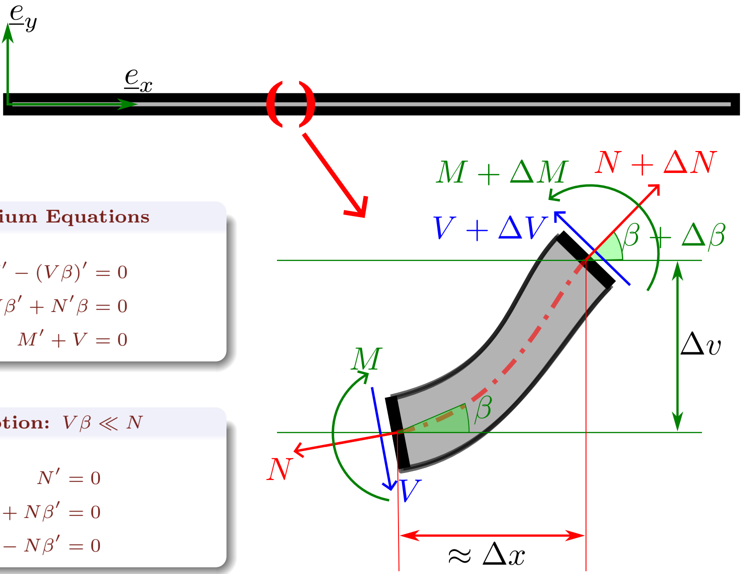
- Our study of buckling will be all about solutions for this with different combinations of boundary conditions.

$$\delta U = \int_0 Tv'\delta v' + EIv''\delta v'' dx = \int_0 -Tv''\delta v - EIv'''\delta v' dx + Tv'\delta v\Big|_{x=0}^{\ell} + EIv''\delta v'\Big|_{x=0}^{\ell}$$

$$= \int_0^{\ell} \left(EIv'''' - Tv''\right)\delta v dx + EIv''\delta v'\Big|_{x=0}^{\ell} + \left(Tv' - EIv'''\right)\delta v\Big|_{x=0}^{\ell}$$

3.3. Equilibrium Equations Through Force Balance

Euler Bernoulli Beam Theory



Equilibrium Equations

$$N' - (V\beta)' = 0$$
$$V' + N\beta' + N'\beta = 0$$
$$M' + V = 0$$

Assumption: $V\beta \ll N$

$$N' = 0$$
$$V' + N\beta' = 0$$
$$M'' - N\beta' = 0$$

3.4. The Euler Buckling Problem

Euler Bernoulli Beam Theory

- Substituting, we are left with,

$$N' = \boxed{EAu'' = 0}, \quad M'' - N\beta' = \boxed{EIv'''' - Nv'' = 0}.$$

Axial Problem

- Boundary conditions representing axial compression:

$$u(x=0) = 0, \quad EAu'(x=\ell) = -P$$

- Solution:

$$\boxed{u(x) = -\frac{P}{EA}x}$$

Transverse Problem

- Substituting $N = -P$ we have,

$$v'''' + k^2 v'' = 0, \quad k^2 = \frac{P}{EI}.$$

- The general solution to this **Homogeneous ODE** are

$$\boxed{v(x) = A_0 + A_1 x + A_2 \cos kx + A_3 \sin kx}$$

- Boundary conditions on the transverse displacement function $v(x)$ are necessary to fix A_0, A_1, A_2, A_3.

3.4.1. The Pinned-Pinned Column

The Euler Buckling Problem

- For a Pinned-pinned beam we have $v = 0$ on the ends and zero reaction moments at the supports:

$$v = 0, \quad x = \{0, \ell\}$$
$$v'' = 0, \quad x = \{0, \ell\}$$

- So the general solution reduces to

$$v(x) = A_3 \sin kx,$$

with the boundary condition

$$\boxed{A_3 \sin k\ell = 0.}$$

- Apart from the trivial solution ($A_3 = 0$) we have

$$k_{(n)}\ell = n\pi \implies k_n = n\frac{\pi}{\ell}$$

or in terms of the compressive load P,

$$\boxed{P_{cr,n} = n^2 \frac{\pi^2 EI}{\ell^2}}$$

- **Interpretation**: If $P \neq P_{cr,n}$, $A_3 = 0$ to satisfy boundary conditions. But for $P = P_{cr,n}$, A_3 **CAN BE ANYTHING!**.

3.4.1. The Pinned-Pinned Column

The Euler Buckling Problem

- For a Pinned- … ion ($A_3 = 0$) on the ends a … the supports: … $= n\frac{\pi}{\ell}$

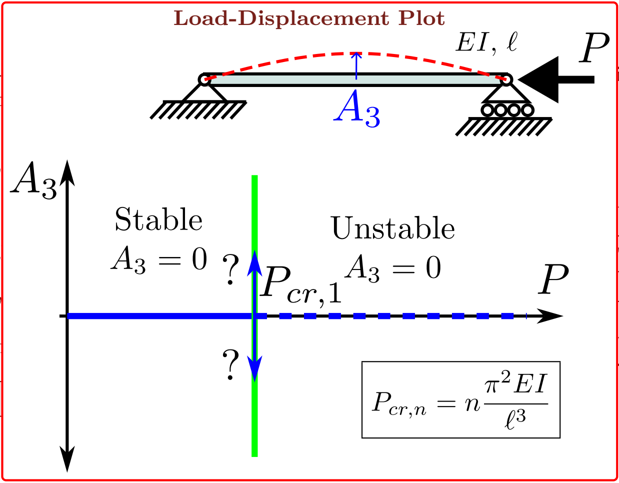
v …

v'' … ve load P,

- So the genera … I

v …

with the boun … $_{r,n}$, $A_3 = 0$ to … . But for



3.4.1. The Pinned-Pinned Column: The Imperfect Case I

The Euler Buckling Problem

- Suppose there are initial imperfections in the beam's neutral axis such that the neutral axis can be written as $v_0(x)$.
- Noting that strains are accumulated only on the *relative displacement* $v(x) - v_0(x)$, we write

$$EI(v - v_0)'''' + Pv'' = 0.$$

  Note that the axial load P acts on the **net rotation** of the deflected beam, so we do not need to use $(v - v_0)''$ here.
- The governing equations become

$$EIv'''' + Pv'' = EIv_0'''',$$

  or, in more convenient notation,

$$\boxed{v'''' + k^2 v'' = v_0''''}.$$

3.4.1. The Pinned-Pinned Column: The Imperfect Case II

The Euler Buckling Problem

- Describing the imperfect neutral axis using an infinite series,

$$v_0 = \sum_n C_n \sin(n\frac{\pi x}{\ell}) \quad \left(\implies v_0'''' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell}) \right),$$

the governing equations become

$$v'''' + k^2 v'' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell}).$$

- This is solved by,

$$\begin{aligned} v(x) &= \sum_n \frac{\left(n\frac{\pi}{\ell}\right)^2}{\left(n\frac{\pi}{\ell}\right)^2 - k^2} C_n \sin(n\frac{\pi x}{\ell}) \\ &= \sum_n \frac{\frac{n^2\pi^2 EI}{\ell^2}}{\frac{n^2\pi^2 EI}{\ell^2} - P} C_n \sin(n\frac{\pi x}{\ell}) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell}) \end{aligned}$$

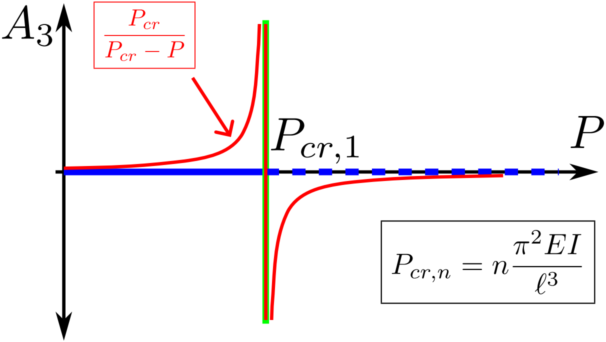
3.4.1. The Pinned-Pinned Column: The Imperfect Case

The Euler Buckling Problem

- Look carefully at the solution

$$v(x) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell}).$$

- Clearly $P \to P_{cr,n}$ are **singularities**. Even for very small C_n, the "blow-up" is huge.



3.4.2. The Southwell Plot

The Euler Buckling Problem

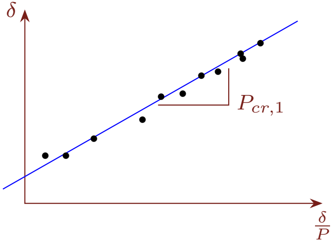
- The relative deformation amplitude at the mid-point is given as (for $P < P_{cr,1}$),

$$\delta \approx \frac{P_{cr,1}}{P_{cr,1} - P} C_1 - C_1 = \frac{C_1}{\frac{P_{cr,1}}{P} - 1}$$

$$\implies \boxed{\delta = P_{cr,1} \frac{\delta}{P} - C_1}$$

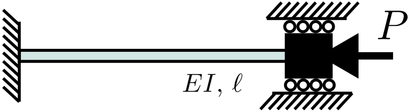
The Southwell Plot

- Plotting δ vs $\frac{\delta}{P}$ allows **Non-Destructive Evaluation of the critical load**
- $P_{cr,1}$ is estimated <u>without having to buckle the column</u>



3.4.3. The Clamped-Clamped Column

The Euler Buckling Problem



$@x = 0$
$u = 0$
$v = v' = 0$

$@x = \ell$
$EAu' = -P$
$v = v' = 0$

- The axial solution is the same as before: $u(x) = -\frac{P}{EA}x$.
- The transverse general solution also has the same form but boundary conditions are different.

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 1 & x & \cos(kx) & \sin(kx) \\ 0 & 1 & -k\sin(kx) & k\cos(kx) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

- The boundary conditions may be expressed as

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & \ell & \cos(k\ell) & \sin(k\ell) \\ 0 & 1 & -k\sin(k\ell) & k\cos(k\ell) \end{bmatrix}}_{\underline{\underline{M}}} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

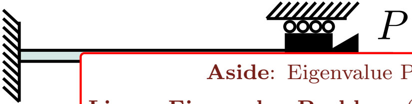
- There can be non-trivial solutions only when $\underline{\underline{M}}$ is singular, i.e., **for choices of k such that $\Delta(\underline{\underline{M}}) = 0$.**

The Eigenvalue Problem

This problem setting of finding k such that $\Delta(\underline{\underline{M}}(k)) = 0$ is known as an **eigenvalue problem**.

3.4.3. The Clamped-Clamped Column

The Euler Buckling Problem



$@x = 0$
$u = 0$
$v = v' = 0$

- The boundary conditions may be expressed as

$$\ldots\Bigg] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Aside: Eigenvalue Problems ($\underline{\underline{M}} \in \mathbf{R}^{d\times d}$)

Linear Eigenvalue Problem (d eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}}$$

Quadratic Eigenvalue Problem ($2d$ eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}} + k^2\underline{\underline{M_2}}$$

- The axia... $u(x) = -$...
- The tran... the same... are differ...

...ns only ...**oices**

...that ...**ue** **problem**.

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \ldots \begin{bmatrix} 0 & 1 & -k\sin(kx) & k\cos(kx) \end{bmatrix} \begin{bmatrix} \vdots \\ A_2 \\ A_3 \end{bmatrix}$$

3.4.3. The Clamped-Clamped Column

The Euler Buckling Problem

$@x = 0$
$u = 0$
$v = v' = 0$

- The boundary conditions may be expressed as

$$\ldots \Big] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- The axia… $u(x) = -$…
- The tran… the same… are differ…

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} \ldots \\ 0 & 1 & -k\sin(kx) & k\cos(kx) \end{bmatrix} \begin{bmatrix} \ldots \\ A_2 \\ A_3 \end{bmatrix}$$

…ns only … **…oices**

… that … **…ue problem**.

Aside: Eigenvalue Problems ($\underline{\underline{M}} \in \mathbf{R}^{d\times d}$)

Linear Eigenvalue Problem (d eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}}$$

Quadratic Eigenvalue Problem ($2d$ eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}} + k^2\underline{\underline{M_2}}$$

Our matrix $\underline{\underline{M}}(k)$ has k-dependency in terms of k, $\sin(k\ell)$, $\cos(k\ell)$, making this a **Nonlinear Eigenvalue Problem**.

- $\Longrightarrow \infty$ eigenvalues here (not always though!)

3.4.3. The Clamped-Clamped Column I

The Euler Buckling Problem

- We proceed to solve this as,

$$\Delta\left(\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & k\\ 1 & \ell & \cos(k\ell) & \sin(k\ell)\\ 0 & 1 & -k\sin(k\ell) & k\cos(k\ell)\end{bmatrix}\right) = -k\left(k\ell\sin(k\ell) + 2\cos(k\ell) - 2\right)$$

- We set it to zero through the following factorizations:

$$\Delta(\underline{\underline{M}}(k)) = -k\left(2k\ell\sin(\frac{k\ell}{2})\cos(\frac{k\ell}{2}) - 4\sin^2(\frac{k\ell}{2})\right)$$
$$= -2k\sin(\frac{k\ell}{2})\left(k\ell\cos(\frac{k\ell}{2}) - 2\sin(\frac{k\ell}{2})\right) = 0$$
$$\implies \boxed{\sin(\frac{k\ell}{2}) = 0}, \quad \text{(or)} \quad \boxed{\tan(\frac{k\ell}{2}) = \frac{k\ell}{2}}.$$

- Two "classes" of solutions emerge:

1. $\sin(\frac{k\ell}{2}) = 0 \implies \frac{k_n\ell}{2} = n\pi \implies \boxed{P_n^{(1)} = 4n^2\frac{\pi^2 EI}{\ell^2}}$
2. $\tan(\frac{k\ell}{2}) = \frac{k\ell}{2} \implies \frac{k_n\ell}{2} \approx 0, 4.49, 7.72, \cdots \implies P_1^{(2)} \approx 8.98\frac{\pi^2 EI}{\ell^2}$

3.4.3. The Clamped-Clamped Column II

The Euler Buckling Problem

- **The smallest critical load** is $P_n^{(1)} = 4\frac{\pi^2 EI}{\ell^2} = \frac{\pi^2 EI}{(\frac{\ell}{2})^2}$.

Concept of "Effective Length"

- **Question**: If the beam were simply supported, what would be the length such that it also has the same first critical load?
- Here it comes out to be $\ell_{eff} = \frac{\ell}{2}$.
- The column clamped on both ends can take the same buckling load as a column that is pinned on both ends with half the length.

3.4.3. The Clamped-Clamped Column III

The Euler Buckling Problem

Boundary conditions	Critical load P_{cr}	Deflection mode shape	Effective length KL
Simple support–simple support	$\frac{\pi^2 EI}{L^2}$		L
Clamped-clamped	$4\frac{\pi^2 EI}{L^2}$		$\frac{1}{2}L$
Clamped–simple support	$2.04\frac{\pi^2 EI}{L^2}$		$0.70L$
Clamped-free	$\frac{1}{4}\frac{\pi^2 EI}{L^2}$		$2L$

Effective lengths of beams with different boundary conditions (Figure from Brush and Almroth 1975)

Self-Study

- Derive the effective length for the clamped-simply supported and clamped-free columns.

3.4.3. The Clamped-Clamped Column: The Mode-shape

The Euler Buckling Problem

- Let us substitute $k_1 = \frac{2\pi}{\ell}$ into the matrix $\underline{\underline{M}}(k_1)$ so that the boundary conditions now read as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \\ 1 & \ell & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- This implies the following:

$$A_1 = 0, \quad A_3 = 0, \quad A_2 = -A_0.$$

- So, if $k = k_1$, the solution has to be the following to satisfy the boundary conditions:

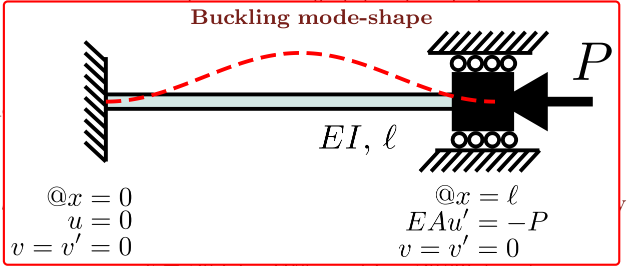
$$v = A_0 \left(1 - \cos(\frac{2\pi x}{\ell})\right) \equiv A_0 \sin^2(\frac{\pi x}{\ell})$$

3.4.3. The Clamped-Clamped Column: The Mode-shape

The Euler Buckling Problem

- Let us substitute $k_1 = \frac{2\pi}{\ell}$ into the matrix $\underline{\underline{M}}(k_1)$ so that the boundary conditions now read as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}$$



- This impli

- So, if $k =$ ... conditions:

$$v = A_0\left(1 - \cos(\ldots \ell)\right) = A_0 \sin(\ldots \ell)$$

4. Energy Perspectives

- Concept of conservative force field.
- Work done by a force field:

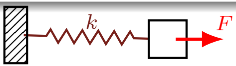
$$W(\underline{x})\Big|_{\underline{x}_1}^{\underline{x}_2} = \int_{\underline{x}_1}^{\underline{x}_2} \underline{f}(\underline{x}) \cdot d\underline{x}.$$

- Introduction to work done.

$$W(\underline{x}) = \underbrace{\Pi(\underline{x})}_{\text{External Work}} - \underbrace{V(\underline{x})}_{\text{Internal Work/Potential Energy}}$$

Example

- Force balance reads: $F = kx$
- Work done expression: $W(x) = Fx - \frac{k}{2}x^2$



4. Energy Perspectives

- Expanding W about some $\underline{x}_s$ we have,

$$W(\underline{x}_s + \delta\underline{x}) = W(\underline{x}_s) + \underline{\nabla} W\big|_{\underline{x}_s} \delta\underline{x} + \mathcal{O}(\delta\underline{x}^2).$$

- Stationarity of work: $\boxed{\delta W = \underline{\nabla} W(\underline{x}_s)\delta\underline{x} = 0, \quad \forall \quad \underline{x} \in \Omega}$, where Ω is the configuration-space.

Example

- For the SDoF system above, we have $W = Fx - \frac{k}{2}x^2$ and

$$\nabla W(x_s) = \frac{dW}{dx} = F - kx_s = 0 \implies x_s = \frac{F}{k}.$$

- Work-stationarity hereby gives a convenient definition for equilibrium.
- What about higher order effects?

4. Energy Perspectives

- Continuing the Taylor expansion (SDoF case) for $W(x)$ we have,

$$W(x) = W(x_s) + \frac{dW}{dx}(x_s)\delta x + \frac{1}{2}\frac{d^2W}{dx^2}(x_s)\delta x^2 + \mathcal{O}(\delta x^3).$$

- At equilibrium, $\frac{dW}{dx}$ is zero. The sign of $\frac{d^2W}{dx^2}$ governs the local tendency of the work around equilibrium.

Example

- For the SDoF example, $\frac{d^2W}{dx^2} = -k$, implying W is maximized.

- If $\frac{d^2W}{dx^2} < 0$, then the second order effect of virtual displacements is to reduce the work scalar: **Stable Equilibrium**.
- The opposite case is **Unstable Equilibrium**.

4. Energy Perspectives

- Continuing the Taylor expansion (SDoF case) for $W(x)$ we have,

$$W(\ldots) \ldots W(\ldots) + \frac{dW}{\ldots}(\ldots)\delta\ldots + \frac{1}{\ldots}\frac{d^2W}{\ldots}(\ldots)\delta\ldots^2 + \mathcal{O}(\delta x^3).$$

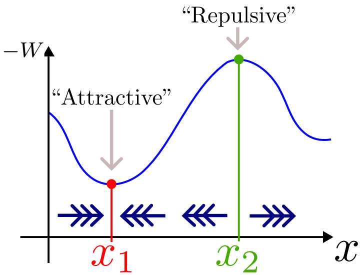
- At equilibrium, $\frac{d}{\ldots}$ … ndency of the work around equilibriu…

Example

- For the SDoF exa…

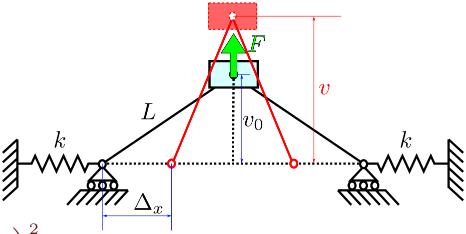
- If $\frac{d^2W}{dx^2} < 0$, then … is to reduce the work scalar: **Stable E…**
- The opposite case…

Hypothetical Example



4.1. Snap-Through Buckling

Energy Perspectives



- We will consider the SDoF model to the right (from Wiebe et al. 2011).
- The strain energy on the springs (two) is

$$U(v) = 2\times\frac{k}{2}\Delta_x^2 = k\left(\sqrt{L^2 - v_0^2} - \sqrt{L^2 - v^2}\right)^2.$$

- The work done by the load (to take the mid-point from v_0 to v) is given by,

$$\Pi(v) = F(v - v_0).$$

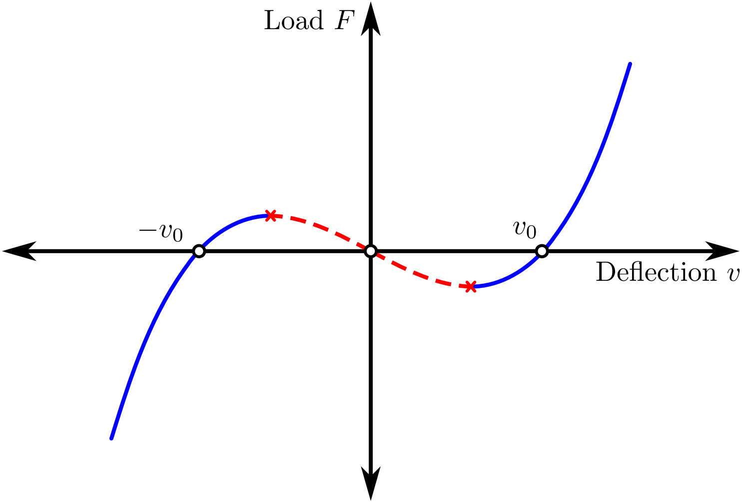
Setting $\frac{dW}{dv} = 0$ we get

$$F = -2kv\left(1 - \sqrt{\frac{L^2 - v_0^2}{L^2 - v^2}}\right).$$

4.1. Snap-Through Buckling

Energy Perspectives

- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.



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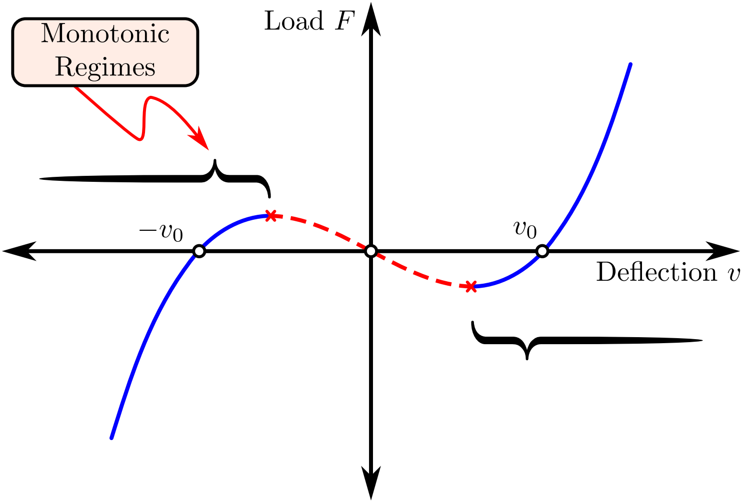

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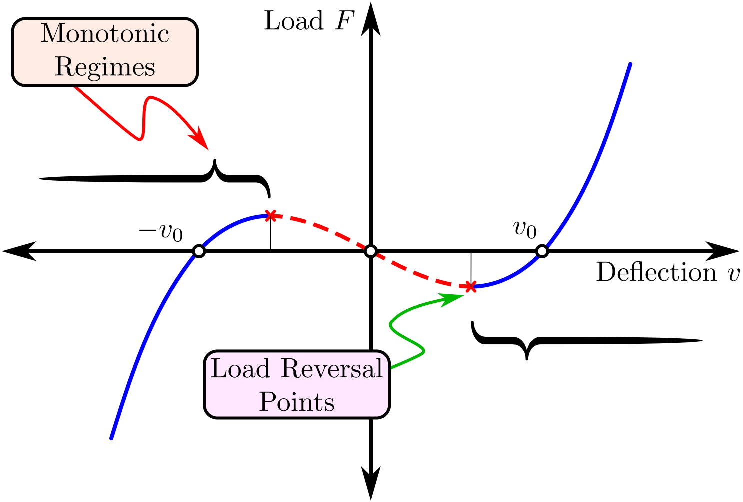

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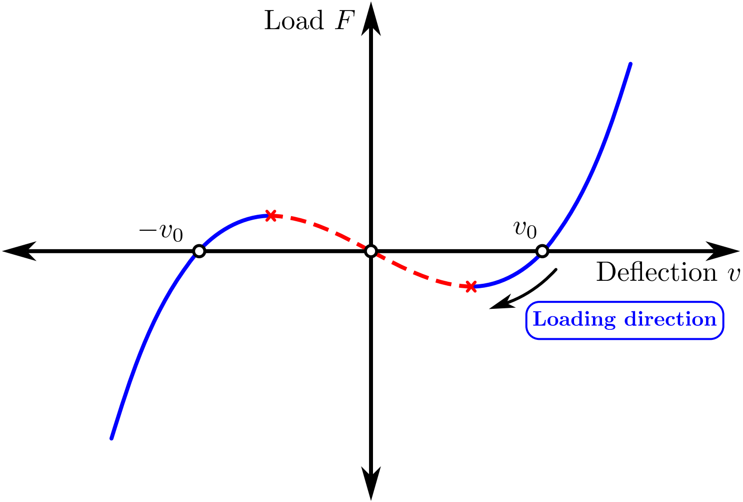

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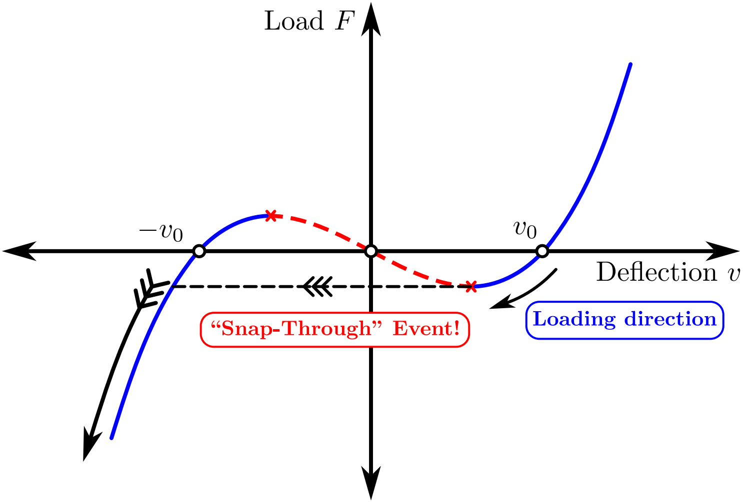

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Energy Perspectives

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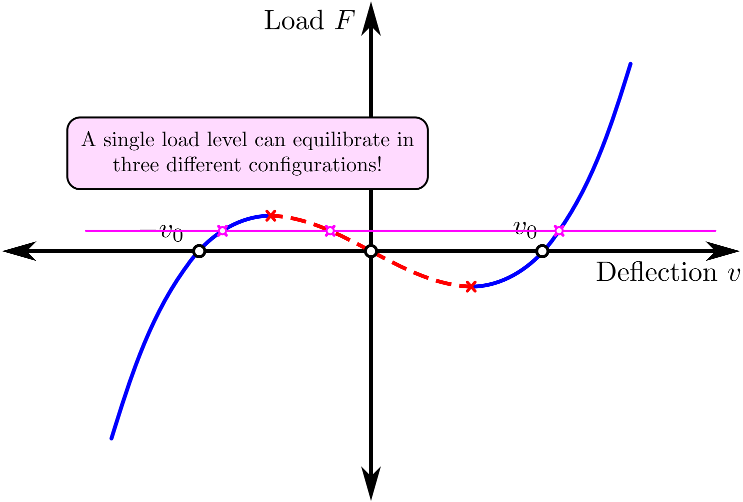
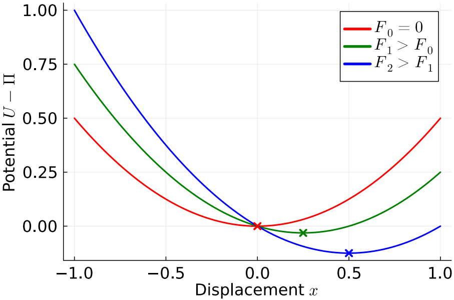
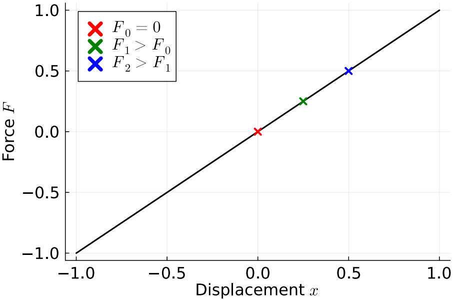


4.1. Snap-Through Buckling: Equilibrium Visualization

Energy Perspectives

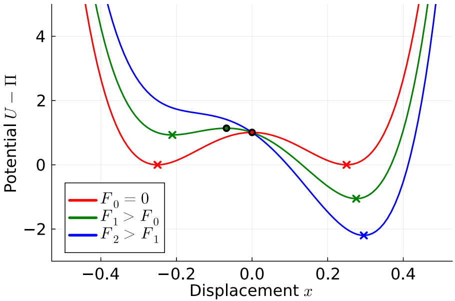


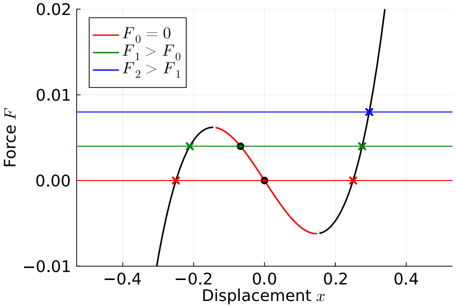


Linear System: $U - \Pi = \frac{k}{2}x^2 - Fx$

4.1. Snap-Through Buckling: Equilibrium Visualization

Energy Perspectives

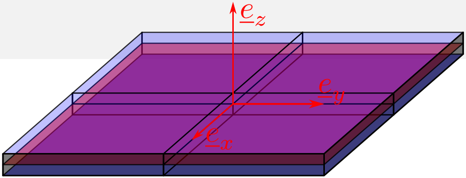




Snap-Through Problem: $U - \Pi = k\left(\sqrt{L^2 - v_0^2} - \sqrt{L^2 - v^2}\right)^2 - Fx$

5.1. Plate Buckling

Governing Equations



- Kichhoff-Love Plate Theory.
- **Kinematic Assumptions**: Lines along section-thickness deform as lines and stay perpendicular to the neutral axis.
- Governing equations written in the form

$$\frac{Et^3}{12(1-\nu^2)}(w_{,xxxx} + w_{,yyyy} + 2w_{,xxyy}) - (N_{xx}w_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}) = 0$$

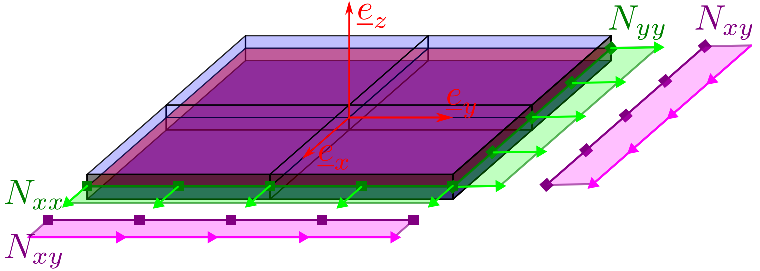
$$\boxed{D\nabla^4 w - (N_{xx}w_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}) = 0}$$

- This is all that is needed to conduct buckling analysis - the procedure is identical as above!
- Before this, however, let us develop intuition on the different reaction force components and their kinematic relationships.

5.2. Reaction-Kinematics Relationships

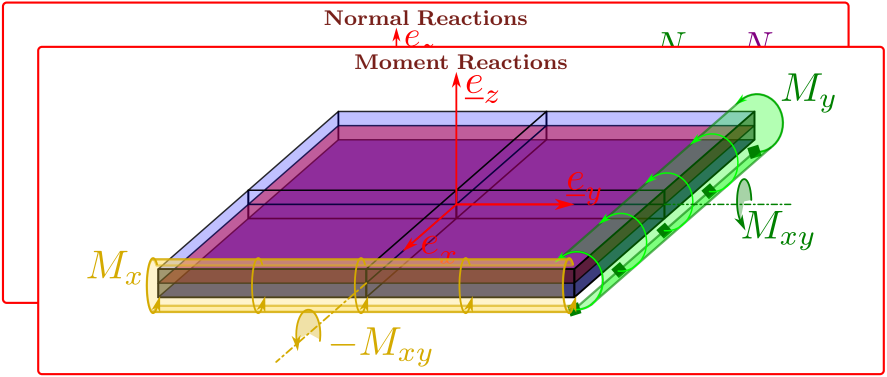
Plate Buckling

Normal Reactions



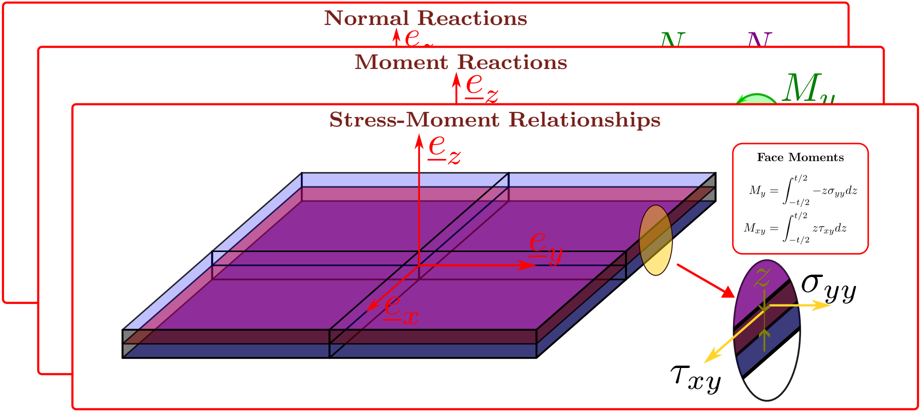
5.2. Reaction-Kinematics Relationships

Plate Buckling



5.2. Reaction-Kinematics Relationships

Plate Buckling



5.2. Reaction-Kinematics Relationships

Plate Buckling

Normal Reactions

Moment Reactions

Stress-Moment Relationships

Equilibrium Equations (Shear Force-Moment Relationships)

$$\left.\begin{aligned}\sigma_{xx,x} + \tau_{xy,y} + \tau_{xz,z} &= 0\\ \tau_{xy,x} + \sigma_{yy,y} + \tau_{yz,z} &= 0\\ \tau_{xz,x} + \tau_{yz,z} + \sigma_{zz,z} &= 0\end{aligned}\right\} \implies \begin{cases} Q_x &= M_{x,x} + M_{xy,y}\\ Q_y &= -M_{y,y} + M_{xy,x}\\ 0 &= Q_{x,x} + Q_{y,y}.\end{cases}$$

Note:

- Although the shear strains γ_{xz} & γ_{yz} are assumed zero by the Kirchhoff kinematic assumptions, and thereby, the stresses τ_{xz} & τ_{yz} are also zero, **the shear forces can not be zero for equilibrium!!**
- They are defined as $Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} dz, \quad Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz.$

5.2. Reaction-Kinematics Relationships

Plate Buckling

- With this background, we are ready to write the following:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_x \\ -M_y \\ M_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \left(\begin{bmatrix} t & 0 \\ 0 & -\frac{t^3}{12} \end{bmatrix} \otimes \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \right) \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \\ w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$

- A moment-free boundary condition (simply supported edge) would imply simply setting the second derivatives $(w_{,xx}, w_{,yy}, w_{,xy})$ to zero at the edge.

5.3. Thin Plates Under Uniaxial Compression

Plate Buckling

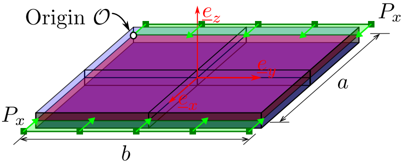


Plate under uniaxial compression

Ansatz (Simply Supported Case)

$$w(x,y) = \sum_{m,n} W_{mn} \sin\left(m\frac{\pi x}{a}\right) \sin\left(n\frac{\pi y}{b}\right)$$

Boundary Conditions:
$w = 0,\ M_x, M_y = 0 \quad \text{on} \quad \Gamma$

Governing Equations

$$D\nabla^4 w + P w_{,xx} = 0$$

$$\implies P_{cr,nm} = \frac{\pi^2 D}{b^2}\left(\frac{m}{a/b} + n^2\frac{a/b}{m}\right)^2$$

(n=1 always for minimum critical load)

$$\implies P_{cr,m} = \frac{\pi^2 D}{b^2}\left(\frac{m}{a/b} + \frac{a/b}{m}\right)^2$$

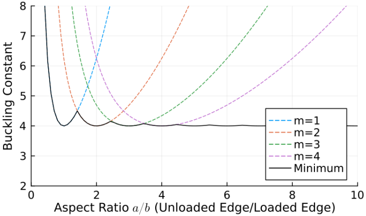
$$P_{cr} = \frac{\pi^2 D}{b^2} \underbrace{\min_{m\in\mathbb{Z}+}\left(\frac{m}{a/b} + \frac{a/b}{m}\right)^2}_{k_{cr}(a/b)}$$

5.3. Thin Plates Under Uniaxial Compression

Plate Buckling

Buckling Constant

$$k_{cr}(r) = \min_{m\in\mathbb{Z}^+} \left(\frac{m}{r} + \frac{r}{m}\right)^2$$

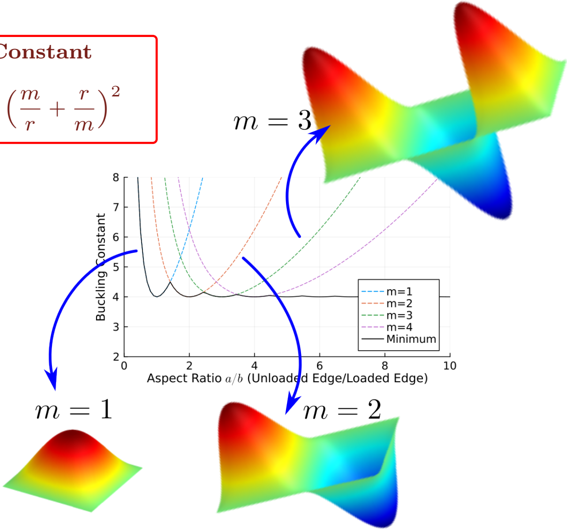


5.3. Thin Plates Under Uniaxial Compression

Plate Buckling

Buckling Constant

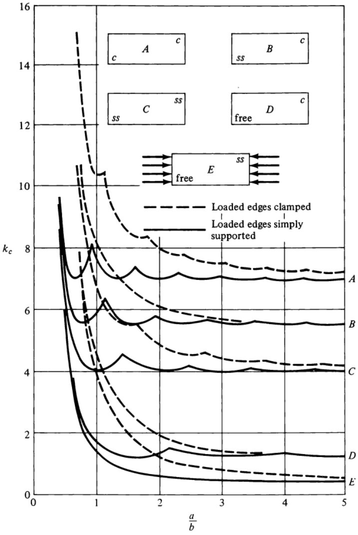
$$k_{cr}(r) = \min_{m\in\mathbb{Z}^+} \left(\frac{m}{r} + \frac{r}{m}\right)^2$$



5.3. Other Boundary Conditions

Thin Plates Under Uniaxial Compression

- It is possible to conduct the same analysis for other (combinations) of boundary conditions.
- The analysis is slightly more tedious (due to the Ansatz not being as simple any more), **but possible along the same lines**.
- The critical plot comes out as shown in your textbook.



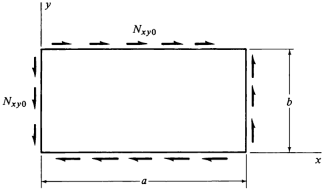
(Figure 3.9 from Brush and Almroth 1975)

5.3. Other Boundary Conditions

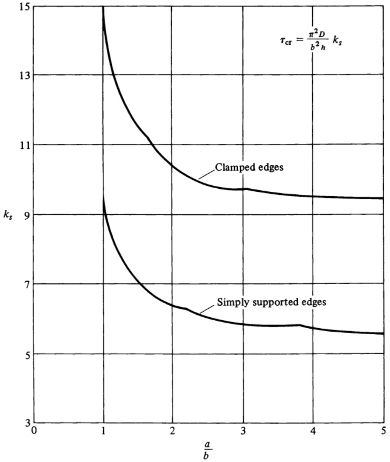
Thin Plates Under Uniaxial Compression

- It is possible to conduct the same analysis for other (combinations) of boundary conditions.
- The analysis is slightly more tedious (due to the Ansatz not being as simple any more), **but possible along the same lines**.
- The critical plot comes out as shown in your textbook.

The same works for shear buckling too!



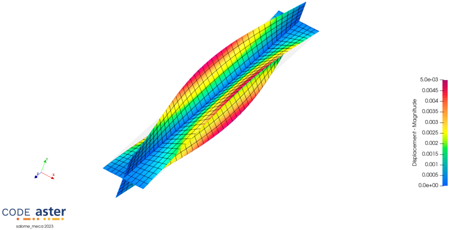
(Fig. 3.10 from Brush and Almroth 1975)



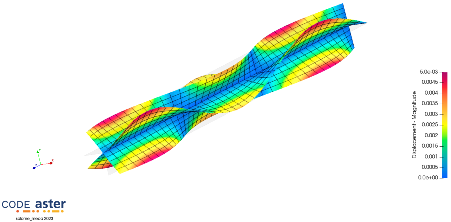
(Figure 3.11 from Brush and Almroth 1975)

6. Food For Thought

- We're not yet ready to handle this (wait one more semester for AS3020), but some types of beam undergo **twisting instability**!
- In the right we have simply supported beams under axial compression - the beams twist before they bend under the instability.



Simply Supported Beam Under Axial Compression



Cantilevered Beam under Axial Compression (2nd mode)

Heads Up: You are designed to see this in your structures lab experiment!

References I

[1] Don Orr Brush and Bo O. Almroth. Buckling of Bars, Plates, and Shells, McGraw-Hill, 1975. ISBN: 978-0-07-008593-0 (cit. on pp. 2, 56, 81, 82).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).

[3] Richard Wiebe et al. "On Snap-Through Buckling". In: *52nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*. Denver, Colorado: American Institute of Aeronautics and Astronautics, Apr. 2011. ISBN: 978-1-60086-951-8. DOI: 10.2514/6.2011-2083. (Visited on 02/18/2025) (cit. on p. 63).

8. Class Discussions (Outside of Slides)

- Ball on a hill. 2D, 3D cases.
- Assumptions behind compression of a bar.

8.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

- Let us use the energy approach to study the post-buckling behavior of a beam.
- We've developed some intuition that buckling blows up the displacement levels. Let us revise our kinematic description to capture this.
- The (simplified) approach we will follow is as follows:
  1. **Write out nonlinear kinematics**, identify normal force $N = \int_{\mathcal{A}} \sigma_{ax} dA$ and moment $M = \int_{\mathcal{A}} -y\sigma_{ax} dA$.
  2. **Assume transverse deformation field** $v = V \sin\left(\frac{\pi x}{\ell}\right)$
  3. **Assume axial tip deflection** u_T **and derive axial deformation field.**
  4. **Express work done in terms of scalars** V **and** u_T**.** $\rightarrow$ **Extremize**.
  5. **Plot force deflection curves, analyze stability**.

8.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

Geometrically Nonlinear Kinematics

- The deformation field is written as $u_x = u - yv', \quad u_y = v$. Consider the deformation of a line from (x, y) to $(x + \Delta x, y)$:

$$(x, y) \to (x + u - yv', y + v),$$
$$(x + \Delta x, y) \to (x + \Delta x + u - yv' + (u' - yv'')\Delta x, y + v + v'\Delta x),$$
$$\Delta S = \Delta x, \quad \Delta s^2 = \Delta x^2((1 + u' - yv'')^2 + v'^2).$$

- We write the axial strain as

$$\epsilon_{ax} = \frac{1}{2}\frac{\Delta s^2 - \Delta S^2}{\Delta S^2} = (u' - yv'') + \frac{1}{2}\left((u' - yv'')^2 + v'^2\right)$$
$$\boxed{\epsilon_{ax} \approx (u' - yv'') + \frac{v'^2}{2}}.$$

- The final assumption is sometimes referred to as Von Karman strain assumptions.

8.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

- Nearly nothing changes in the equilibrium equations. We first write out the area-normal stresses and moments:

$$N = \int_{\mathcal{A}} E\epsilon_{ax} dA = EA(u' + \frac{v'^2}{2}), \quad M = \int_{\mathcal{A}} -yE\epsilon_{ax} dA = EIv''.$$

- The axial force balance reads:

$$N' = EA\frac{d}{dx}\left(u' + \frac{v'^2}{2}\right) = 0, \quad u(x)|_{x=0} = 0, \quad u|_{x=\ell} = u_T.$$

8.1. Post-Buckling Behavior (Out of Syllabus): Axial Problem

Class Discussions (Outside of Slides)

- We next **impose the transverse deformation field** $v(x) = V\sin\left(\frac{\pi x}{\ell}\right)$ on the axial problem. Solving this, we get

$$u(x) = -\frac{\pi V^2}{8\ell}\sin\left(\frac{2\pi x}{\ell}\right) + C_1 x + C_2.$$

- Boundary conditioned are imposed by setting $C_1 = \frac{u_T}{\ell}$ and $C_2 = 0$.
- The parameterized axial deformation field, therefore, is

$$\boxed{u(x; V, u_T) = \frac{u_T}{\ell}x - \frac{\pi V^2}{8\ell}\sin\left(\frac{2\pi x}{\ell}\right)}.$$

- Note that we have not said anything about V or u_T so far.

8.1. Post-Buckling Behavior (Out of Syllabus): Strain Energy Density

Class Discussions (Outside of Slides)

- The strain energy density (per unit length) is written as,

$$\begin{aligned}\mathcal{V} &= \int_{\mathcal{A}} \frac{E\epsilon_{ax}^2}{2} dA = \frac{E}{2}\int_{\mathcal{A}} (u' - yv'' + \frac{v'^2}{2})^2 dx \\ &= \frac{EA}{2}\left(u' + \frac{v'^2}{2}\right)^2 + \frac{EI}{2}v''^2 \approx \frac{EI}{2}v''^2 + \frac{EA}{2}\frac{v'^4}{4}.\end{aligned}$$

- Note that we have assumed $u_T \to 0$, i.e., providing negligible influence on the overall potential energy.
- Substituting the assumed deformation field $v = V\sin(\frac{\pi x}{\ell})$ and integrating over $(0, \ell)$ we have,

$$\begin{aligned}\mathcal{V}_{tot} &= \int_0^\ell \mathcal{V}(x)dx = \frac{\pi^4 EI}{4\ell^3}V^2 + \frac{3\Pi^4 EA}{64\ell^3}V^4 \\ &= \frac{\pi^2 P_{cr}}{4\ell}V^2 + \frac{3\pi^2 AP_{cr}}{64I\ell}V^4.\end{aligned}$$

8.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)

- The work done by an axial compressive load P is given by

$$\Pi = \int_0^\ell \int_{\mathcal{A}} \frac{P}{A} \varepsilon_{ax} dA dx = \int_0^\ell \int_{\mathcal{A}} \frac{P}{A}(u' - yv'' + \frac{v'^2}{2}) dA dx$$

$$= P \int_0^\ell u' dx + \frac{P}{2} \int_0^\ell v'^2 dx$$

$$\boxed{\Pi = P u_T + \frac{\pi^2 P}{4\ell} V^2}.$$

- So the total work scalar ($W = \Pi - \mathcal{V}_{tot}$) is given as (we ignore u_T here)

$$W(V) = \frac{\pi^2}{4\ell}(P - P_{cr})V^2 - \frac{3\pi^2 A}{64 I \ell} P_{cr} V^4.$$

8.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)

- Stationarizing the work we get,

$$\frac{dW}{dV} = \frac{\pi^2 P_{cr}}{2\ell} V \left(\left(\frac{P}{P_{cr}} - 1 \right) - \frac{3A}{8I} V^2 \right) \implies V = 0, \pm\sqrt{\frac{8I}{3A}\left(\frac{P}{P_{cr}} - 1\right)}.$$

  Note that the non-trivial solution is only active for $P >= P_{cr}$.

- We can next estimate u_T easily by applying the boundary conditions.

8.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)

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Post-Buckling Solution