



# AS2070: Aerospace Structural Mechanics (V1)

## Module 1: Elastic Stability

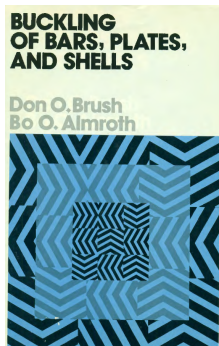
**Instructor: Nidish Narayanaa Balaji**

Dept. of Aerospace Engg., IIT Madras, Chennai

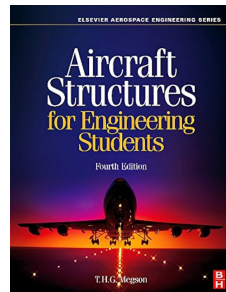
January 19, 2026

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*Chapters 1-3 in Brush and Almroth (1975).*

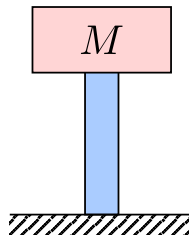


*Chapters 7-9 in Megson (2013)*

# 1. Introduction

Structural Stability: What?

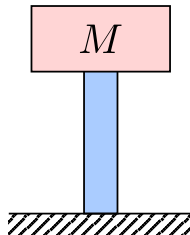
- Consider supporting a mass  $M$  on the top of a rod.



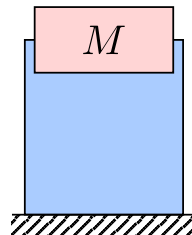
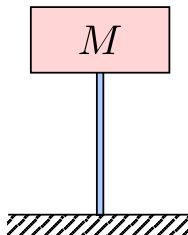
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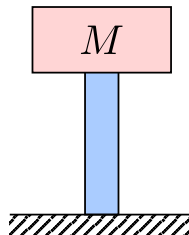
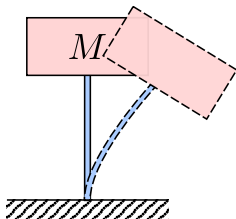
**Two Extreme Cases:**



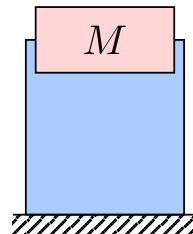
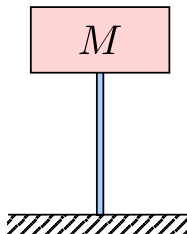
# 1. Introduction

Structural Stability: What?

- Consider supporting a mass  $M$  on the top of a rod.
- Collapse is imminent on at least one!



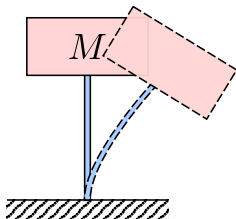
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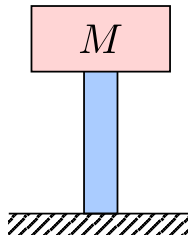
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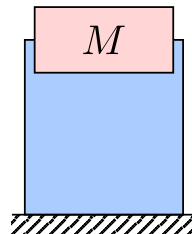
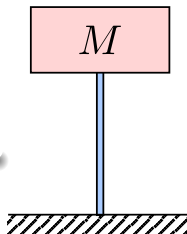
- Consider supporting a mass  $M$  on the top of a rod.
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How can we mathematically describe this?



Two Extreme Cases:



# 1. Introduction

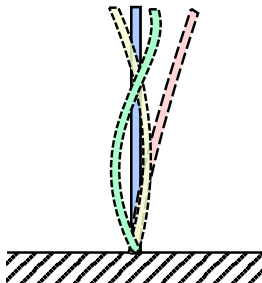
## Structural Stability: Perturbation Behavior

### Perturbation Behavior

Key insight we will invoke is behavior under **perturbation**:

*How would the system respond if I slightly perturb it?*

- Mathematically, by perturbation we mean *any change to the system's configuration*.
- In this case, this could be different deflection shapes.



# 1. Introduction

Structural Stability: Perturbation Behavior

## Perturbation Behavior

Key insight we will invoke is behavior under **perturbation**:

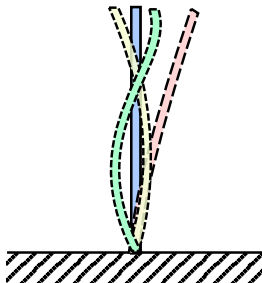
*How would the system respond if I slightly perturb it?*

- Mathematically, by perturbation we mean *any change to the system's configuration*.
- In this case, this could be different deflection shapes.

### Question (Slightly more specific)

What will the system tend to do if an arbitrarily small magnitude of perturbation is introduced?

- Will it tend to **return to its original configuration**?
- Will it **blow up**?
- Will it do **something else entirely**?





# 1.1. Elastic Stability

## Introduction

What do these words mean?

Elastic  $\rightarrow$  Reversible  $\rightarrow$  Conservative

### Conservative System

- The restoring force of a conservative system can be written using a gradient of a **potential function**:

$$\underline{F} = -\nabla U.$$

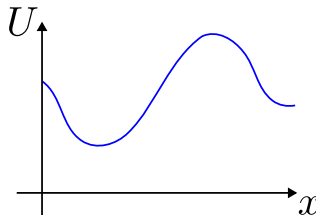
### Equilibrium

- System achieves equilibrium when  $\underline{F} = \underline{0}$ , i.e.,

$$\nabla U = 0.$$

### 1D Example

Consider a system whose configuration is expressed by the scalar  $x$  and the potential is as shown.



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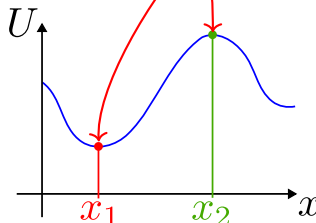
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Consider a system expressed by the scalar potential is as shown.

These are the equilibria



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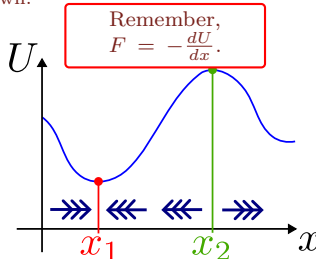
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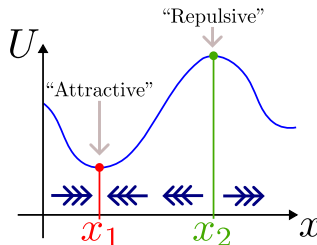
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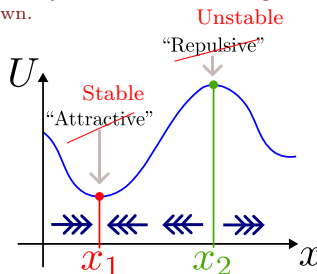
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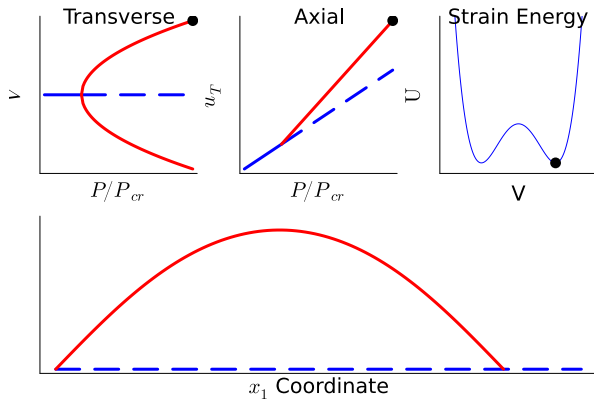
Consider a system whose configuration is expressed by the scalar  $x$  and the potential is as shown.



## 1.2. Bifurcation

### Introduction

A system is said to have **undergone a bifurcation** if its state of stability has changed due to the variation of some parameter.

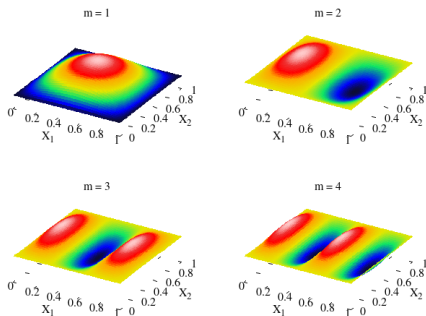


*Example: A pinned-pinned beam undergoing axial loading.*

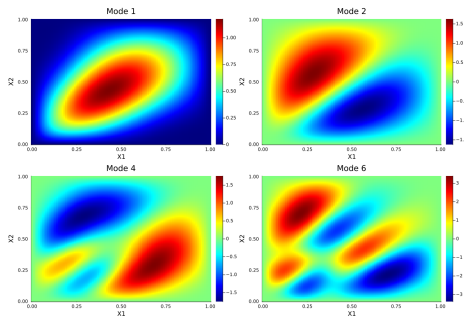
# 1.3. Modes of Stability Loss

## Introduction

The **configuration** that a system can assume as it undergoes a bifurcation is the *mode* of the stability loss.



*Example: Thin plate (pinned) under axial loading*



*Example: Thin plate (pinned) under shear loading*

## 2.1. Equilibrium Equations

### Euler Buckling of Columns

#### Equilibrium Equations

$$N' - (V\beta)' = 0$$

$$V' + N\beta' + N'\beta = 0$$

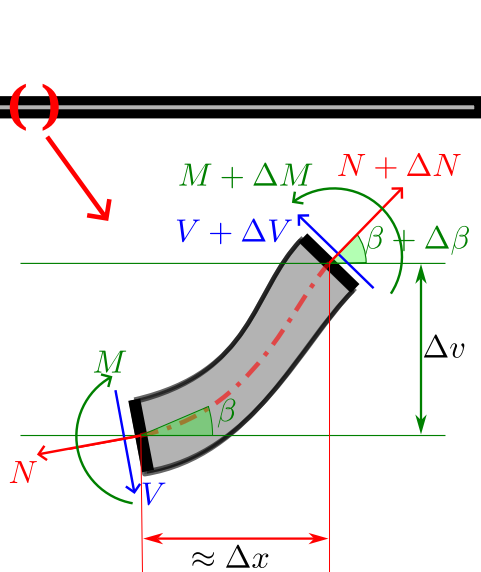
$$M' + V = 0$$

#### Assumption: $V\beta \ll N$

$$N' = 0$$

$$V' + N\beta' = 0$$

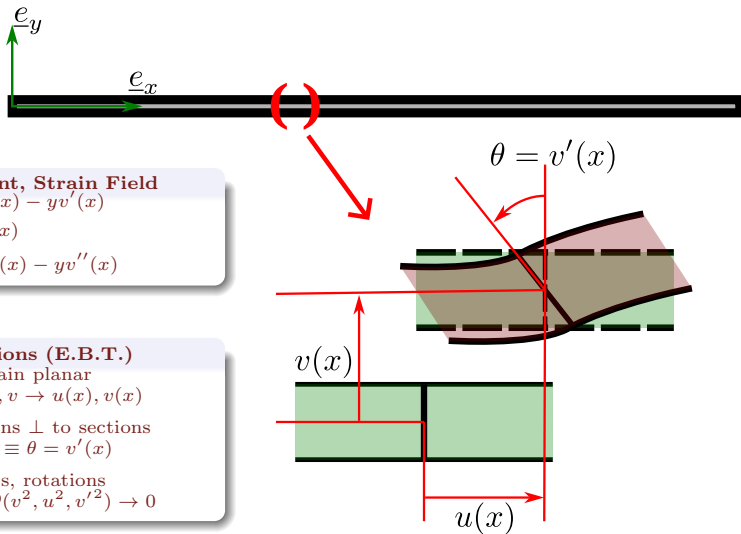
$$M'' - N\beta' = 0$$





## 2.2. Kinematic Description

### Euler Buckling of Columns



#### Displacement, Strain Field

$$u_x = u(x) - yv'(x)$$

$$u_y = v(x)$$

$$\epsilon_{xx} = u'(x) - yv''(x)$$

#### Assumptions (E.B.T.)

Plane sections remain planar

$$u, v \rightarrow u(x), v(x)$$

Neutral Axis remains  $\perp$  to sections

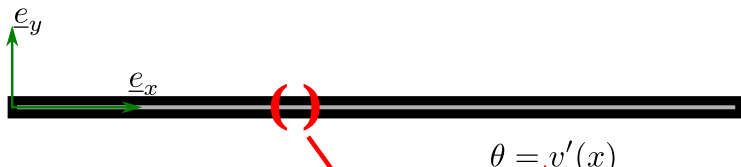
$$\beta \equiv \theta = v'(x)$$

Small displacements, rotations

$$\mathcal{O}(v^2, u^2, v'^2) \rightarrow 0$$

## 2.2. Kinematic Description

### Euler Buckling of Columns



#### Displacement, Strain

$$u_x = u(x) - y\theta$$

$$u_y = v(x)$$

$$\varepsilon_{xx} = u'(x) - y\theta'$$

#### Constitutive Modeling

$$\sigma_{xx} = E\varepsilon_{xx} = Eu' - yE\theta'$$

$$N = \int_{\mathcal{A}} \sigma_{xx} = EAu'$$

$$M = \int_{\mathcal{A}} -y\sigma_{xx} = EI\theta'$$

$$\theta = v'(x)$$

#### Assumptions

Plane sections remain plane

$$u, v \rightarrow$$

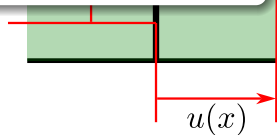
**Note:**  $y$  measured in Centroidal coordinates s.t.  $\int_{\mathcal{A}} y = 0$ .

Neutral Axis remains  $\perp$  to sections

$$\beta \equiv \theta = v'(x)$$

Small displacements, rotations

$$\mathcal{O}(v^2, u^2, v'^2) \rightarrow 0$$



## 2.3. The Linear Buckling Problem

### Euler Buckling of Columns

- Substituting, we are left with,

$$N' = \boxed{EAu'' = 0}, \quad M'' - N\beta' = \boxed{EIv'''' - Nv'' = 0}.$$

#### Axial Problem

- Boundary conditions representing axial compression:

$$u(x=0) = 0, \quad EAu'(x=\ell) = -P$$

- Solution:

$$\boxed{u(x) = -\frac{P}{EA}x}$$

#### Transverse Problem

- Substituting  $N = -P$  we have,

$$v'''' + k^2 v'' = 0, \quad k^2 = \frac{P}{EI}.$$

- The general solution to this **Homogeneous ODE** are

$$\boxed{v(x) = A_0 + A_1 x + A_2 \cos kx + A_3 \sin kx}$$

- Boundary conditions on the transverse displacement function  $v(x)$  are necessary to fix  $A_0, A_1, A_2, A_3$ .

## 2.3.1. The Pinned-Pinned Column

### The Linear Buckling Problem

- For a Pinned-pinned beam we have  $v = 0$  on the ends and zero reaction moments at the supports:

$$v = 0, \quad x = \{0, \ell\}$$

$$v'' = 0, \quad x = \{0, \ell\}$$

- So the general solution reduces to

$$v(x) = A_3 \sin kx,$$

with the boundary condition

$$A_3 \sin k\ell = 0.$$

- Apart from the trivial solution ( $A_3 = 0$ ) we have

$$k_{(n)}\ell = n\pi \implies k_n = n\frac{\pi}{\ell}$$

or in terms of the compressive load  $P$ ,

$$P_{cr,n} = n^2 \frac{\pi^2 EI}{\ell^2}$$

- Interpretation:** If  $P \neq P_{cr,n}$ ,  $A_3 = 0$  to satisfy boundary conditions. But for  $P = P_{cr,n}$ ,  $A_3$  **CAN BE ANYTHING!**

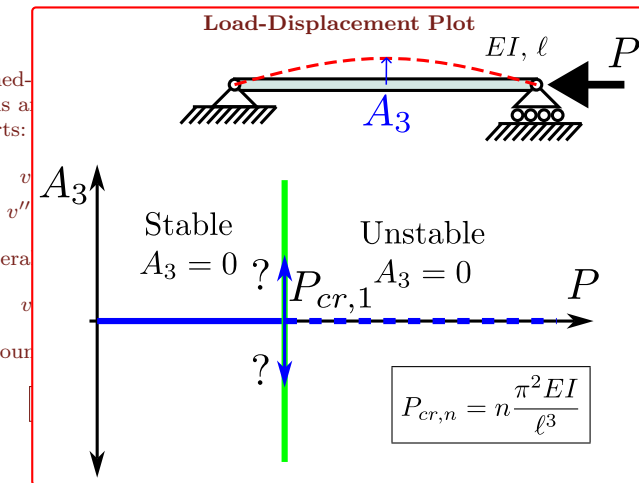
## 2.3.1. The Pinned-Pinned Column

### The Linear Buckling Problem

- For a Pinned-Pinned column, the boundary conditions on the ends are:

- So the general solution is:

with the boundary conditions:



ion ( $A_3 = 0$ )

$$= n \frac{\pi}{\ell}$$

ve load  $P$ ,

$$\left[ \frac{I}{\ell^3} \right]$$

,  $n$ ,  $A_3 = 0$  to

But for

## 2.3.1. The Pinned-Pinned Column: The Imperfect Case I

### The Linear Buckling Problem

- Suppose there are initial imperfections in the beam's neutral axis such that the neutral axis can be written as  $v_0(x)$ .
- Noting that strains are accumulated only on the *relative displacement*  $v(x) - v_0(x)$ , we write

$$EI(v - v_0)'''' + Pv'' = 0.$$

Note that the axial load  $P$  acts on the **net rotation** of the deflected beam, so we do not need to use  $(v - v_0)''$  here.

- The governing equations become

$$EIv'''' + Pv'' = EIv_0'''' ,$$

or, in more convenient notation,

$$\boxed{v'''' + k^2 v'' = v_0''''}.$$

## 2.3.1. The Pinned-Pinned Column: The Imperfect Case II

### The Linear Buckling Problem

- Describing the imperfect neutral axis using an infinite series,

$$v_0 = \sum_n C_n \sin\left(n \frac{\pi x}{\ell}\right) \quad \left( \Rightarrow v_0'''' = \sum_n \left(n \frac{\pi}{\ell}\right)^4 C_n \sin\left(n \frac{\pi x}{\ell}\right) \right),$$

the governing equations become

$$v'''' + k^2 v'' = \sum_n \left(n \frac{\pi}{\ell}\right)^4 C_n \sin\left(n \frac{\pi x}{\ell}\right).$$

- This is solved by,

$$\begin{aligned} v(x) &= \sum_n \frac{\left(n \frac{\pi}{\ell}\right)^2}{\left(n \frac{\pi}{\ell}\right)^2 - k^2} C_n \sin\left(n \frac{\pi x}{\ell}\right) \\ &= \sum_n \frac{\frac{n^2 \pi^2 EI}{\ell^2}}{\frac{n^2 \pi^2 EI}{\ell^2} - P} C_n \sin\left(n \frac{\pi x}{\ell}\right) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin\left(n \frac{\pi x}{\ell}\right) \end{aligned}$$

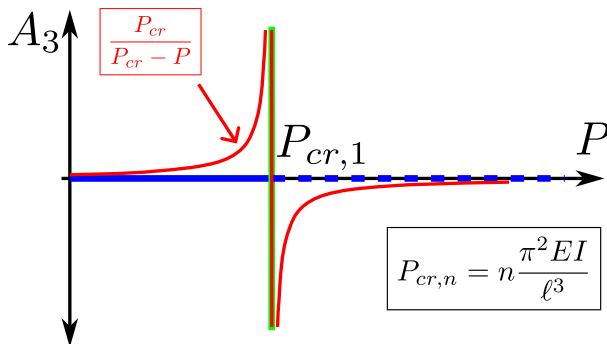
## 2.3.1. The Pinned-Pinned Column: The Imperfect Case

### The Linear Buckling Problem

- Look carefully at the solution

$$v(x) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n \frac{\pi x}{\ell}).$$

- Clearly  $P \rightarrow P_{cr,n}$  are **singularities**. Even for very small  $C_n$ , the “blow-up” is huge.





## 2.3.2. The Southwell Plot

### The Linear Buckling Problem

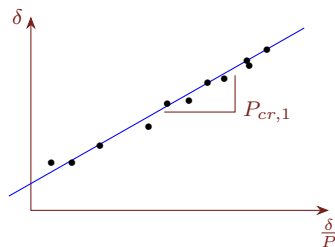
- The relative deformation amplitude at the mid-point is given as (for  $P < P_{cr,1}$ ),

$$\delta \approx \frac{P_{cr,1}}{P_{cr,1} - P} C_1 - C_1 = \frac{C_1}{\frac{P_{cr,1}}{P} - 1}$$

$$\Rightarrow \boxed{\delta = P_{cr,1} \frac{\delta}{P} - C_1}$$

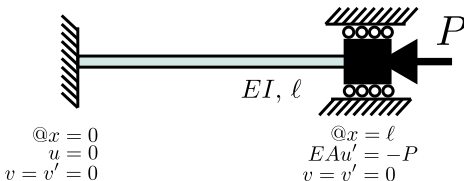
### The Southwell Plot

- Plotting  $\delta$  vs  $\frac{\delta}{P}$  allows **Non-Destructive Evaluation of the critical load**
- $P_{cr,1}$  is estimated without having to buckle the column



## 2.3.3. The Clamped-Clamped Column

### The Linear Buckling Problem



- The axial solution is the same as before:  

$$u(x) = -\frac{P}{EA}x.$$
- The transverse general solution also has the same form but boundary conditions are different.

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 1 & x & \cos(kx) & \sin(kx) \\ 0 & 1 & -k \sin(kx) & k \cos(kx) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

- The boundary conditions may be expressed as

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & \ell & \cos(k\ell) & \sin(k\ell) \\ 0 & 1 & -k \sin(k\ell) & k \cos(k\ell) \end{bmatrix}}_{\underline{\underline{M}}} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

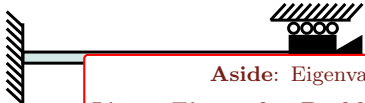
- There can be non-trivial solutions only when  $\underline{\underline{M}}$  is singular, i.e., **for choices of  $k$  such that  $\Delta(\underline{\underline{M}}) = 0$ .**

### The Eigenvalue Problem

This problem setting of finding  $k$  such that  $\Delta(\underline{\underline{M}}(k)) = 0$  is known as an **eigenvalue problem**.

## 2.3.3. The Clamped-Clamped Column

### The Linear Buckling Problem



• The boundary conditions may be expressed as

**Aside: Eigenvalue Problems ( $\underline{\underline{M}} \in \mathbf{R}^{d \times d}$ )**

**Linear Eigenvalue Problem** ( $d$  eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M}}_0 + k\underline{\underline{M}}_1$$

**Quadratic Eigenvalue Problem** ( $2d$  eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M}}_0 + k\underline{\underline{M}}_1 + k^2\underline{\underline{M}}_2$$

• The axial displacement  $u(x) = 0$

• The transverse displacement  $v(x)$  and its derivative  $v'(x)$  are different at the clamped ends.

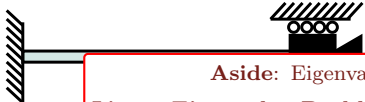
that  
ue

**problem.**

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -k \sin(kx) & k \cos(kx) \end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \end{bmatrix}$$

## 2.3.3. The Clamped-Clamped Column

### The Linear Buckling Problem



• The boundary conditions may be expressed as

**Aside: Eigenvalue Problems ( $\underline{\underline{M}} \in \mathbf{R}^{d \times d}$ )**

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$$\underline{\underline{M}}(k) = \underline{\underline{M}}_0 + k\underline{\underline{M}}_1$$

**Quadratic Eigenvalue Problem** ( $2d$  eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M}}_0 + k\underline{\underline{M}}_1 + k^2\underline{\underline{M}}_2$$

Our matrix  $\underline{\underline{M}}(k)$  has  $k$ -dependency in terms of  $k$ ,  $\sin(k\ell)$ ,  $\cos(k\ell)$ , making this a **Nonlinear Eigenvalue Problem**.

•  $\Rightarrow \infty$  eigenvalues here (not always though!)

problem.

• The axial displacement  $u(x) = 0$

• The transverse displacement  $v(x)$  and its derivative  $v'(x)$  are different at the two ends.

At  $x = 0$ :

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -k \sin(kx) & k \cos(kx) \end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \end{bmatrix}$$

At  $x = \ell$ :

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \end{bmatrix}$$

## 2.3.3. The Clamped-Clamped Column I

### The Linear Buckling Problem

- We proceed to solve this as,

$$\Delta \left( \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & \ell & \cos(k\ell) & \sin(k\ell) \\ 0 & 1 & -k \sin(k\ell) & k \cos(k\ell) \end{bmatrix} \right) = -k (k\ell \sin(k\ell) + 2 \cos(k\ell) - 2)$$

- We set it to zero through the following factorizations:

$$\begin{aligned} \Delta(\underline{\underline{M}}(k)) &= -k \left( 2k\ell \sin\left(\frac{k\ell}{2}\right) \cos\left(\frac{k\ell}{2}\right) - 4 \sin^2\left(\frac{k\ell}{2}\right) \right) \\ &= -2k \sin\left(\frac{k\ell}{2}\right) \left( k\ell \cos\left(\frac{k\ell}{2}\right) - 2 \sin\left(\frac{k\ell}{2}\right) \right) = 0 \\ \implies \boxed{\sin\left(\frac{k\ell}{2}\right) = 0}, \quad (\text{or}) \quad \boxed{\tan\left(\frac{k\ell}{2}\right) = \frac{k\ell}{2}}. \end{aligned}$$

- Two “classes” of solutions emerge:

$$\begin{aligned} \textcircled{1} \quad \sin\left(\frac{k\ell}{2}\right) = 0 &\implies \frac{k_n \ell}{2} = n\pi \implies \boxed{P_n^{(1)} = 4n^2 \frac{\pi^2 EI}{\ell^2}} \\ \textcircled{2} \quad \tan\left(\frac{k\ell}{2}\right) = \frac{k\ell}{2} &\implies \frac{k_n \ell}{2} \approx 0, 4.49, 7.72, \dots \implies P_1^{(2)} \approx 8.98 \frac{\pi^2 EI}{\ell^2} \end{aligned}$$

## 2.3.3. The Clamped-Clamped Column II

### The Linear Buckling Problem





- The smallest critical load is  $P_n^{(1)} = 4 \frac{\pi^2 EI}{\ell^2} = \frac{\pi^2 EI}{(\frac{\ell}{2})^2}$ .

#### Concept of “Effective Length”

- **Question:** If the beam were simply supported, what would be the length such that it also has the same first critical load?
- Here it comes out to be  $\ell_{eff} = \frac{\ell}{2}$ .
- The column clamped on both ends can take the same buckling load as a column that is pinned on both ends with half the length.

## 2.3.3. The Clamped-Clamped Column III

### The Linear Buckling Problem

Boundary conditions	Critical load $P_{cr}$	Deflection mode shape	Effective length $KL$
Simple support– simple support	$\frac{\pi^2 EI}{L^2}$		$L$
Clamped-clamped	$4 \frac{\pi^2 EI}{L^2}$		$\frac{1}{2}L$
Clamped–simple support	$2.04 \frac{\pi^2 EI}{L^2}$		$0.70L$
Clamped-free	$\frac{1}{4} \frac{\pi^2 EI}{L^2}$		$2L$

*Effective lengths of beams with different boundary conditions (Figure from Brush and Almroth 1975)*

### Self-Study

- Derive the effective length for the clamped-simply supported and clamped-free columns.

## 2.3.3. The Clamped-Clamped Column: The Mode-shape

### The Linear Buckling Problem

- Let us substitute  $k_1 = \frac{2\pi}{\ell}$  into the matrix  $\underline{\underline{M}}(k_1)$  so that the boundary conditions now read as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \\ 1 & \ell & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- This implies the following:

$$A_1 = 0, \quad A_3 = 0, \quad A_2 = -A_0.$$

- So, if  $k = k_1$ , the solution has to be the following to satisfy the boundary conditions:

$$v = A_0 \left( 1 - \cos\left(\frac{2\pi x}{\ell}\right) \right) \equiv A_0 \sin^2\left(\frac{\pi x}{\ell}\right)$$



## 2.3.3. The Clamped-Clamped Column: The Mode-shape

### The Linear Buckling Problem

- Let us substitute  $k_1 = \frac{2\pi}{\ell}$  into the matrix  $\underline{\underline{M}}(k_1)$  so that the boundary conditions now read as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}$$

Buckling mode-shape

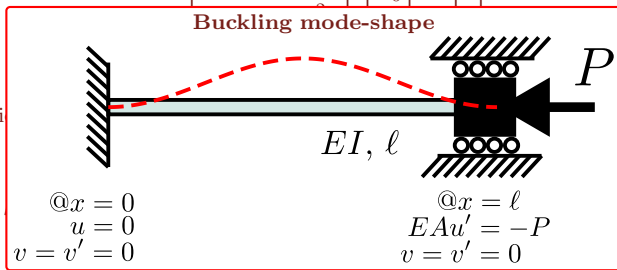
- This implies

- So, if  $k =$

$$\begin{aligned} @x = 0 \\ u = 0 \\ v = v' = 0 \end{aligned}$$

$$\begin{aligned} @x = \ell \\ EAu' = -P \\ v = v' = 0 \end{aligned}$$

boundary conditions:



### 3. Energy Perspectives

- Concept of conservative force field.
- Work done by a force field:

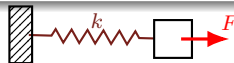
$$W(\underline{x}) \Big|_{\underline{x}_1}^{\underline{x}_2} = \int_{\underline{x}_1}^{\underline{x}_2} \underline{f}(\underline{x}) \cdot d\underline{x}.$$

- Introduction to work done.

$$W(\underline{x}) = \underbrace{\Pi(\underline{x})}_{\text{External Work}} - \underbrace{V(\underline{x})}_{\text{Internal Work/Potential Energy}}$$

#### Example

- Force balance reads:  $F = kx$
- Work done expression:  $W(x) = Fx - \frac{k}{2}x^2$



### 3. Energy Perspectives

- Expanding  $W$  about some  $\underline{x}_s$  we have,

$$W(\underline{x}_s + \delta \underline{x}) = W(\underline{x}_s) + \underline{\nabla} W|_{\underline{x}_s} \delta \underline{x} + \mathcal{O}(\delta \underline{x}^2).$$

- Stationarity of work:  $\delta W = \underline{\nabla} W(\underline{x}_s) \delta \underline{x} = 0, \quad \forall \quad \underline{x} \in \Omega$ , where  $\Omega$  is the configuration-space.

#### Example

- For the SDoF system above, we have  $W = Fx - \frac{k}{2}x^2$  and

$$\nabla W(x_s) = \frac{dW}{dx} = F - kx_s = 0 \implies x_s = \frac{F}{k}.$$

- Work-stationarity hereby gives a convenient definition for equilibrium.
- What about higher order effects?

### 3. Energy Perspectives

- Continuing the Taylor expansion (SDoF case) for  $W(x)$  we have,

$$W(x) = W(x_s) + \frac{dW}{dx}(x_s)\delta x + \frac{1}{2} \frac{d^2W}{dx^2}(x_s)\delta x^2 + \mathcal{O}(\delta x^3).$$

- At equilibrium,  $\frac{dW}{dx}$  is zero. The sign of  $\frac{d^2W}{dx^2}$  governs the local tendency of the work around equilibrium.

#### Example

- For the SDoF example,  $\frac{d^2W}{dx^2} = -k$ , implying  $W$  is maximized.
- If  $\frac{d^2W}{dx^2} < 0$ , then the second order effect of virtual displacements is to reduce the work scalar: **Stable Equilibrium**.
- The opposite case is **Unstable Equilibrium**.

### 3. Energy Perspectives

- Continuing the Taylor expansion (SDoF case) for  $W(x)$  we have,

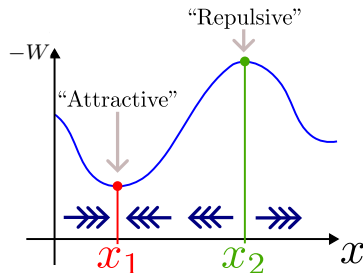
$$W(x) = W(x_0) + \frac{dW}{dx}\bigg|_{x_0} \delta x + \frac{1}{2} \frac{d^2W}{dx^2}\bigg|_{x_0} (\delta x)^2 + \mathcal{O}(\delta x^3).$$

Hypothetical Example

- At equilibrium,  $\frac{dW}{dx} = 0$   
around equilibrium

#### Example

- For the SDoF exar
- If  $\frac{d^2W}{dx^2} < 0$ , then  
scalar: **Stable E**
- The opposite case



tendency of the work

is to reduce the work

### 3.1. Snap-Through Buckling

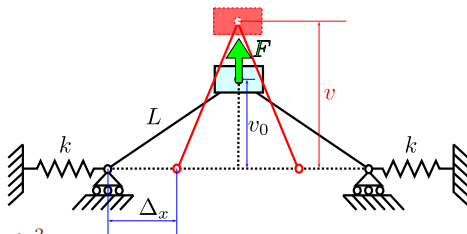
#### Energy Perspectives

- We will consider the SDoF model to the right (from Wiebe et al. 2011).
- The strain energy on the springs (two) is

$$U(v) = 2 \times \frac{k}{2} \Delta_x^2 = k \left( \sqrt{L^2 - v_0^2} - \sqrt{L^2 - v^2} \right)^2.$$

- The work done by the load (to take the mid-point from  $v_0$  to  $v$ ) is given by,

$$\Pi(v) = F(v - v_0).$$



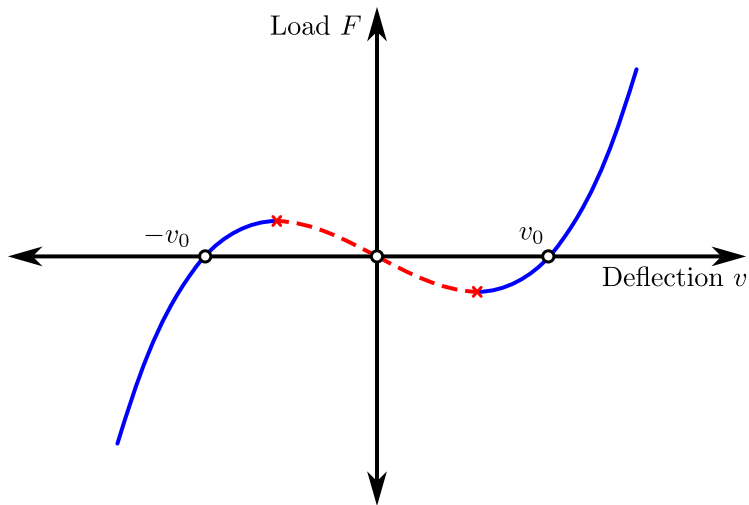
Setting  $\frac{dW}{dv} = 0$  we get

$$F = -2kv \left( 1 - \sqrt{\frac{L^2 - v_0^2}{L^2 - v^2}} \right).$$

## 3.1. Snap-Through Buckling

### Energy Perspectives

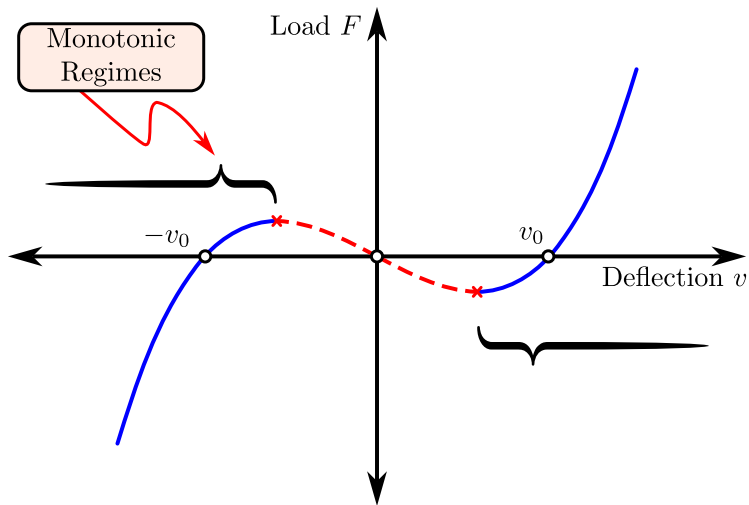
- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.



## 3.1. Snap-Through Buckling

### Energy Perspectives

- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.

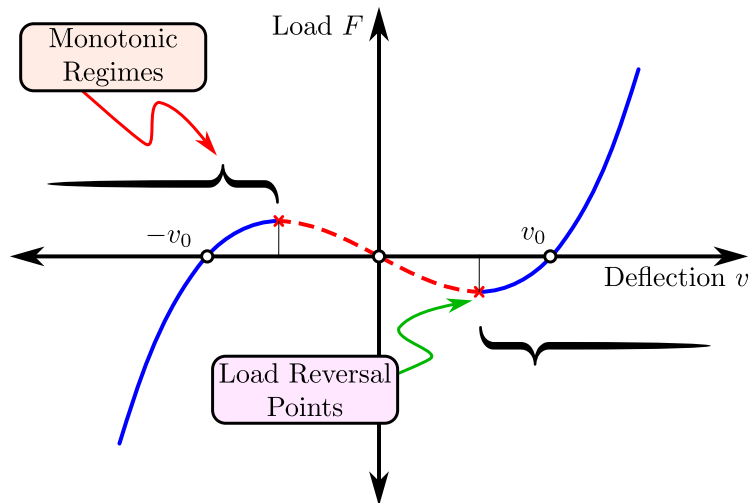




### 3.1. Snap-Through Buckling

#### Energy Perspectives

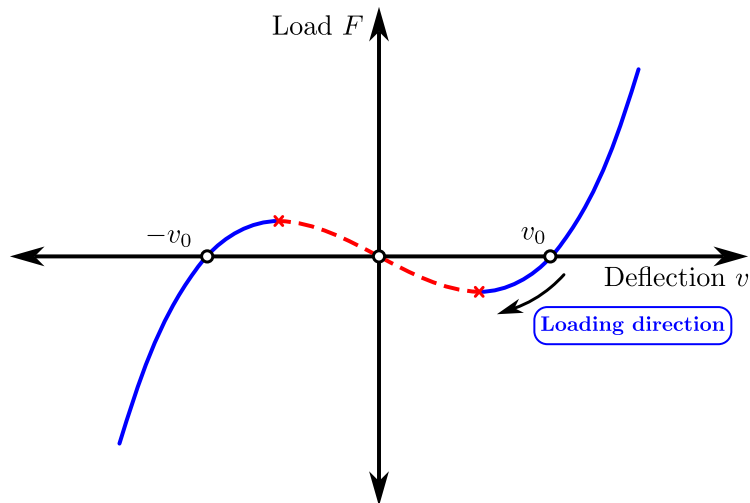
- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.



## 3.1. Snap-Through Buckling

### Energy Perspectives

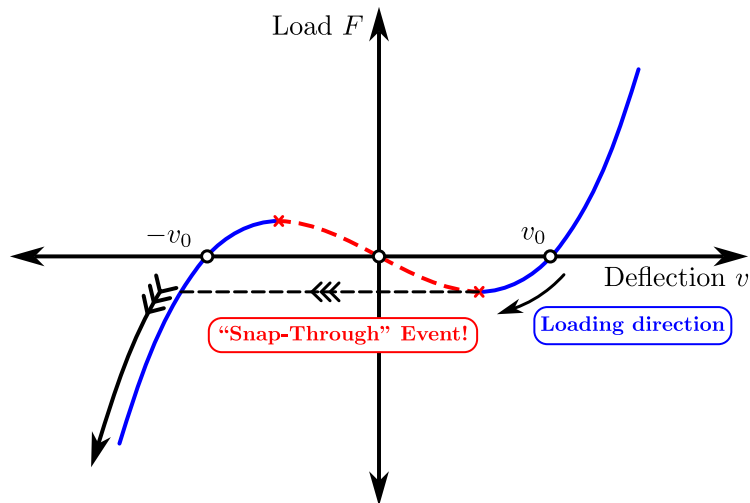
- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.



### 3.1. Snap-Through Buckling

#### Energy Perspectives

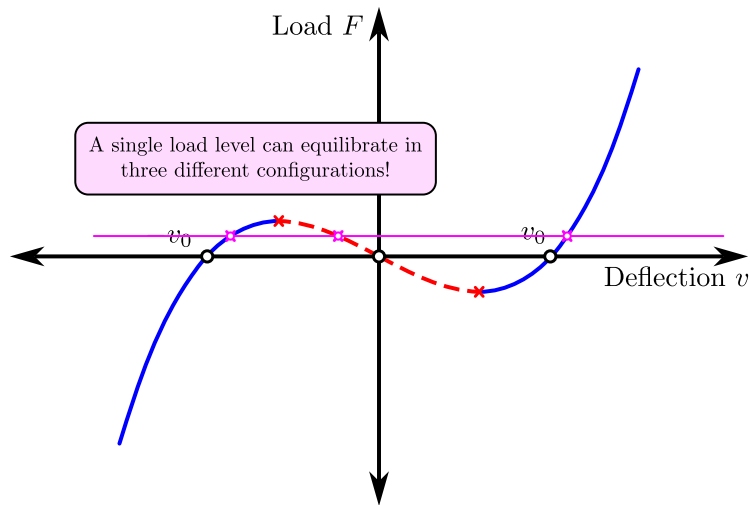
- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.



### 3.1. Snap-Through Buckling

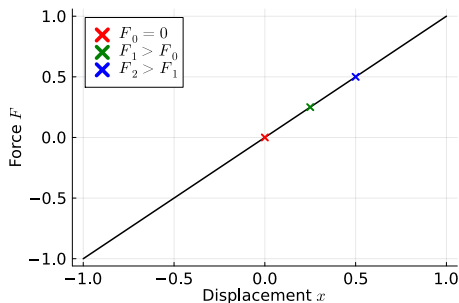
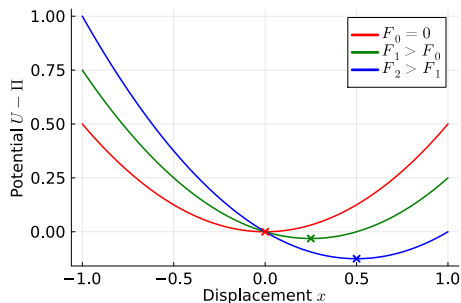
#### Energy Perspectives

- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.



# 3.1. Snap-Through Buckling: Equilibrium Visualization

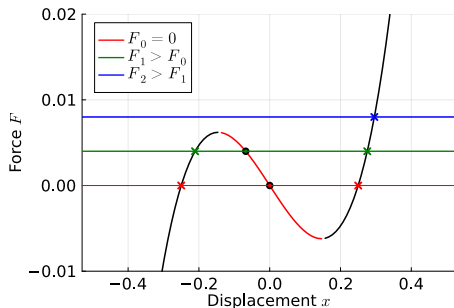
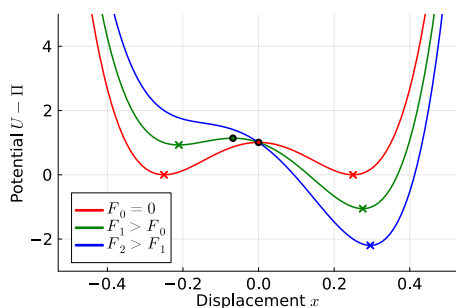
## Energy Perspectives



$$\text{Linear System: } U - \Pi = \frac{k}{2}x^2 - Fx$$

# 3.1. Snap-Through Buckling: Equilibrium Visualization

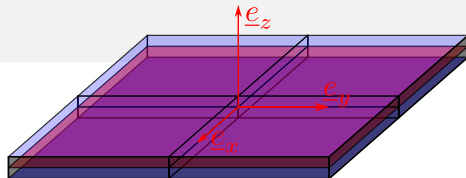
## Energy Perspectives



$$\text{Snap-Through Problem: } U - \Pi = k(\sqrt{L^2 - v_0^2} - \sqrt{L^2 - v^2})^2 - Fx$$

## 4.1. Plate Buckling

### Governing Equations



- Kirchhoff-Love Plate Theory.
- **Kinematic Assumptions:** Lines along section-thickness deform as lines and stay perpendicular to the neutral axis.
- Governing equations written in the form

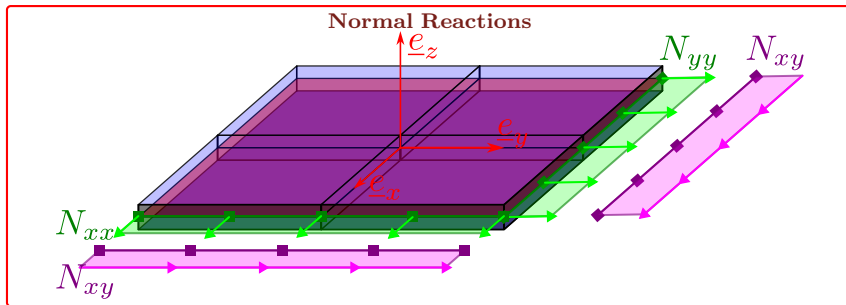
$$\frac{Et^3}{12(1-\nu^2)}(w_{,xxxx} + w_{,yyyy} + 2w_{,xxyy}) - (N_{xx}w_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}) = 0$$

$$D\nabla^4 w - (N_{xx}w_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}) = 0$$

- This is all that is needed to conduct buckling analysis - the procedure is identical as above!
- Before this, however, let us develop intuition on the different reaction force components and their kinematic relationships.

## 4.2. Reaction-Kinematics Relationships

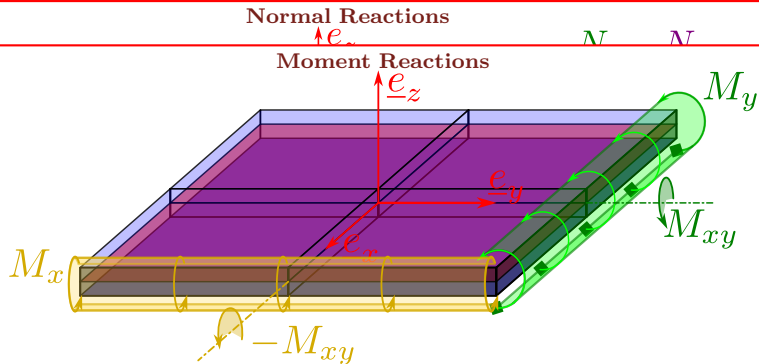
### Plate Buckling





## 4.2. Reaction-Kinematics Relationships

### Plate Buckling



## 4.2. Reaction-Kinematics Relationships

### Plate Buckling

Normal Reactions

$\underline{e}_z$

$N$

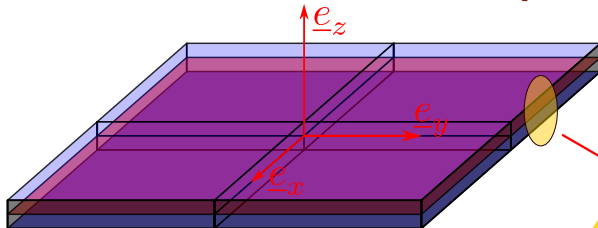
$N$

Moment Reactions

$\underline{e}_z$

$M_{yy}$

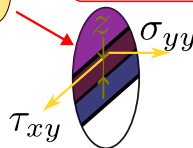
Stress-Moment Relationships



Face Moments

$$M_y = \int_{-t/2}^{t/2} -z\sigma_{yy}dz$$

$$M_{xy} = \int_{-t/2}^{t/2} z\tau_{xy}dz$$



## 4.2. Reaction-Kinematics Relationships

Plate Buckling

Normal Reactions

$\uparrow e_x$

$N$

$N$

Moment Reactions

$\uparrow e_z$

$M_y$

Stress-Moment Relationships

**Equilibrium Equations** (Shear Force-Moment Relationships)

$$\left. \begin{aligned} \sigma_{xx,x} + \tau_{xy,y} + \tau_{xz,z} &= 0 \\ \tau_{xy,x} + \sigma_{yy,y} + \tau_{yz,z} &= 0 \\ \tau_{xz,x} + \tau_{yz,z} + \sigma_{zz,z} &= 0 \end{aligned} \right\} \Rightarrow \begin{cases} Q_x &= M_{x,x} + M_{xy,y} \\ Q_y &= -M_{y,y} + M_{xy,x} \\ 0 &= Q_{x,x} + Q_{y,y} \end{cases}$$

**Note:**

- Although the shear strains  $\gamma_{xz}$  &  $\gamma_{yz}$  are assumed zero by the Kirchhoff kinematic assumptions, and thereby, the stresses  $\tau_{xz}$  &  $\tau_{yz}$  are also zero, **the shear forces can not be zero for equilibrium!!**
- They are defined as  $Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} dz$ ,  $Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz$ .

## 4.2. Reaction-Kinematics Relationships

### Plate Buckling

- With this background, we are ready to write the following:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_x \\ -M_y \\ M_{xy} \end{bmatrix} = \frac{E}{1 - \nu^2} \left( \begin{bmatrix} t & 0 \\ 0 & -\frac{t^3}{12} \end{bmatrix} \otimes \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \right) \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \\ w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$

- A moment-free boundary condition (simply supported edge) would imply simply setting the second derivatives ( $w_{,xx}, w_{,yy}, w_{,xy}$ ) to zero at the edge.

## 4.3. Thin Plates Under Uniaxial Compression

### Plate Buckling

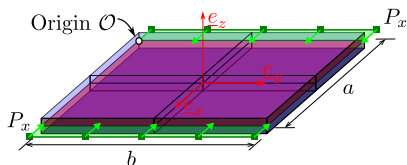


Plate under uniaxial compression

### Governing Equations

$$D\nabla^4 w + Pw_{,xx} = 0$$

$$\Rightarrow P_{cr,nm} = \frac{\pi^2 D}{b^2} \left( \frac{m}{a/b} + n^2 \frac{a/b}{m} \right)^2$$

(n=1 always for minimum critical load)

$$\Rightarrow P_{cr,m} = \frac{\pi^2 D}{b^2} \left( \frac{m}{a/b} + \frac{a/b}{m} \right)^2$$

$$P_{cr} = \frac{\pi^2 D}{b^2} \underbrace{\min_{m \in \mathbb{Z}^+} \left( \frac{m}{a/b} + \frac{a/b}{m} \right)^2}_{k_{cr}(a/b)}$$

### Ansatz (Simply Supported Case)

$$w(x, y) = \sum_{m,n} W_{mn} \sin \left( m \frac{\pi x}{a} \right) \sin \left( n \frac{\pi y}{b} \right)$$

### Boundary Conditions:

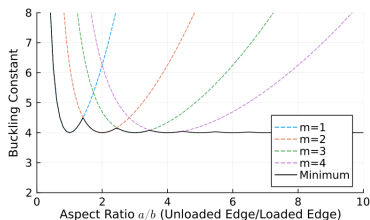
$$w = 0, M_x, M_y = 0 \quad \text{on} \quad \Gamma$$

## 4.3. Thin Plates Under Uniaxial Compression

### Plate Buckling

#### Buckling Constant

$$k_{cr}(r) = \min_{m \in \mathbb{Z}^+} \left( \frac{m}{r} + \frac{r}{m} \right)^2$$

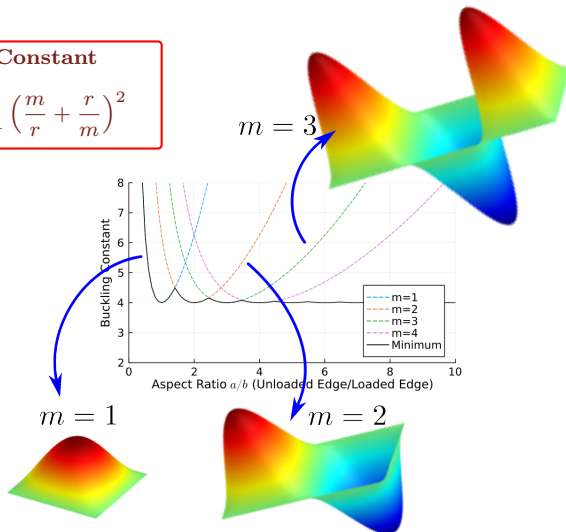


## 4.3. Thin Plates Under Uniaxial Compression

### Plate Buckling

#### Buckling Constant

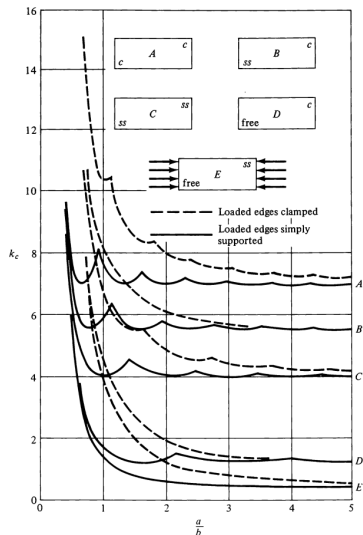
$$k_{cr}(r) = \min_{m \in \mathbb{Z}^+} \left( \frac{m}{r} + \frac{r}{m} \right)^2$$



## 4.3. Other Boundary Conditions

### Thin Plates Under Uniaxial Compression

- It is possible to conduct the same analysis for other (combinations) of boundary conditions.
- The analysis is slightly more tedious (due to the Ansatz not being as simple any more), **but possible along the same lines.**
- The critical plot comes out as shown in your textbook.



(Figure 3.9 from Brush and Almroth 1975)

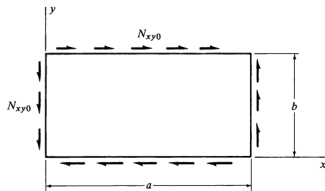


## 4.3. Other Boundary Conditions

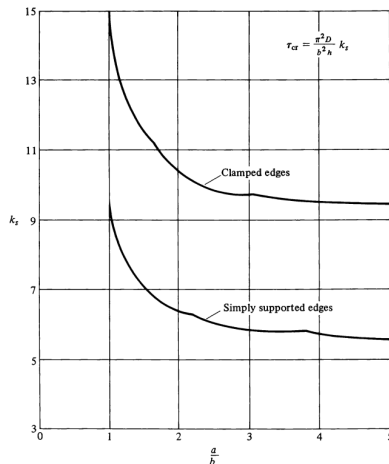
### Thin Plates Under Uniaxial Compression

- It is possible to conduct the same analysis for other (combinations) of boundary conditions.
- The analysis is slightly more tedious (due to the Ansatz not being as simple any more), **but possible along the same lines.**
- The critical plot comes out as shown in your textbook.

The same works for shear buckling too!



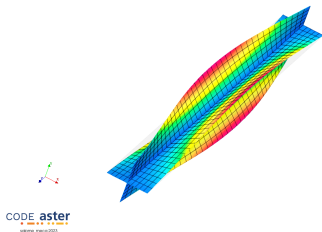
(Fig. 3.10 from Brush and Almroth 1975)



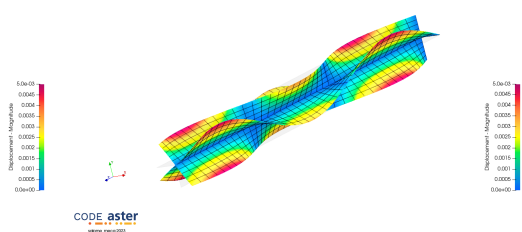
(Figure 3.11 from Brush and Almroth 1975)

## 5. Food For Thought

- We're not yet ready to handle this (wait one more semester for AS3020), but some types of beam undergo **twisting instability**!
- In the right we have simply supported beams under axial compression - the beams twist before they bend under the instability.



*Simply Supported Beam Under Axial Compression*



*Cantilevered Beam under Axial Compression (2nd mode)*

**Heads Up:** You are designed to see this in your structures lab experiment!

# References I

- [1] Don Orr Brush and Bo O. Almroth. **Buckling of Bars, Plates, and Shells**, McGraw-Hill, 1975. ISBN: 978-0-07-008593-0 (cit. on pp. [2](#), [31](#), [56](#), [57](#)).
- [2] T. H. G. Megson. **Aircraft Structures for Engineering Students**, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. [2](#)).
- [3] Richard Wiebe et al. “On Snap-Through Buckling”. In: *52nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*. Denver, Colorado: American Institute of Aeronautics and Astronautics, Apr. 2011. ISBN: 978-1-60086-951-8. DOI: [10.2514/6.2011-2083](#). (Visited on 02/18/2025) (cit. on p. [38](#)).

## 7. Class Discussions (Outside of Slides)

- Ball on a hill. 2D, 3D cases.
- Assumptions behind compression of a bar.

## 7.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

- Let us use the energy approach to study the post-buckling behavior of a beam.
- We've developed some intuition that buckling blows up the displacement levels. Let us revise our kinematic description to capture this.
- The (simplified) approach we will follow is as follows:
  - ➊ **Write out nonlinear kinematics**, identify normal force  $N = \int_{\mathcal{A}} \sigma_{ax} dA$  and moment  $M = \int_{\mathcal{A}} -y \sigma_{ax} dA$ .
  - ➋ **Assume transverse deformation field**  $v = V \sin\left(\frac{\pi x}{\ell}\right)$
  - ➌ **Assume axial tip deflection**  $u_T$  **and derive axial deformation field.**
  - ➍ **Express work done in terms of scalars**  $V$  **and**  $u_T$ .  $\rightarrow$  Extremize.
  - ➎ **Plot force deflection curves, analyze stability.**

## 7.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

### Geometrically Nonlinear Kinematics

- The deformation field is written as  $u_x = u - yv'$ ,  $u_y = v$ . Consider the deformation of a line from  $(x, y)$  to  $(x + \Delta x, y)$ :

$$(x, y) \rightarrow (x + u - yv', y + v),$$

$$(x + \Delta x, y) \rightarrow (x + \Delta x + u - yv' + (u' - yv'')\Delta x, y + v + v'\Delta x),$$

$$\Delta S = \Delta x, \quad \Delta s^2 = \Delta x^2((1 + u' - yv'')^2 + v'^2).$$

- We write the axial strain as

$$\epsilon_{ax} = \frac{1}{2} \frac{\Delta s^2 - \Delta S^2}{\Delta S^2} = (u' - yv'') + \frac{1}{2} \left( (u' - yv'')^2 + v'^2 \right)$$

$$\epsilon_{ax} \approx (u' - yv'') + \frac{v'^2}{2}.$$

- The final assumption is sometimes referred to as Von Karman strain assumptions.

## 7.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

- Nearly nothing changes in the equilibrium equations. We first write out the area-normal stresses and moments:

$$N = \int_{\mathcal{A}} E\epsilon_{ax} dA = EA(u' + \frac{v'^2}{2}), \quad M = \int_{\mathcal{A}} -yE\epsilon_{ax} dA = EIv''.$$

- The axial force balance reads:

$$N' = EA \frac{d}{dx} \left( u' + \frac{v'^2}{2} \right) = 0, \quad u(x)|_{x=0} = 0, \quad u|_{x=\ell} = u_T.$$

## 7.1. Post-Buckling Behavior (Out of Syllabus): Axial Problem

Class Discussions (Outside of Slides)

- We next **impose the transverse deformation field**  $v(x) = V \sin\left(\frac{\pi x}{\ell}\right)$  on the axial problem. Solving this, we get

$$u(x) = -\frac{\pi V^2}{8\ell} \sin\left(\frac{2\pi x}{\ell}\right) + C_1 x + C_2.$$

- Boundary conditioned are imposed by setting  $C_1 = \frac{u_T}{\ell}$  and  $C_2 = 0$ .
- The parameterized axial deformation field, therefore, is

$$u(x; V, u_T) = \frac{u_T}{\ell} x - \frac{\pi V^2}{8\ell} \sin\left(\frac{2\pi x}{\ell}\right).$$

- Note that we have not said anything about  $V$  or  $u_T$  so far.



## 7.1. Post-Buckling Behavior (Out of Syllabus): Strain Energy Density

Class Discussions (Outside of Slides)

- The strain energy density (per unit length) is written as,

$$\begin{aligned}\mathcal{V} &= \int_{\mathcal{A}} \frac{E\epsilon_{ax}^2}{2} dA = \frac{E}{2} \int_{\mathcal{A}} \left(u' - yv'' + \frac{v'^2}{2}\right)^2 dx \\ &= \frac{EA}{2} \left(u' + \frac{v'^2}{2}\right)^2 + \frac{EI}{2} v''^2 \approx \frac{EI}{2} v''^2 + \frac{EA}{2} \frac{v'^4}{4}.\end{aligned}$$

- Note that we have assumed  $u_T \rightarrow 0$ , i.e., providing negligible influence on the overall potential energy.
- Substituting the assumed deformation field  $v = V \sin(\frac{\pi x}{\ell})$  and integrating over  $(0, \ell)$  we have,

$$\begin{aligned}\mathcal{V}_{tot} &= \int_0^{\ell} \mathcal{V}(x) dx = \frac{\pi^4 EI}{4\ell^3} V^2 + \frac{3\pi^4 EA}{64\ell^3} V^4 \\ &= \frac{\pi^2 P_{cr}}{4\ell} V^2 + \frac{3\pi^2 AP_{cr}}{64I\ell} V^4.\end{aligned}$$

## 7.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)

- The work done by an axial compressive load  $P$  is given by

$$\begin{aligned}\Pi &= \int_0^\ell \int_{\mathcal{A}} \frac{P}{A} \varepsilon_{ax} dA dx = \int_0^\ell \int_{\mathcal{A}} \frac{P}{A} (u' - yv'' + \frac{v'^2}{2}) dA dx \\ &= P \int_0^\ell u' dx + \frac{P}{2} \int_0^\ell v'^2 dx\end{aligned}$$

$$\boxed{\Pi = Pu_T + \frac{\pi^2 P}{4\ell} V^2}.$$

- So the total work scalar ( $W = \Pi - \mathcal{V}_{tot}$ ) is given as (we ignore  $u_T$  here)

$$W(V) = \frac{\pi^2}{4\ell} (P - P_{cr}) V^2 - \frac{3\pi^2 A}{64I\ell} P_{cr} V^4.$$

## 7.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)

- Stationarizing the work we get,

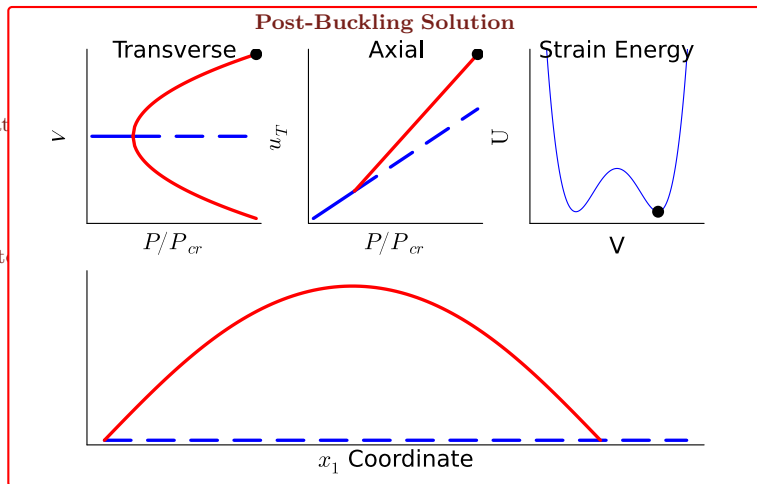
$$\frac{dW}{dV} = \frac{\pi^2 P_{cr}}{2\ell} V \left( \left( \frac{P}{P_{cr}} - 1 \right) - \frac{3A}{8I} V^2 \right) \implies V = 0, \pm \sqrt{\frac{8I}{3A} \left( \frac{P}{P_{cr}} - 1 \right)}.$$

Note that the non-trivial solution is only active for  $P \geq P_{cr}$ .

- We can next estimate  $u_T$  easily by applying the boundary conditions.

# 7.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)



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