AS2070: Aerospace Structural Mechanics (V2)
Module 1: Elastic Stability
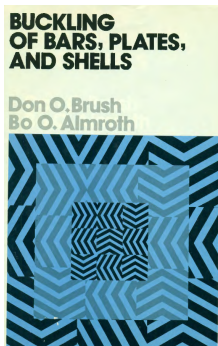
Instructor: Nidish Narayanaa Balaji

Dept. of Aerospace Engg., IIT Madras, Chennai
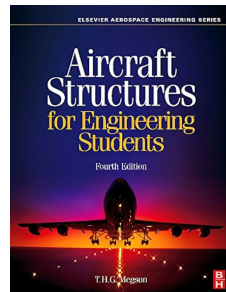
January 27, 2026

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BUCKLING OF BARS, PLATES, AND SHELLS

Don O. Brush
Bo O. Almroth

Chapters 1-3 in Brush and Almroth (1975).

ELSEVIER AEROSPACE ENGINEERING SERIES

Aircraft Structures for Engineering Students

Fourth Edition

T.H.G. Megson

Chapters 7-9 in Megson (2013)

1. Introduction
Structural Stability: What?

- Consider supporting a mass M on the top of a rod.

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Two Extreme Cases:

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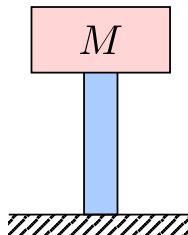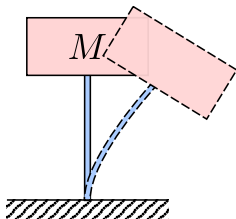
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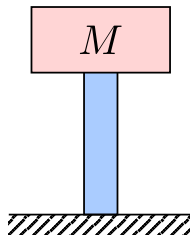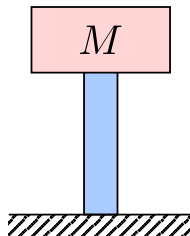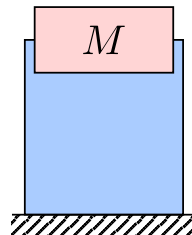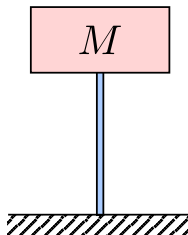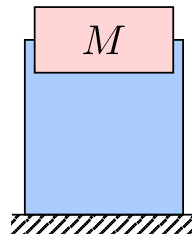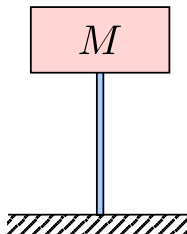
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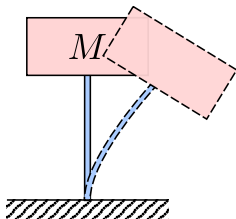
Two Extreme Cases:

How can we mathematically describe this?
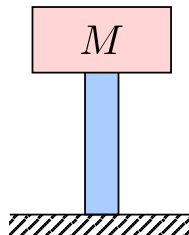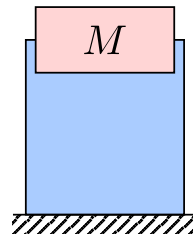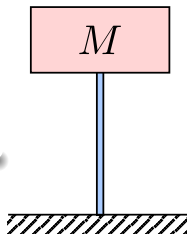
1. Introduction
Structural Stability: Perturbation Behavior

Perturbation Behavior

Key insight we will invoke is behavior under **perturbation**:

How would the system respond if I slightly perturb it?

- Mathematically, by perturbation we mean *any change to the system's configuration.*
- In this case, this could be different deflection shapes.
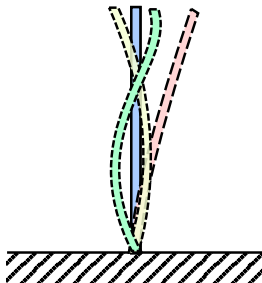
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How would the system respond if I slightly perturb it?

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Question (Slightly more specific)

What will the system tend to do if an arbitrarily small magnitude of perturbation is introduced?

- Will it tend to **return to its original configuration**?
- Will it **blow up**?
- Will it do **something else entirely**?

1.1. Elastic Stability
Introduction

What do these words mean?

Elastic → Reversible → Conservative

Conservative System

- The restoring force of a conservative system can be written using a gradient of a **potential function**:

$$\underline{F} = -\nabla U.$$

Equilibrium

- System achieves equilibrium when $\underline{F} = \underline{0}$, i.e.,

$$\nabla U = 0.$$

1D Example

Consider a system whose configuration is expressed by the scalar x and the potential is as shown.

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These are the equilibria

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Remember,
$F = -\frac{dU}{dx}$.

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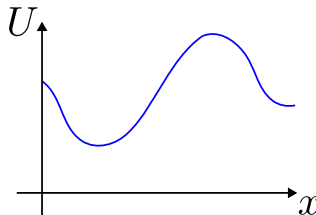
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1.2. Bifurcation
Introduction

A system is said to have **undergone a bifurcation** if its state of stability has changed due to the variation of some parameter.
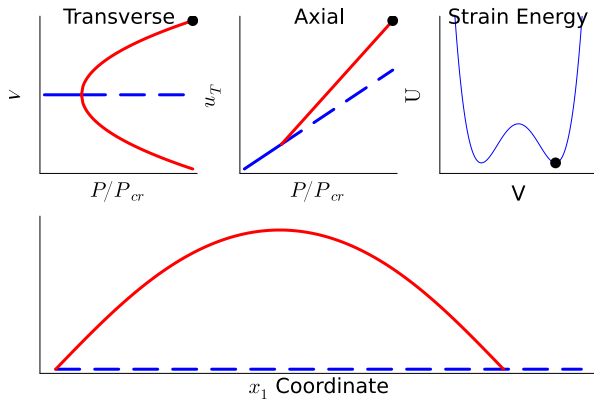
Example: A pinned-pinned beam undergoing axial loading.

1.3. Modes of Stability Loss
Introduction

The **configuration** that a system can assume as it undergoes a bifurcation is the *mode* of the stability loss.
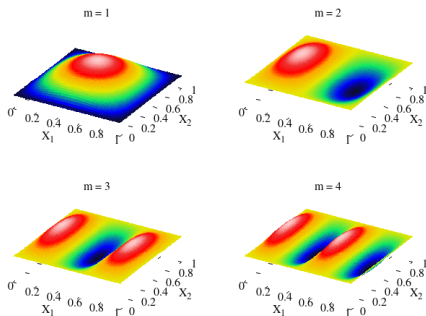
Example: Thin plate (pinned) under axial loading
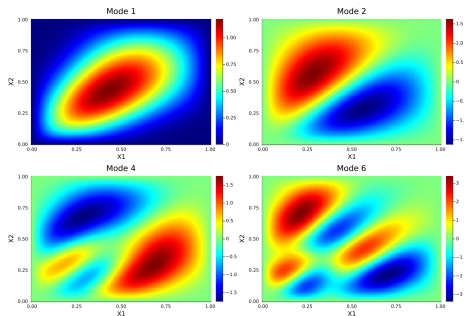
Example: Thin plate (pinned) under shear loading

2. The Principle of Virtual Work
An Optimization Framework for Mechanics

- Let us consider a point supported by a linear spring acted upon by a force.

Kinematics	**Equilibrium Condition**	**Constitutive Modeling**
Assume displacement to be restricted to the $\underline{e}_x$ axis: $$\underline{u} = u_x \underline{e}_x.$$	Net forces acting on the object is zero at equilibrium: $$\sum F_x = F - F_{spring} = 0.$$	Spring reaction is linearly proportional to displacement. $$F_{spring} = ku_x.$$

- While the above three already allow us to solve the problem of finding the equilibrium configuration of the system, we turn our attention to what happens in the vicinity of the equilibrium now.

- We define **Virtual Displacement** as an *infinitesimal displacement that is consistent with constraints.* (we read kinematic assumptions as constraints here) And we denote this by $\delta \underline{u}$.

- Since this is infinitesimal by definition, the associated work done, i.e., **Virtual Work** is written as

$$\delta W = \left(\sum F_x \right) \delta u_x = (F - ku_x)\delta u_x.$$

- Being a small quantity, regular calculus rules apply to the $\delta(\cdot)$ if considered as an infinitesimal operator just as they do for $d(\cdot)$, so the above simplifies as:

$$\delta W = \delta \left(Fu_x - \frac{k}{2}u_x^2 \right).$$

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> We have now defined a new quantity: **Work Potential**, with units of energy. This is written as $\boxed{W = \Pi - U}$.

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e_y e_x k F u_x

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  > Load Contribution $\Pi = \mathbf{F} u_x$

  > Elastic Contribution $U = \frac{1}{2} \mathbf{k} u_x^2$

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2. The Principle of Virtual Work
An Optimization Framework for Mechanics

- The **Work Potential** is quite a helpful quantity for us because
  1. **the first derivative $\partial W/\partial u_x$ represents the overall force** $\sum F_x$ acting on the system (static equilibrium is the same as stating $\partial W/\partial u_x = 0$), and
  2. **the second derivative $\partial^2 W/\partial u_x^2$ at equilibrium represents the surplus force as we move away** from the equilibrium (because this represents $\partial(\sum F_x)/\partial u_x$).
- An equilibrium can be sought by finding u_x for $\partial W/\varepsilon u_x = 0$.
- For classifying the equilibria that have been found, we use the following principles (based on arguments about what is happening to the surplus force as we move away):
  - If $\partial^2 W/\partial u_x^2 < 0$, the equilibrium is stable.
  - If $\partial^2 W/\partial u_x^2 > 0$, the equilibrium is unstable.
  - If $\partial^2 W/\partial u_x^2 = 0$, the equilibrium is neutrally stable (up to second order).
- The above is <u>mathematically identical</u> to an optimization problem posed as

$$\underset{u_x \in \mathbb{R}}{\text{extremize}} \quad W,$$

and the **maxima of this optimization problem are stable equilibria**, the **minima are unstable**, and **the saddles are neutral**.

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For the Spring Example,

$$W = F u_x - \frac{k}{2} u_x^2, \quad \frac{\partial W}{\partial u_x} = F - k u_x, \quad \frac{\partial^2 W}{\partial u_x^2} = -k.$$

- There exists exactly only equilibrium point for fixed F, k: $u_x^* = \frac{F}{k}$, and
- this equilibrium is <u>unconditionally</u> stable (always stable).

The above is mathematically identical to an optimization problem posed as

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and the **maxima of this optimization problem are stable equilibria**, the **minima are unstable**, and **the saddles are neutral**.

2.1. A Rigid Column Under Axial Load
The Principle of Virtual Work

- Now let us consider a <u>rigid column supported by a pin</u> as shown.
- It is also supported by a <u>torsional spring</u> which is governed by a linear constitutive law ($M_{spring} = k\theta$).
- Let's count the work done by each member (<u>only once per member</u>)

Elastic Contributions

- Rotational angle is the work-conjugate of moment.

$$M_{spring}\delta\theta = k\theta\delta\theta$$
$$= \delta\left(\frac{1}{2}k\theta^2\right),$$
$$\implies \boxed{U = \frac{k}{2}\theta^2}.$$

T

$\ell\sin\theta$

$\ell(1-\cos\theta)$

$(\cos\theta - 1)\ell$

θ

ℓ
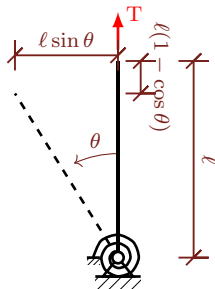
Load Contributions

- The load at the tip is $T\underline{e_y}$.
- The displacement of the tip is $\boxed{-\ell\sin\theta\underline{e_x} - \ell(1-\cos\theta)\underline{e_y}}$.
  So we have:
  $$\Pi = T\ell(\cos\theta - 1).$$
- We **do not** count moment contributions separately.

- So the work potential is written as: $\Pi - U = \boxed{W = T\ell(\cos\theta - 1) - \frac{k}{2}\theta^2}$.

2.1. A Rigid Column Under Axial Load
The Principle of Virtual Work

Work Potential

$$W = T\ell(\cos\theta - 1) - \frac{k}{2}\theta^2, \quad \frac{\partial W}{\partial \theta} = -(T\ell\sin\theta + k\theta), \quad \frac{\partial^2 W}{\partial \theta^2} = -(T\ell\cos\theta + k).$$

- Under leading term small angle assumptions ($\sin\theta \approx \theta, \cos\theta \approx 1$) we have,

$$\theta_{eq} = 0 \quad (\text{always}), \frac{\partial^2 W}{\partial \theta^2}\bigg|_{\theta_{eq}} = -(T\ell + k).$$

  The equilibrium is stable as long as $T > -\frac{k}{\ell}$.
  - Our prediction for $T < -\frac{k}{\ell}$ is just that there exists a **trivial equilibrium ($\theta_{eq} = 0$) that is unstable**.
- Under two-term small angle assumptions ($\sin\theta \approx \theta - \frac{\theta^3}{6}$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$) we have,

$$\theta_{eq} = 0, \pm\sqrt{6\left(1 + \frac{k}{T\ell}\right)}, \quad \text{with} \quad \frac{\partial^2 W}{\partial \theta^2} = -(T\ell + k), 2(T\ell + k),$$

  where we now predict two additional solutions.
  - Here we have a more precise prediction for $T < -\frac{k}{\ell}$: the non-trivial θ solutions are stable.

2.1. A Rigid Column Under Axial Load
The Principle of Virtual Work

The "Forced Response Curve" ($k = 1\,\text{N/m}, \ell = 1\,\text{m}$)

W = ⋯ + k).

- Under l⋯

The equ⋯
- Our⋯ ($\theta_{eq} = 0$)
  tha⋯
- Under t⋯ e have,
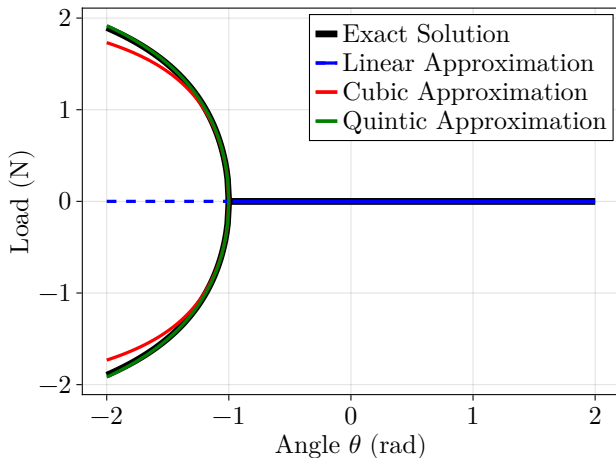
where w⋯
- Her⋯ ons are stable.

3.1. Equilibrium Equations
Euler Buckling of Columns

Equilibrium Equations
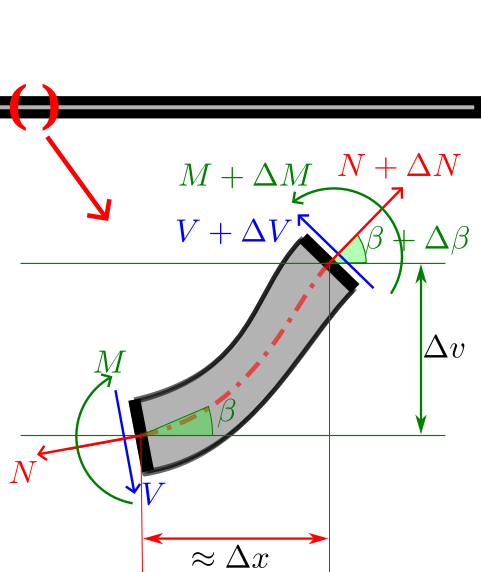$$N' - (V\beta)' = 0$$
$$V' + N\beta' + N'\beta = 0$$
$$M' + V = 0$$

Assumption: $V\beta \ll N$
$$N' = 0$$
$$V' + N\beta' = 0$$
$$M'' - N\beta' = 0$$

3.2. Kinematic Description
Euler Buckling of Columns
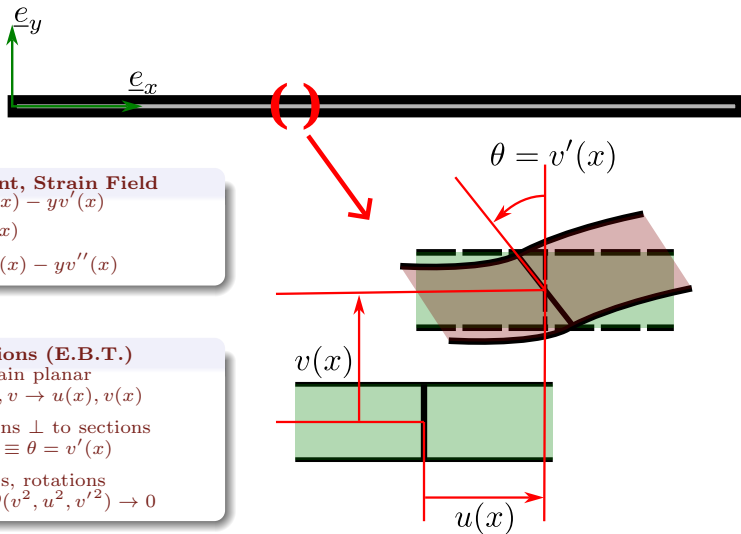
Displacement, Strain Field

$$u_x = u(x) - yv'(x)$$

$$u_y = v(x)$$

$$\varepsilon_{xx} = u'(x) - yv''(x)$$

Assumptions (E.B.T.)

Plane sections remain planar
$$u, v \rightarrow u(x), v(x)$$

Neutral Axis remains $\perp$ to sections
$$\beta \equiv \theta = v'(x)$$

Small displacements, rotations
$$\mathcal{O}(v^2, u^2, v'^2) \rightarrow 0$$

$$\theta = v'(x)$$

$$v(x)$$

$$u(x)$$

3.2. Kinematic Description
Euler Buckling of Columns

e_y

e_x

$\theta = v'(x)$

Displacement, St...

$u_x = u(x) - $

$u_y = v(x)$

$\varepsilon_{xx} = u'(x) - $

Constitutive Modeling

$$\sigma_{xx} = E\varepsilon_{xx} = Eu' - yEv''$$

$$N = \int_{\mathcal{A}} \sigma_{xx} = EAu'$$

$$M = \int_{\mathcal{A}} -y\sigma_{xx} = EIv''$$

Note: y measured in Centroidal coordinates s.t. $\int_{\mathcal{A}} y = 0$.

Assumptions (

Plane sections remain pl...

$u, v \to$

Neutral Axis remains $\perp$ to sections

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Small displacements, rotations

$\mathcal{O}(v^2, u^2, v'^2) \to 0$

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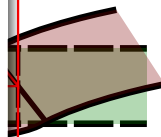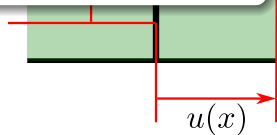
3.3. The Linear Buckling Problem
Euler Buckling of Columns

- Substituting, we are left with,

$$N' = \boxed{EAu'' = 0}, \quad M'' - N\beta' = \boxed{EIv'''' - Nv'' = 0}.$$

Axial Problem

- Boundary conditions representing axial compression:

$$u(x=0) = 0, \quad EAu'(x=\ell) = -P$$

- Solution:

$$\boxed{u(x) = -\frac{P}{EA}x}$$

Transverse Problem

- Substituting $N = -P$ we have,

$$v'''' + k^2 v'' = 0, \quad k^2 = \frac{P}{EI}.$$

- The general solution to this **Homogeneous ODE** are

$$v(x) = A_0 + A_1 x + A_2 \cos kx + A_3 \sin kx$$

- Boundary conditions on the transverse displacement function $v(x)$ are necessary to fix A_0, A_1, A_2, A_3.

3.3.1. The Pinned-Pinned Column
The Linear Buckling Problem

- For a Pinned-pinned beam we have $v = 0$ on the ends and zero reaction moments at the supports:

$$v = 0, \quad x = \{0, \ell\}$$
$$v'' = 0, \quad x = \{0, \ell\}$$

- So the general solution reduces to

$$v(x) = A_3 \sin kx,$$

with the boundary condition

$$\boxed{A_3 \sin k\ell = 0.}$$

- Apart from the trivial solution ($A_3 = 0$) we have

$$k_{(n)}\ell = n\pi \implies k_n = n\frac{\pi}{\ell}$$

or in terms of the compressive load P,

$$\boxed{P_{cr,n} = n^2 \frac{\pi^2 EI}{\ell^2}}$$

- **Interpretation**: If $P \neq P_{cr,n}$, $A_3 = 0$ to satisfy boundary conditions. But for $P = P_{cr,n}$, A_3 **CAN BE ANYTHING!**.

3.3.1. The Pinned-Pinned Column
The Linear Buckling Problem

Load-Displacement Plot

EI, ℓ

P

A_3

A_3

Stable
$A_3 = 0$? $P_{cr,1}$

Unstable
$A_3 = 0$

P

?

$$P_{cr,n} = n\frac{\pi^2 EI}{\ell^3}$$

- For a Pinned-... ion ($A_3 = 0$) on the ends a... the supports:

$= n\frac{\pi}{\ell}$

ve load P,

- So the genera...

with the boun... $_{r,n}$, $A_3 = 0$ to ... But for

3.3.1. The Pinned-Pinned Column: The Imperfect Case I
The Linear Buckling Problem

- Suppose there are initial imperfections in the beam's neutral axis such that the neutral axis can be written as $v_0(x)$.

- Noting that strains are accumulated only on the *relative displacement* $v(x) - v_0(x)$, we write

$$EI(v - v_0)'''' + Pv'' = 0.$$

  Note that the axial load P acts on the **net rotation** of the deflected beam, so we do not need to use $(v - v_0)''$ here.

- The governing equations become

$$EIv'''' + Pv'' = EIv_0'''',$$

  or, in more convenient notation,

$$\boxed{v'''' + k^2 v'' = v_0''''}.$$

3.3.1. The Pinned-Pinned Column: The Imperfect Case II
The Linear Buckling Problem

- Describing the imperfect neutral axis using an infinite series,

$$v_0 = \sum_n C_n \sin(n\frac{\pi x}{\ell}) \quad \left(\implies v_0'''' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell}) \right),$$

the governing equations become

$$v'''' + k^2 v'' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell}).$$

- This is solved by,

$$v(x) = \sum_n \frac{\left(n\frac{\pi}{\ell}\right)^2}{\left(n\frac{\pi}{\ell}\right)^2 - k^2} C_n \sin(n\frac{\pi x}{\ell})$$

$$= \sum_n \frac{\frac{n^2 \pi^2 EI}{\ell^2}}{\frac{n^2 \pi^2 EI}{\ell^2} - P} C_n \sin(n\frac{\pi x}{\ell}) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell})$$
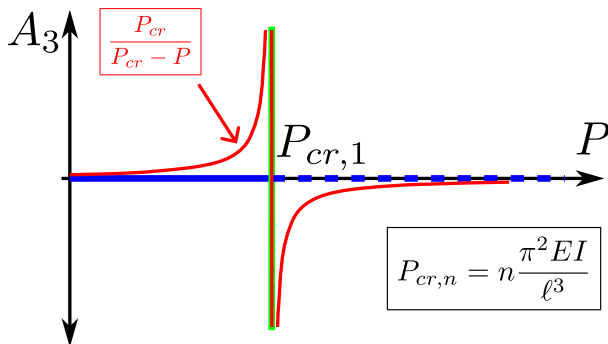
3.3.1. The Pinned-Pinned Column: The Imperfect Case
The Linear Buckling Problem

- Look carefully at the solution

$$v(x) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell}).$$

- Clearly $P \to P_{cr,n}$ are **singularities**. Even for very small C_n, the "blow-up" is huge.

$$P_{cr,n} = n\frac{\pi^2 EI}{\ell^3}$$

3.3.2. The Southwell Plot
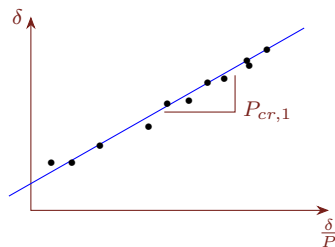The Linear Buckling Problem

- The relative deformation amplitude at the mid-point is given as (for $P < P_{cr,1}$),

$$\delta \approx \frac{P_{cr,1}}{P_{cr,1} - P} C_1 - C_1 = \frac{C_1}{\frac{P_{cr,1}}{P} - 1}$$

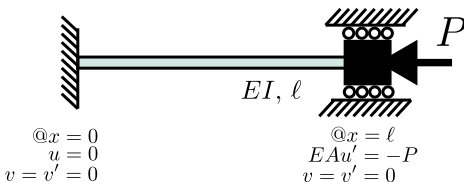$$\implies \boxed{\delta = P_{cr,1} \frac{\delta}{P} - C_1}$$

The Southwell Plot

- Plotting δ vs $\frac{\delta}{P}$ allows **Non-Destructive Evaluation of the critical load**
- $P_{cr,1}$ is estimated <u>without having to buckle the column</u>

3.3.3. The Clamped-Clamped Column
The Linear Buckling Problem

EI, ℓ

@$x = 0$
$u = 0$
$v = v' = 0$

@$x = \ell$
$EAu' = -P$
$v = v' = 0$

P

- The axial solution is the same as before: $u(x) = -\frac{P}{EA}x$.

- The transverse general solution also has the same form but boundary conditions are different.

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 1 & x & \cos(kx) & \sin(kx) \\ 0 & 1 & -k\sin(kx) & k\cos(kx) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

- The boundary conditions may be expressed as

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & \ell & \cos(k\ell) & \sin(k\ell) \\ 0 & 1 & -k\sin(k\ell) & k\cos(k\ell) \end{bmatrix}}_{\underline{M}} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- There can be non-trivial solutions only when $\underline{M}$ is singular, i.e., **for choices of k such that $\Delta(\underline{M}) = 0$.**

The Eigenvalue Problem

This problem setting of finding k such that $\Delta(\underline{M}(k)) = 0$ is known as an **eigenvalue problem**.

3.3.3. The Clamped-Clamped Column
The Linear Buckling Problem

@$x = 0$
$u = 0$
$v = v' = 0$

P

- The boundary conditions may be expressed as

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- The axia...
$u(x) = -$

- The tran...
the same...
are differ...

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -k\sin(kx) & k\cos(kx) \end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \end{bmatrix}$$

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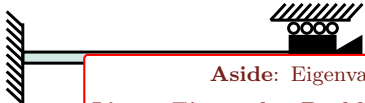
that
ue

problem.

Aside: Eigenvalue Problems ($\underline{\underline{M}} \in \mathbf{R}^{d \times d}$)

Linear Eigenvalue Problem (d eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}}$$

Quadratic Eigenvalue Problem ($2d$ eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}} + k^2\underline{\underline{M_2}}$$

3.3.3. The Clamped-Clamped Column
The Linear Buckling Problem

- The boundary conditions may be expressed as

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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- The axia...
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$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -k\sin(kx) & k\cos(kx) \end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \end{bmatrix}$$

ns only
oices

that
ue
problem.

Aside: Eigenvalue Problems ($\underline{M} \in \mathbf{R}^{d\times d}$)

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$$\underline{M}(k) = \underline{M_0} + k\underline{M_1}$$

Quadratic Eigenvalue Problem ($2d$ eigenvalues)

$$\underline{M}(k) = \underline{M_0} + k\underline{M_1} + k^2\underline{M_2}$$

Our matrix $\underline{M}(k)$ has k-dependency in terms of k, $\sin(k\ell)$, $\cos(k\ell)$, making this a **Nonlinear Eigenvalue Problem**.

$\implies \infty$ eigenvalues here (not always though!)

3.3.3. The Clamped-Clamped Column I
The Linear Buckling Problem

- We proceed to solve this as,

$$\Delta \left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & \ell & \cos(k\ell) & \sin(k\ell) \\ 0 & 1 & -k\sin(k\ell) & k\cos(k\ell) \end{bmatrix} \right) = -k\left(k\ell\sin(k\ell) + 2\cos(k\ell) - 2\right)$$

- We set it to zero through the following factorizations:

$$\Delta(\underline{\underline{M}}(k)) = -k\left(2k\ell\sin(\frac{k\ell}{2})\cos(\frac{k\ell}{2}) - 4\sin^2(\frac{k\ell}{2})\right)$$

$$= -2k\sin(\frac{k\ell}{2})\left(k\ell\cos(\frac{k\ell}{2}) - 2\sin(\frac{k\ell}{2})\right) = 0$$

$$\implies \boxed{\sin(\frac{k\ell}{2}) = 0}, \quad \text{(or)} \quad \boxed{\tan(\frac{k\ell}{2}) = \frac{k\ell}{2}}.$$

- Two "classes" of solutions emerge:

❶ $\sin(\frac{k\ell}{2}) = 0 \implies \frac{k_n\ell}{2} = n\pi \implies \boxed{P_n^{(1)} = 4n^2\,\dfrac{\pi^2 EI}{\ell^2}}$

❷ $\tan(\frac{k\ell}{2}) = \frac{k\ell}{2} \implies \frac{k_n\ell}{2} \approx 0, 4.49, 7.72, \cdots \implies P_1^{(2)} \approx 8.98\frac{\pi^2 EI}{\ell^2}$

3.3.3. The Clamped-Clamped Column II
The Linear Buckling Problem

- **The smallest critical load** is $P_n^{(1)} = 4\frac{\pi^2 EI}{\ell^2} = \frac{\pi^2 EI}{(\frac{\ell}{2})^2}$.

Concept of "Effective Length"

- **Question**: If the beam were simply supported, what would be the length such that it also has the same first critical load?
- Here it comes out to be $\ell_{eff} = \frac{\ell}{2}$.
- The column clamped on both ends can take the same buckling load as a column that is pinned on both ends with half the length.

3.3.3. The Clamped-Clamped Column III
The Linear Buckling Problem

Boundary conditions	Critical load P_{cr}	Deflection mode shape	Effective length KL
Simple support–simple support	$\dfrac{\pi^2 EI}{L^2}$		L
Clamped-clamped	$4\,\dfrac{\pi^2 EI}{L^2}$		$\frac{1}{2}L$
Clamped–simple support	$2.04\,\dfrac{\pi^2 EI}{L^2}$		$0.70L$
Clamped-free	$\dfrac{1}{4}\dfrac{\pi^2 EI}{L^2}$		$2L$

Effective lengths of beams with different boundary conditions (Figure from Brush and Almroth 1975)

Self-Study

- Derive the effective length for the clamped-simply supported and clamped-free columns.

3.3.3. The Clamped-Clamped Column: The Mode-shape
The Linear Buckling Problem

- Let us substitute $k_1 = \frac{2\pi}{\ell}$ into the matrix $\underline{\underline{M}}(k_1)$ so that the boundary conditions now read as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \\ 1 & \ell & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- This implies the following:
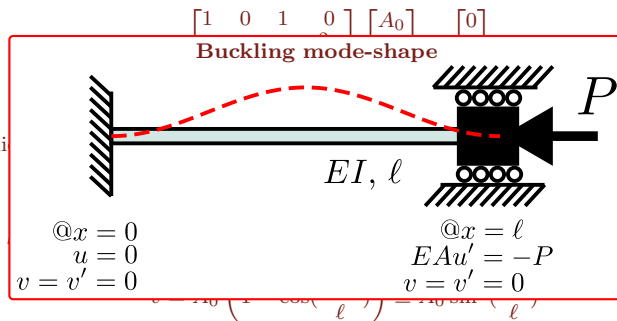
$$A_1 = 0, \quad A_3 = 0, \quad A_2 = -A_0.$$

- So, if $k = k_1$, the solution has to be the following to satisfy the boundary conditions:

$$v = A_0 \left(1 - \cos(\frac{2\pi x}{\ell}) \right) \equiv A_0 \sin^2(\frac{\pi x}{\ell})$$

3.3.3. The Clamped-Clamped Column: The Mode-shape
The Linear Buckling Problem

- Let us substitute $k_1 = \frac{2\pi}{\ell}$ into the matrix $\underline{\underline{M}}(k_1)$ so that the boundary conditions now read as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}$$

Buckling mode-shape

- This impli

- So, if $k =$

$$@x = 0 \qquad\qquad @x = \ell$$
$$u = 0 \qquad\qquad EAu' = -P$$
$$v = v' = 0 \qquad\qquad v = v' = 0$$

conditions:

P

EI, ℓ

4. Energy Perspectives

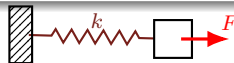- Concept of conservative force field.
- Work done by a force field:

$$W(\underline{x})\Big|_{\underline{x_1}}^{\underline{x_2}} = \int_{\underline{x_1}}^{\underline{x_2}} \underline{f}(\underline{x}) \cdot d\underline{x}.$$

- Introduction to work done.

$$W(\underline{x}) = \underbrace{\Pi(\underline{x})}_{\text{External Work}} - \underbrace{V(\underline{x})}_{\text{Internal Work/Potential Energy}}$$

Example

- Force balance reads: $F = kx$
- Work done expression: $W(x) = Fx - \frac{k}{2}x^2$

4. Energy Perspectives

- Expanding W about some $\underline{x}_s$ we have,

$$W(\underline{x}_s + \delta\underline{x}) = W(\underline{x}_s) + \underline{\nabla}W\big|_{\underline{x}_s}\delta\underline{x} + \mathcal{O}(\delta\underline{x}^2).$$

- Stationarity of work: $\boxed{\delta W = \underline{\nabla}W(\underline{x}_s)\delta\underline{x} = 0, \quad \forall \quad \underline{x} \in \Omega}$, where Ω is the configuration-space.

Example

- For the SDoF system above, we have $W = Fx - \frac{k}{2}x^2$ and

$$\nabla W(x_s) = \frac{dW}{dx} = F - kx_s = 0 \implies x_s = \frac{F}{k}.$$

- Work-stationarity hereby gives a convenient definition for equilibrium.
- What about higher order effects?

4. Energy Perspectives

- Continuing the Taylor expansion (SDoF case) for $W(x)$ we have,

$$W(x) = W(x_s) + \frac{dW}{dx}(x_s)\delta x + \frac{1}{2}\frac{d^2W}{dx^2}(x_s)\delta x^2 + \mathcal{O}(\delta x^3).$$

- At equilibrium, $\frac{dW}{dx}$ is zero. The sign of $\frac{d^2W}{dx^2}$ governs the <u>local tendency of the work</u> around equilibrium.

Example

- For the SDoF example, $\frac{d^2W}{dx^2} = -k$, implying W is maximized.

- If $\frac{d^2W}{dx^2} < 0$, then the second order effect of virtual displacements is to reduce the work scalar: **Stable Equilibrium**.

- The opposite case is **Unstable Equilibrium**.

4. Energy Perspectives

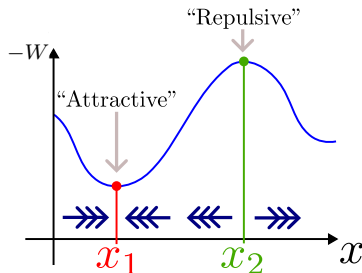- Continuing the Taylor expansion (SDoF case) for $W(x)$ we have,

$$W(x) = W(x) + \frac{dW}{dx}(x)\delta x + \frac{1}{2}\frac{d^2W}{dx^2}(x)\delta x^2 + \mathcal{O}(\delta x^3).$$

- At equilibrium, $\frac{d}{dx}$ [...] dency of the work around equilibrium [...]

Example

- For the SDoF exam[...]

- If $\frac{d^2W}{dx^2} < 0$, then [...] is to reduce the work scalar: **Stable E[...]**

- The opposite case [...]

Hypothetical Example
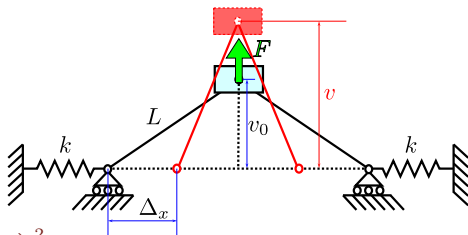
4.1. Snap-Through Buckling
Energy Perspectives

- We will consider the SDoF model to the right (from Wiebe et al. 2011).
- The strain energy on the springs (two) is

$$U(v) = 2 \times \frac{k}{2} \Delta_x^2 = k \left(\sqrt{L^2 - v_0^2} - \sqrt{L^2 - v^2} \right)^2.$$

- The work done by the load (to take the mid-point from v_0 to v) is given by,
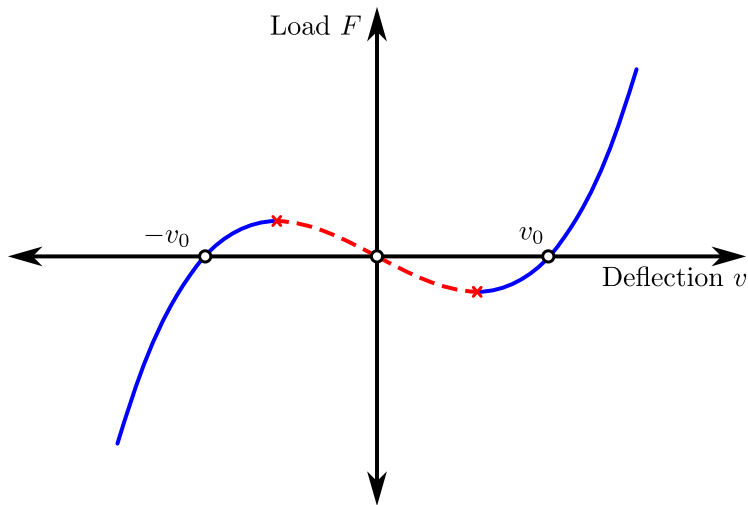
$$\Pi(v) = F(v - v_0).$$

Setting $\frac{dW}{dv} = 0$ we get

$$F = -2kv \left(1 - \sqrt{\frac{L^2 - v_0^2}{L^2 - v^2}} \right).$$

4.1. Snap-Through Buckling
Energy Perspectives

- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.

4.1. Snap-Through Buckling
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4.1. Snap-Through Buckling
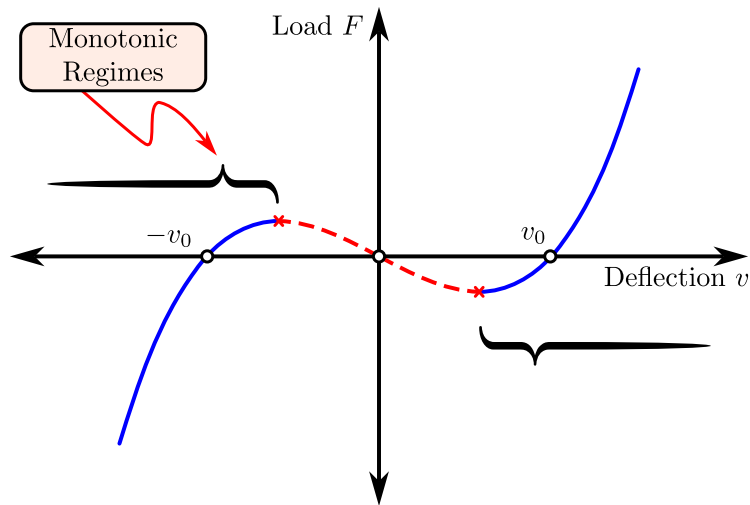### Energy Perspectives

- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.

4.1. Snap-Through Buckling
Energy Perspectives
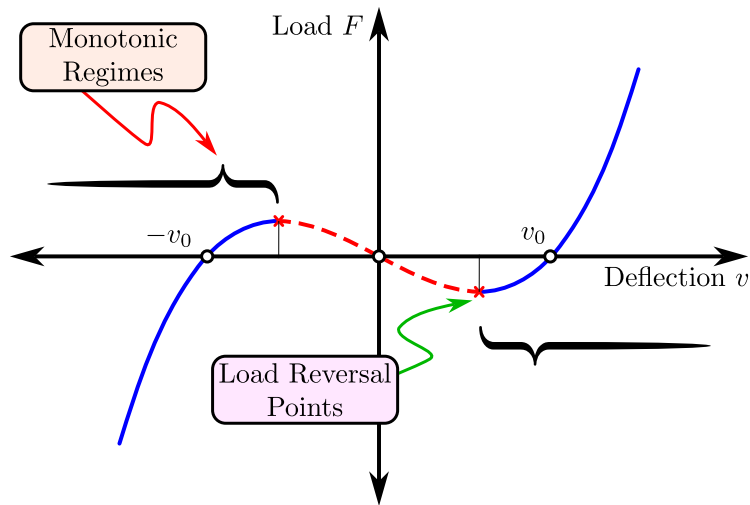
- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.

4.1. Snap-Through Buckling
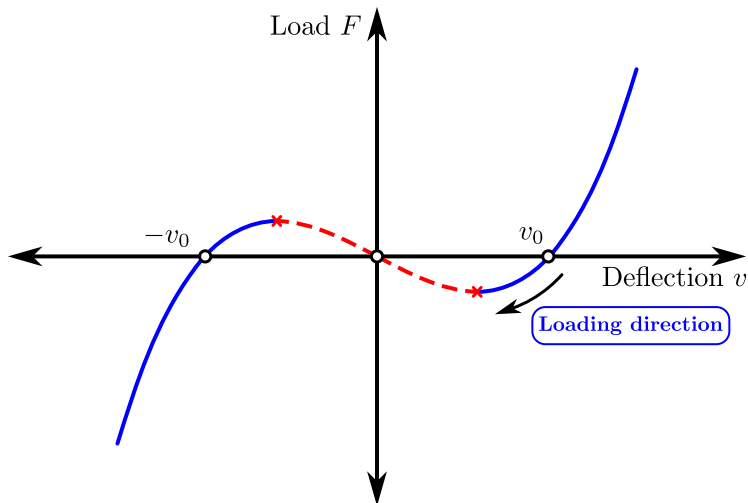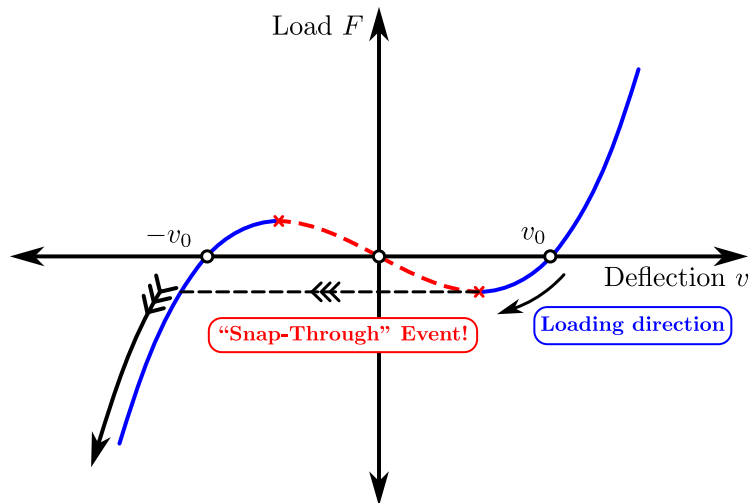**Energy Perspectives**

- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.

4.1. Snap-Through Buckling
Energy Perspectives

- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.

Load F

A single load level can equilibrate in three different configurations!

v_0 v_0

Deflection v

4.1. Snap-Through Buckling: Equilibrium Visualization
Energy Perspectives

Linear System: $U - \Pi = \frac{k}{2} x^2 - Fx$

4.1. Snap-Through Buckling: Equilibrium Visualization

Energy Perspectives
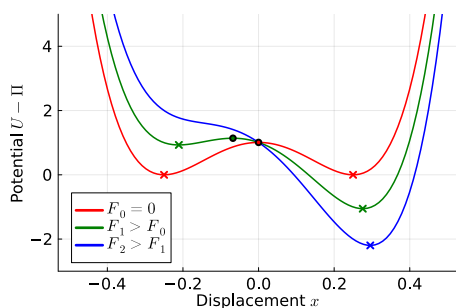
Snap-Through Problem: $U - \Pi = k(\sqrt{L^2 - v_0^2} - \sqrt{L^2 - v^2})^2 - Fx$

5.1. Plate Buckling
Governing Equations
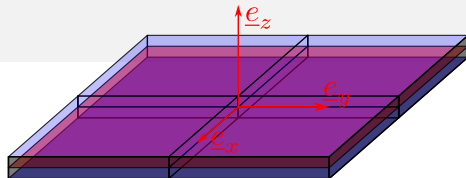
- Kichhoff-Love Plate Theory.
- **Kinematic Assumptions**: Lines along section-thickness <u>deform as lines</u> and <u>stay perpendicular to the neutral axis</u>.
- Governing equations written in the form

$$\frac{Et^3}{12(1-\nu^2)}(w_{,xxxx} + w_{,yyyy} + 2w_{,xxyy}) - (N_{xx}w_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}) = 0$$
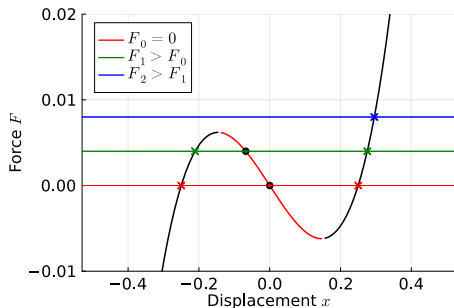
$$\boxed{D\nabla^4 w - (N_{xx}w_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}) = 0}$$

- This is all that is needed to conduct buckling analysis - the procedure is identical as above!
- Before this, however, let us develop intuition on the different reaction force components and their kinematic relationships.

5.2. Reaction-Kinematics Relationships
Plate Buckling
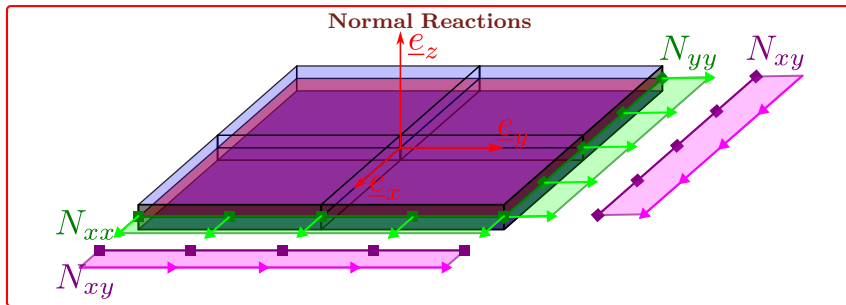
Normal Reactions

5.2. Reaction-Kinematics Relationships
Plate Buckling

5.2. Reaction-Kinematics Relationships
Plate Buckling
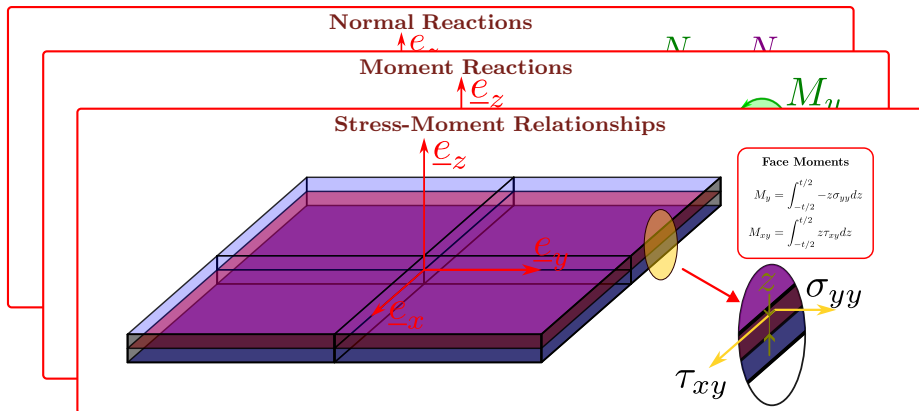
Normal Reactions

$\underline{e}_z$ N N

Moment Reactions

$\underline{e}_z$ M_y

Stress-Moment Relationships

$\underline{e}_z$

$\underline{e}_y$

$\underline{e}_x$

Face Moments

$M_y = \int_{-t/2}^{t/2} -z\sigma_{yy}dz$

$M_{xy} = \int_{-t/2}^{t/2} z\tau_{xy}dz$

σ_{yy}

τ_{xy}

5.2. Reaction-Kinematics Relationships
Plate Buckling

Normal Reactions $\uparrow e_z$ N N

Moment Reactions $\uparrow \underline{e}_z$ M_y

Stress-Moment Relationships

Equilibrium Equations (Shear Force-Moment Relationships)

$$
\left.\begin{aligned}
\sigma_{xx,x} + \tau_{xy,y} + \tau_{xz,z} &= 0 \\
\tau_{xy,x} + \sigma_{yy,y} + \tau_{yz,z} &= 0 \\
\tau_{xz,x} + \tau_{yz,z} + \sigma_{zz,z} &= 0
\end{aligned}\right\}
\implies
\begin{cases}
Q_x &= M_{x,x} + M_{xy,y} \\
Q_y &= -M_{y,y} + M_{xy,x} \\
0 &= Q_{x,x} + Q_{y,y}.
\end{cases}
$$

Note:

- Although the shear strains γ_{xz} & γ_{yz} are assumed zero by the Kirchhoff kinematic assumptions, and thereby, the stresses τ_{xz} & τ_{yz} are also zero, **the shear forces can not be zero for equilibrium!!**

- They are defined as $Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} dz$, $Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz$.

5.2. Reaction-Kinematics Relationships
Plate Buckling

- With this background, we are ready to write the following:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_x \\ -M_y \\ M_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \left(\begin{bmatrix} t & 0 \\ 0 & -\frac{t^3}{12} \end{bmatrix} \otimes \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \right) \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \\ w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$

- A moment-free boundary condition (simply supported edge) would imply simply setting the second derivatives $(w_{,xx}, w_{,yy}, w_{,xy})$ to zero at the edge.

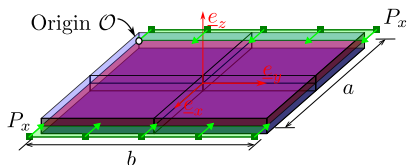# 5.3. Thin Plates Under Uniaxial Compression
Plate Buckling

Origin $\mathcal{O}$

P_x

e_z

e_x

a

P_x

b

Plate under uniaxial compression

Governing Equations

$$D\nabla^4 w + P w_{,xx} = 0$$

$$\implies P_{cr,nm} = \frac{\pi^2 D}{b^2}\left(\frac{m}{a/b} + n^2 \frac{a/b}{m}\right)^2$$

(n=1 always for minimum critical load)

$$\implies P_{cr,m} = \frac{\pi^2 D}{b^2}\left(\frac{m}{a/b} + \frac{a/b}{m}\right)^2$$

$$P_{cr} = \frac{\pi^2 D}{b^2}\underbrace{\min_{m\in\mathbb{Z}^+}\left(\frac{m}{a/b} + \frac{a/b}{m}\right)^2}_{k_{cr}(a/b)}$$

Ansatz (Simply Supported Case)

$$w(x,y) = \sum_{m,n} W_{mn}\sin\left(m\frac{\pi x}{a}\right)\sin\left(n\frac{\pi y}{b}\right)$$

Boundary Conditions:
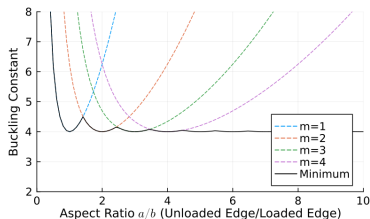$w = 0, M_x, M_y = 0$ on Γ

5.3. Thin Plates Under Uniaxial Compression
Plate Buckling

Buckling Constant

$$k_{cr}(r) = \min_{m \in \mathbb{Z}^+} \left(\frac{m}{r} + \frac{r}{m} \right)^2$$

5.3. Thin Plates Under Uniaxial Compression
Plate Buckling

Buckling Constant

$$k_{cr}(r) = \min_{m \in \mathbb{Z}^+} \left(\frac{m}{r} + \frac{r}{m} \right)^2$$

$m = 3$

$m = 1$

$m = 2$
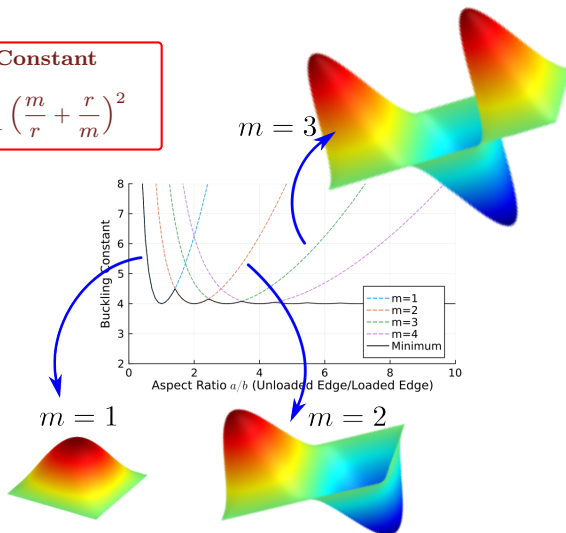
5.3. Other Boundary Conditions
Thin Plates Under Uniaxial Compression

- It is possible to conduct the same analysis for other (combinations) of boundary conditions.
- The analysis is slightly more tedious (due to the Ansatz not being as simple any more), **but possible along the same lines**.
- The critical plot comes out as shown in your textbook.
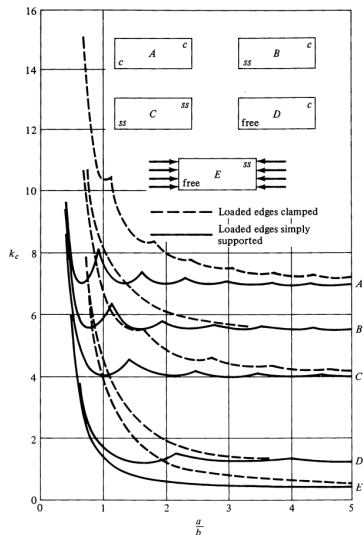
(Figure 3.9 from Brush and Almroth 1975)

5.3. Other Boundary Conditions
Thin Plates Under Uniaxial Compression

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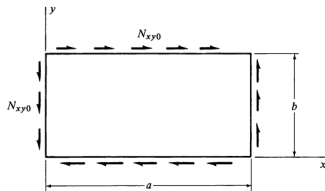
The same works for shear buckling too!

(Fig. 3.10 from Brush and Almroth 1975)

$$\tau_{cr} = \frac{\pi^2 D}{b^2 h} k_s$$

Clamped edges

Simply supported edges

k_s

$\frac{a}{b}$
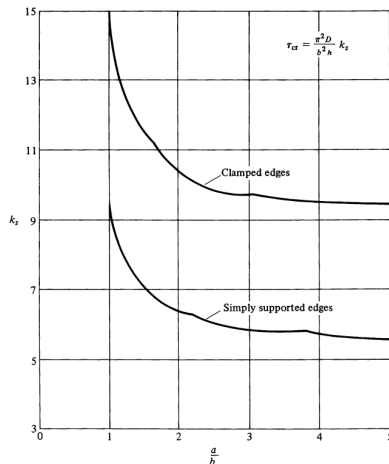
(Figure 3.11 from Brush and Almroth 1975)

6. Food For Thought

- We're not yet ready to handle this (wait one more semester for AS3020), but some types of beam undergo **twisting instability**!

- In the right we have simply supported beams under axial compression - the beams twist before they bend under the instability.
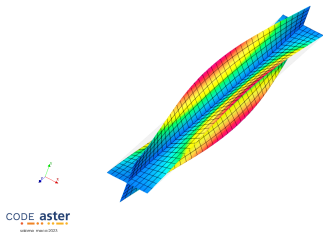
Simply Supported Beam Under Axial Compression
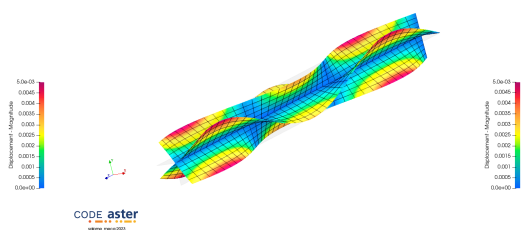
Cantilevered Beam under Axial Compression (2nd mode)

Heads Up: You are <u>designed</u> to see this in your structures lab experiment!

References I

[1]   Don Orr Brush and Bo O. Almroth. Buckling of Bars, Plates, and Shells, McGraw-Hill, 1975. ISBN: 978-0-07-008593-0 (cit. on pp. 2, 39, 64, 65).

[2]   T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).

[3]   Richard Wiebe et al. "On Snap-Through Buckling". In: 52nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference. Denver, Colorado: American Institute of Aeronautics and Astronautics, Apr. 2011. ISBN: 978-1-60086-951-8. DOI: 10.2514/6.2011-2083. (Visited on 02/18/2025) (cit. on p. 46).

8. Class Discussions (Outside of Slides)

- Ball on a hill. 2D, 3D cases.
- Assumptions behind compression of a bar.

8.1. Post-Buckling Behavior (Out of Syllabus)
Class Discussions (Outside of Slides)

- Let us use the energy approach to study the post-buckling behavior of a beam.
- We've developed some intuition that buckling blows up the displacement levels. Let us revise our kinematic description to capture this.
- The (simplified) approach we will follow is as follows:
  1. **Write out nonlinear kinematics**, identify normal force $N = \int_{\mathcal{A}} \sigma_{ax} dA$ and moment $M = \int_{\mathcal{A}} -y\sigma_{ax} dA$.
  2. **Assume transverse deformation field** $v = V \sin\left(\frac{\pi x}{\ell}\right)$
  3. **Assume axial tip deflection** u_T **and derive axial deformation field.**
  4. **Express work done in terms of scalars** V **and** u_T. $\rightarrow$ <u>**Extremize**</u>.
  5. **Plot force deflection curves, analyze stability**.

8.1. Post-Buckling Behavior (Out of Syllabus)
Class Discussions (Outside of Slides)

Geometrically Nonlinear Kinematics

- The deformation field is written as $u_x = u - yv'$, $\quad u_y = v$. Consider the deformation of a line from (x, y) to $(x + \Delta x, y)$:

$$(x, y) \to (x + u - yv', y + v),$$
$$(x + \Delta x, y) \to (x + \Delta x + u - yv' + (u' - yv'')\Delta x, y + v + v'\Delta x),$$
$$\Delta S = \Delta x, \quad \Delta s^2 = \Delta x^2((1 + u' - yv'')^2 + v'^2).$$

- We write the axial strain as

$$\epsilon_{ax} = \frac{1}{2}\frac{\Delta s^2 - \Delta S^2}{\Delta S^2} = (u' - yv'') + \frac{1}{2}\left((u' - yv'')^2 + v'^2\right)$$

$$\boxed{\epsilon_{ax} \approx (u' - yv'') + \frac{v'^2}{2}}.$$

- The final assumption is sometimes referred to as Von Karman strain assumptions.

8.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

- Nearly nothing changes in the equilibrium equations. We first write out the area-normal stresses and moments:

$$N = \int_{\mathcal{A}} E\epsilon_{ax} dA = EA(u' + \frac{v'^2}{2}), \quad M = \int_{\mathcal{A}} -yE\epsilon_{ax} dA = EIv''.$$

- The axial force balance reads:

$$N' = EA\frac{d}{dx}\left(u' + \frac{v'^2}{2}\right) = 0, \quad u(x)|_{x=0} = 0, \quad u|_{x=\ell} = u_T.$$

8.1. Post-Buckling Behavior (Out of Syllabus): Axial Problem
Class Discussions (Outside of Slides)

- We next **impose the transverse deformation field** $v(x) = V \sin\left(\frac{\pi x}{\ell}\right)$ on the axial problem. Solving this, we get

$$u(x) = -\frac{\pi V^2}{8\ell} \sin\left(\frac{2\pi x}{\ell}\right) + C_1 x + C_2.$$

- Boundary conditioned are imposed by setting $C_1 = \frac{u_T}{\ell}$ and $C_2 = 0$.

- The parameterized axial deformation field, therefore, is

$$\boxed{u(x; V, u_T) = \frac{u_T}{\ell} x - \frac{\pi V^2}{8\ell} \sin\left(\frac{2\pi x}{\ell}\right)}.$$

- Note that we have not said anything about V or u_T so far.

8.1. Post-Buckling Behavior (Out of Syllabus): Strain Energy Density
Class Discussions (Outside of Slides)

- The strain energy density (per unit length) is written as,

$$\mathcal{V} = \int_{\mathcal{A}} \frac{E\epsilon_{ax}^2}{2} dA = \frac{E}{2} \int_{\mathcal{A}} (u' - yv'' + \frac{v'^2}{2})^2 dx$$

$$= \frac{EA}{2} \left(u' + \frac{v'^2}{2}\right)^2 + \frac{EI}{2} v''^2 \approx \frac{EI}{2} v''^2 + \frac{EA}{2} \frac{v'^4}{4}.$$

- Note that we have assumed $u_T \to 0$, i.e., providing negligible influence on the overall potential energy.

- Substituting the assumed deformation field $v = V \sin(\frac{\pi x}{\ell})$ and integrating over $(0, \ell)$ we have,

$$\mathcal{V}_{tot} = \int\limits_{0}^{\ell} \mathcal{V}(x)dx = \frac{\pi^4 EI}{4\ell^3} V^2 + \frac{3\Pi^4 EA}{64\ell^3} V^4$$

$$= \frac{\pi^2 P_{cr}}{4\ell} V^2 + \frac{3\pi^2 A P_{cr}}{64 I \ell} V^4.$$

8.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity
Class Discussions (Outside of Slides)

- The work done by an axial compressive load P is given by

$$\Pi = \int\limits_0^\ell \int_{\mathcal{A}} \frac{P}{A} \varepsilon_{ax} dA dx = \int\limits_0^\ell \int_{\mathcal{A}} \frac{P}{A}(u' - yv'' + \frac{v'^2}{2}) dA dx$$

$$= P \int\limits_0^\ell u' dx + \frac{P}{2} \int\limits_0^\ell v'^2 dx$$

$$\boxed{\Pi = P u_T + \frac{\pi^2 P}{4\ell} V^2}.$$

- So the total work scalar ($W = \Pi - \mathcal{V}_{tot}$) is given as (we ignore u_T here)

$$W(V) = \frac{\pi^2}{4\ell}(P - P_{cr})V^2 - \frac{3\pi^2 A}{64 I \ell} P_{cr} V^4.$$

8.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity
Class Discussions (Outside of Slides)

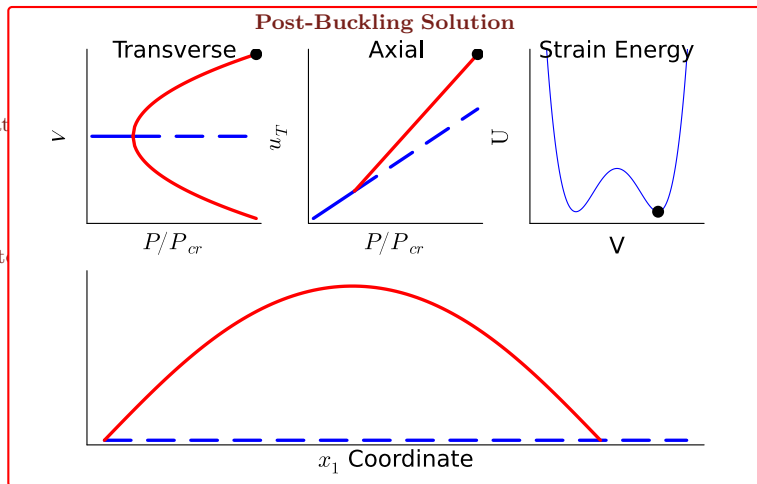- Stationarizing the work we get,

$$\frac{dW}{dV} = \frac{\pi^2 P_{cr}}{2\ell} V \left(\left(\frac{P}{P_{cr}} - 1 \right) - \frac{3A}{8I} V^2 \right) \implies V = 0, \pm \sqrt{\frac{8I}{3A} \left(\frac{P}{P_{cr}} - 1 \right)}.$$

  Note that the non-trivial solution is only active for $P >= P_{cr}$.
- We can next estimate u_T easily by applying the boundary conditions.

8.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity
Class Discussions (Outside of Slides)

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