# AS2070: Aerospace Structural Mechanics

## Module 3: Introduction to Fatigue and Failure

**Instructor: Nidish Narayanaa Balaji**

**Department of Aerospace Engineering, IIT Madras**

April 28, 2025

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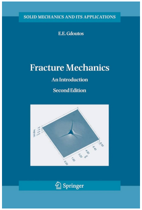
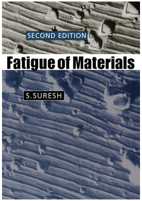
Also see https://www.fracturemechanics.org/

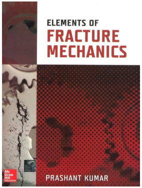




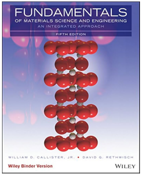

*Chapters 1,4 in Gdoutos (2005)*



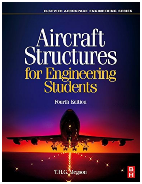

*Chapters 1,7,9 in Suresh (1998)*




*Chapters 1-3 in Kumar (2009)*



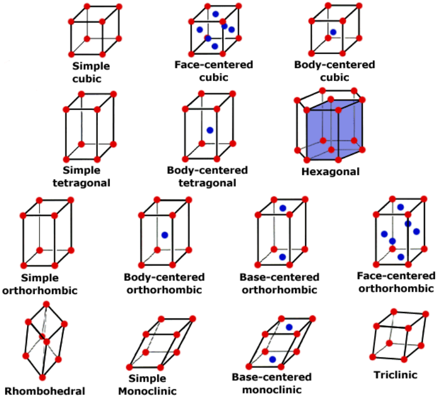

*Chapter 3 in Jr and Rethwisch (2012)*




*Chapter 15 in Megson (2013)*

# 1.1. Structure of Materials

Introduction




*Types of crystal structures in metals Sparky (2013)*

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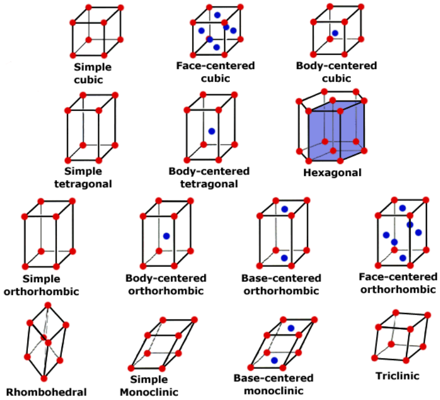


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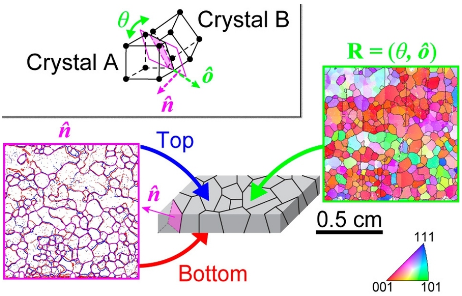


*Crystal and Grain Structures New Technique Provides Detailed Views of Metals' Crystal Structure (2016). "Polycrystallinity"*

# 1.1. Structure of Materials

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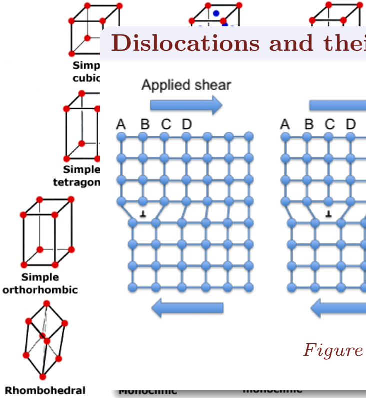



## Dislocations and their “motion” under load

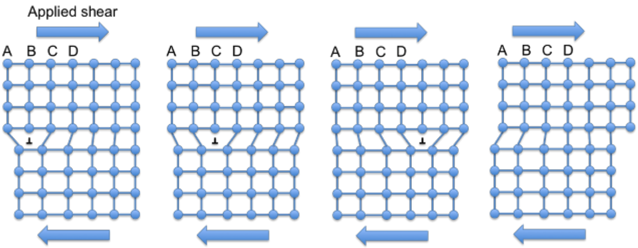
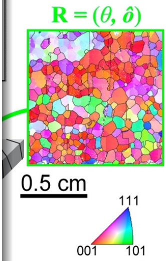



*Figure from Sparky (2013)*

*Types of crystal structures in metals Sparky (2013)*




*New Technique Provides Detailed Views of Metals' Crystal Structure (2016). “Polycrystallinity”*

# 1.2. Understanding the Stress-Strain Curve

Introduction

**The Uniaxial Tensile Test**

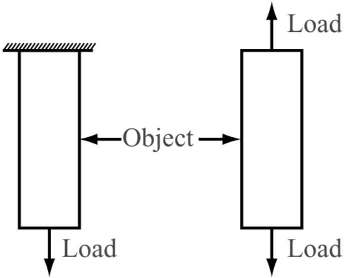



*Figure from Rajendran 2011*

# 1.2. Understanding the Stress-Strain Curve

Introduction

## Terminology

1. Proportionality Limit;
2. Elastic Limit;
3. Yield Point;
4. Ultimate Strength;
5. Fracture Point;
6. Elongation at Failure;

## Ductile Fracture

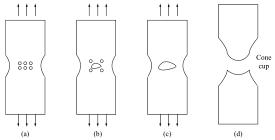



*Figure from Rajendran 2011*

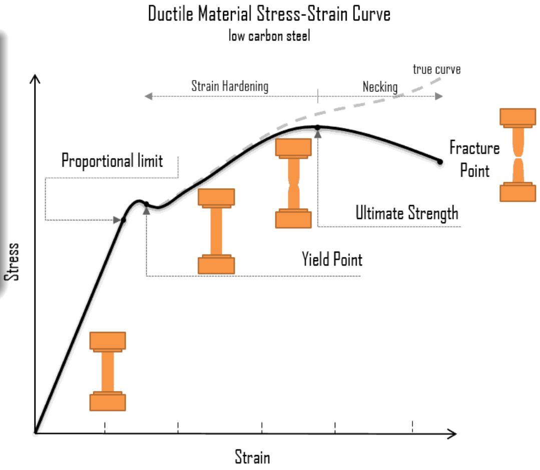



*Figure from Connor 2020*

# 1.3. Failure Mechanisms: Fracture

1. Introduction

**"Griffith Theory" of brittle fracture**

- Theoretical fracture stress $\sim \frac{E}{5} - \frac{E}{30}$ (steel $\sim \frac{E}{1000}$)
- Fracture occurs when $E_{strain} = E_{surface}$
- Crack propagates when $\frac{dE_{strain}}{dL} = \frac{dE_{surface}}{dL}$

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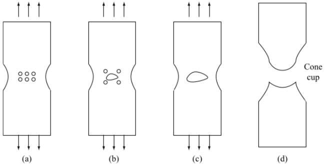


*Ductile Fracture Rajendran 2011*

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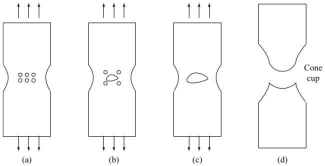



*Ductile Fracture Rajendran 2011*

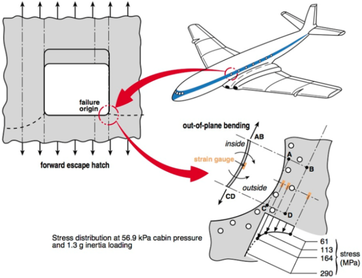
| *Sr. No* | *Brittle Fracture* | *Ductile Fracture* |
|---|---|---|
| 1. | It occurs with no or little plastic deformation. | It occurs with large plastic deformation. |
| 2. | The rate of propagation of the crack is fast. | The rate of propagation of the crack is slow. |
| 3. | It occurs suddenly without any warning. | It occurs slowly. |
| 4. | The fractured surface is flat. | The fractured surface has rough contour and the shape is similar to cup and cone arrangement. |
| 5. | The fractured surface appears shiny. | The fractured surface is dull when viewed with naked eye and the surface has dimpled appearance when viewed with scanning electron microscope. |
| 6. | It occurs where micro crack is larger. | It occurs in localised region where the deformation is larger. |

*Ductile vs Brittle Fracture Rajendran 2011*

# 1.3. Failure Mechanisms: Fatigue

1. Introduction

..over 90% of mechanical failures are caused because of metal fatigue *What Is Metal Fatigue?* 2021...




*The De Havilland Comet The deHavilland Comet Disaster 2019 [lecture]*

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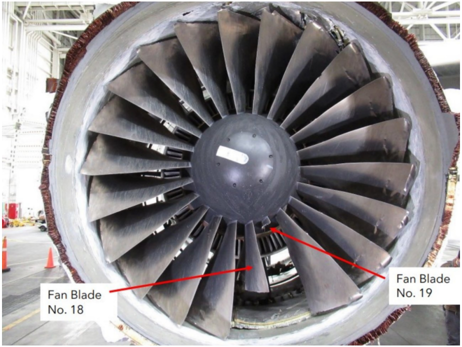
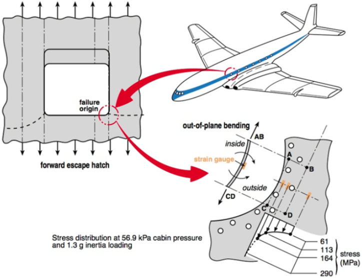



*A more recent example (2021 United Airlines Boeing 777)* **DCA21FA085Aspx**. *[video]*

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*A more recent exam*
*Boeing 777)* **DCA**

## Fatigue Crack Propagation: Beech Marks

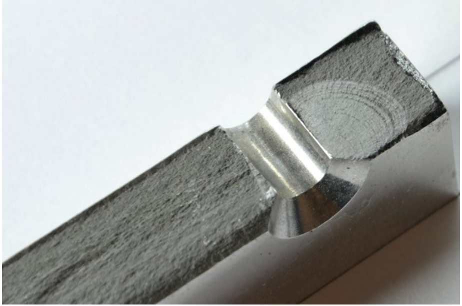
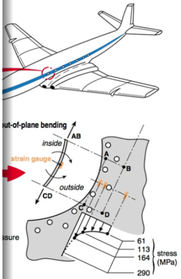


*Figure from Fatigue Physics 2024*




*met The deHavilland*
*2019 [lecture]*

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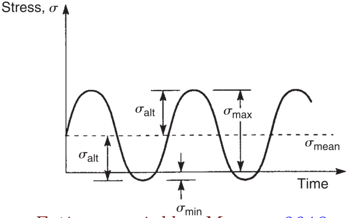
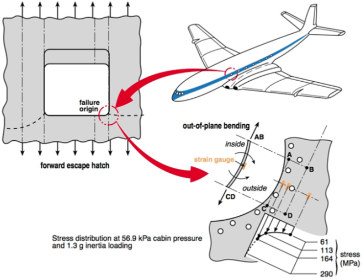



*Fatigue variables* Megson 2013




*The De Havilland Comet* *The deHavilland Comet Disaster* 2019 [lecture]

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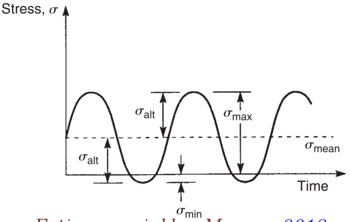
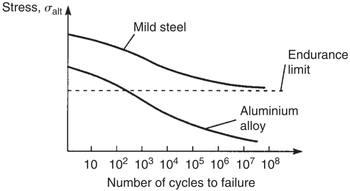
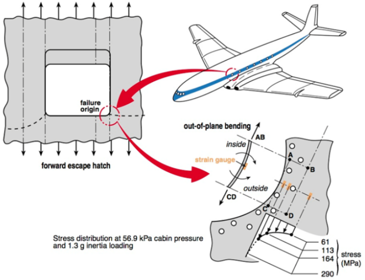



*Fatigue variables Megson 2013*




*The S-n Diagram Megson 2013*




*The De Havilland Comet The deHavilland Comet Disaster 2019 [lecture]*

# 1.3. Failure Mechanisms: Fatigue

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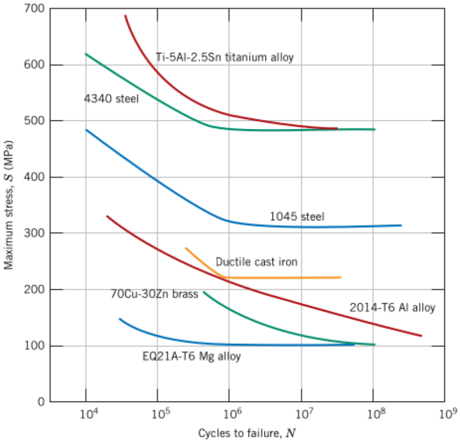
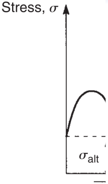
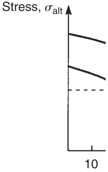
## S-N Curves for Common Metals (Jr and Rethwisch 2012)

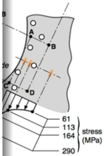



*Fatigue*




*The S-n*

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*ecture]*

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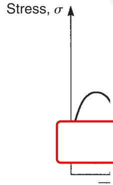
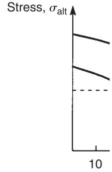
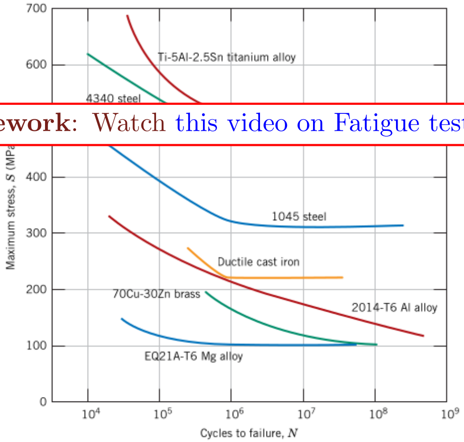



*Fatigue*

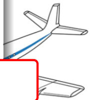


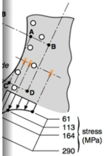

*The S-n*

**S-N Curves for Common Metals** ( **Jr and Rethwisch 2012**)




**Homework**: Watch this video on Fatigue testing.



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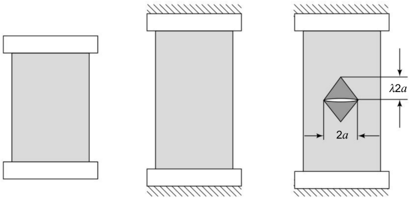
*ecture]*

# 1.4. Energy Release Rate: Griffith's Analysis

## Introduction



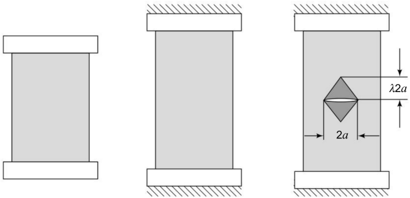

*Simplistic picture of the introduction of a crack in a stretched specimen(Figure from sec 2.5 in Kumar 2009)*

- Because of the crack, we assume $\sigma \approx 0$ in the triangles.
- Corresponding energy loss:

$$E_R = V_\Delta \times (\frac{\sigma^2}{2E}) = \frac{2a^2\lambda t\sigma^2}{E}.$$

# 1.4. Energy Release Rate: Griffith's Analysis

Introduction

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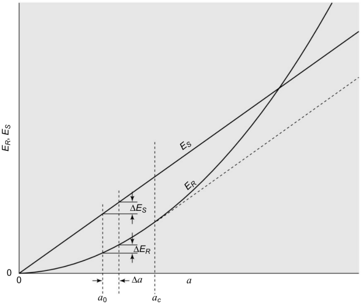
- For thin plates, $\lambda = \frac{\pi}{2}$. So,

$$E_R = \frac{\pi a^2 t \sigma^2}{E}.$$

- The "creation" of a surface takes energy. We write this as,

$$E_S = 2(2at)\gamma = 4at\gamma.$$



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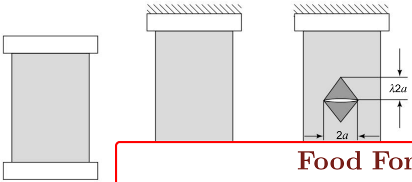


*Simplistic pict*
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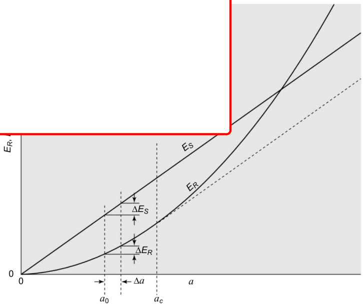
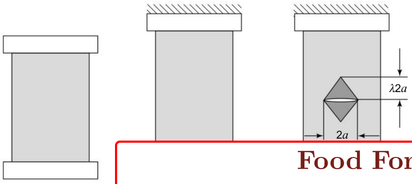



**Food For Thought**

- What would a "safe size" of crack be, for a given loading condition? *Hint: Think incrementally*

# 1.4. Energy Release Rate: Griffith's Analysis

Introduction




*Simplistic pict... in a stretched ... 2009)*

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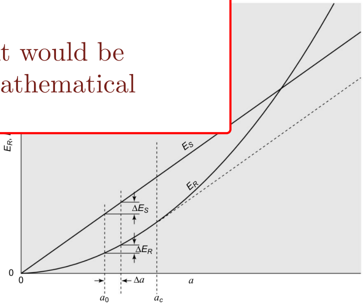



**Food For Thought**

- What would a "safe size" of crack be, for a given loading condition? *Hint: Think incrementally*
- What type of an experiment would be necessary to confirm this mathematical framework?

- Becaus... $\sigma \approx 0$ i...
- Corresponding energy loss:

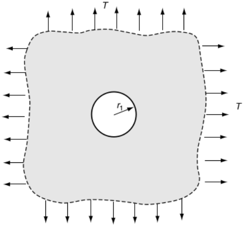
$$E_R = V_\Delta \times (\frac{\sigma^2}{2E}) = \frac{2a^2\lambda t\sigma^2}{E}.$$

# 1.5. Linear Elastic Fracture Mechanics

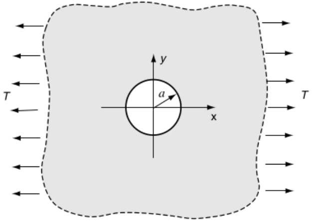
Introduction (Ref: Sec. 8.4.2 in Sadd 2009)

Consider the following two cases.
**Question**: Where will the point of peak stress occur? And which will have higher maximum stress?

**Case 1**




**Case 2**



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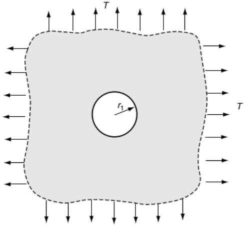


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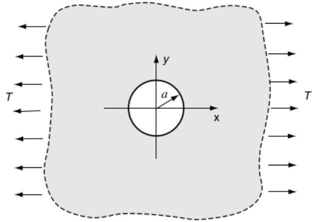


**Analytical Solution**

$\sigma_r = T(1 - \frac{r_1^2}{r^2})$, $\sigma_\theta = T(1 + \frac{r_1^2}{r^2})$

$\implies \boxed{\sigma_{\max} = 2T}$

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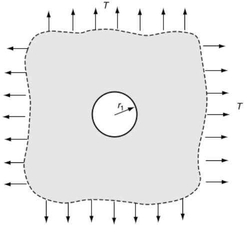
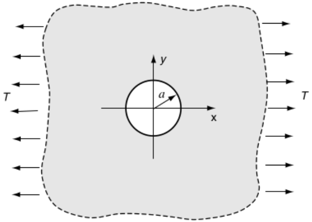



### Case 2




### Analytical Solution

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### Analytical Solution

$\sigma_r = T(1 - \frac{r_1^2}{r^2}) + (\cdot)\cos(2\theta)$, $\sigma_\theta = \dots$

$\implies \boxed{\sigma_{\max} = 3T}$

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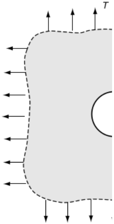
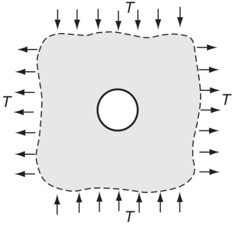
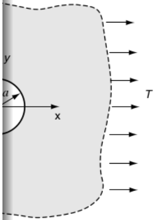



### Case 3




$\boxed{\sigma_{\max} = 4T}$




### Analytical Soluti

$\sigma_r = T(1 - \frac{r_1^2}{r^2})$, $\sigma_\theta = T(1 + \frac{r_1^2}{r^2})$

$\implies \boxed{\sigma_{\max} = 2T}$

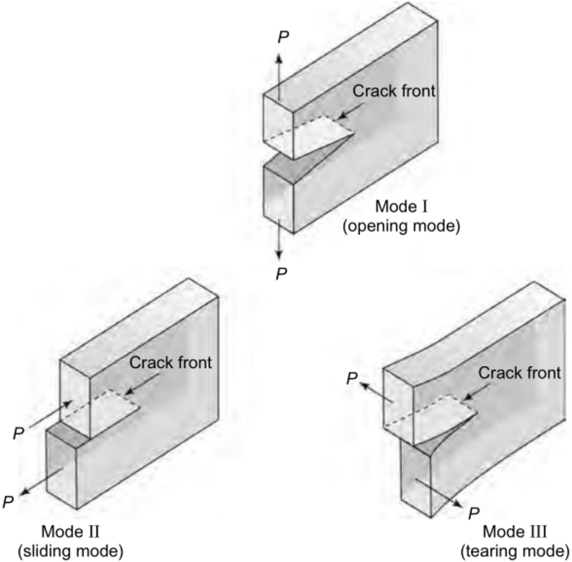
### n

$\sigma_r = T(1 - \frac{r_1^2}{r^2}) + (\cdot)\cos(2\theta)$, $\sigma_\theta = \dots$

$\implies \boxed{\sigma_{\max} = 3T}$

# 1.6. Modes of Fracture

Introduction

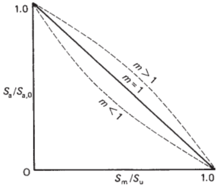


# 2. Introduction to Fatigue

## Concepts

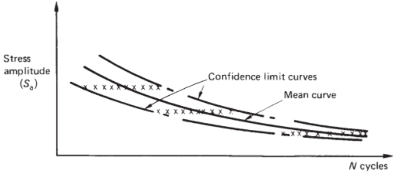
- Safe Life: RUL
- Fail-Safe: Redundancy

## Tensile Stresses: The Goodman Diagram



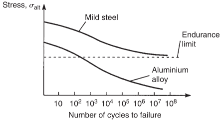

(Figure 15.2 from Megson 2013)

$$\frac{S_a}{S_{a,0}} = 1 - \left(\frac{S_m}{S_u}\right)^m$$




(Figure 15.1 from Megson 2013)

## The S-N Curve




(Figure from Megson 2013)

$$\sigma_{alt} = \sigma_\infty \left(1 + \frac{C}{\sqrt{N}}\right), \quad N \propto \frac{1}{\sigma_{mean}}.$$

# 2.1. The deHavilland Comet

Introduction to Fatigue

> *No aircraft has contributed more to safety in the jet age than the Comet. The lessons it taught the world of aeronautics live in every jet airliner flying today. – D.D. Dempster, 1959, in The Tale of the Comet*

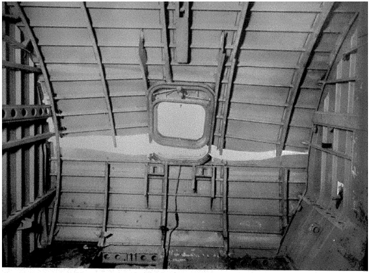



FIG. 7. VIEW FROM INSIDE OF FAILURE AT THE FORWARD ESCAPE HATCH ON THE PORT SIDE—COMET G-ALYU

*(Figures from "De Havilland Comet" 2025)*

# 2.1. The deHavilland Comet

Introduction to Fatigue

*No aircraft has contributed more to safety in the jet age than the Comet. The lessons it taught the world of aeronautics live in every jet airliner flying*

*The Tale of the Comet*

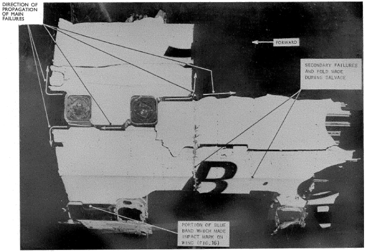
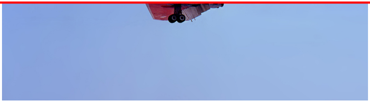



FIG. 12. PHOTOGRAPH OF WRECKAGE AROUND ADF AERIAL WINDOWS—G-ALYP.



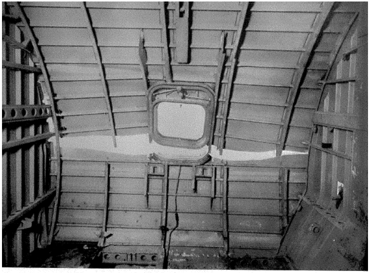


FIG. 7. VIEW FROM INSIDE OF FAILURE AT THE FORWARD ESCAPE HATCH ON THE PORT SIDE—COMET G-ALYU

*(Figures from "De Havilland Comet" 2025)*

# 2.1. The

Introduction to

*No air ... Comet.*
*The les ... r flying*

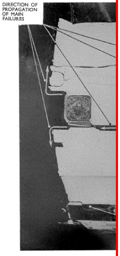
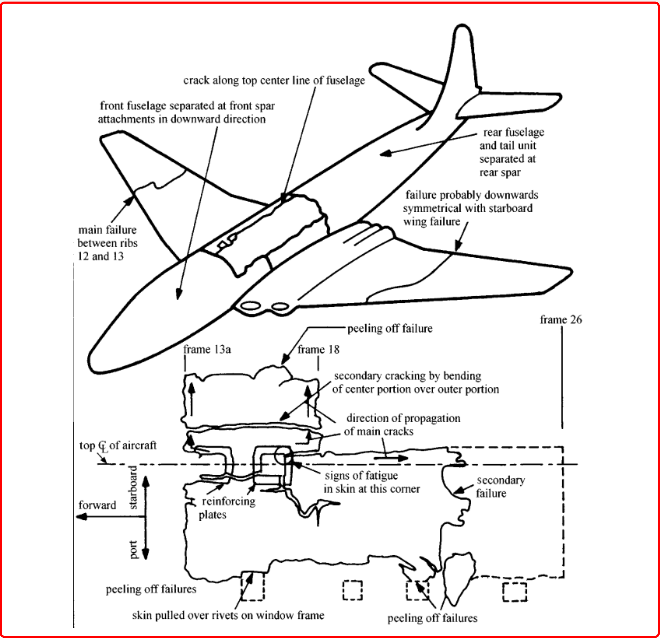



FIG. 12. PHOTO




TCH ON THE

*t” 2025)*

# 2.2. Miner's Rule

Introduction to Fatigue

- Suppose at an operation level of $\sigma_m, \sigma_a$, the fatigue life is $N$ and the structure undergoes $n$ cycles, Miner's rule posits that $\frac{n}{N}$ is the fraction of life that has been consumed.
- Suppose a structure undergoes multiple stress levels in its loading history, the total fraction of fatigue life that has been consumed is written as

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots.$$

- The structure is expected to fail when this sum becomes 1.0..

# 3.1. Griffith's Analysis and Energy Release Rate

Linear Elastic Fracture Mechanics

- The total energy of a loaded elastic body is written as

$$\Pi = \underbrace{U}_{\text{elastic}} - \underbrace{W}_{\text{external}}.$$

- **Griffith's principle**: The **energy lost due to the creation** of a cracked surface must be equal to the **energy required for the creation** of the cracked surface.
- Surface energy is usually expressed as $E_S = \mathcal{A}\gamma$.
- This is a general principle agnostic of the exact structure under consideration.

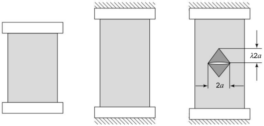
$$\boxed{G = -\frac{d\Pi}{d\mathcal{A}} = 2\gamma}.$$

(note: $2\mathcal{A}$ is the effective total "new" surface area that has been created)

# 3.1. Griffith's Analysis and Energy Release Rate: Examples

Linear Elastic Fracture Mechanics

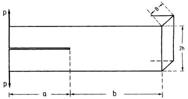
## Crack in Stretched Specimen




(Figure from sec 2.5 in Kumar 2009)

- Crack: $\mathcal{A} = 2at$, $\partial_{\mathcal{A}} = \frac{1}{2t}\partial_a$
- $\Pi = U = \frac{\sigma^2 t}{2E'}(\mathcal{A}_{tot} - 4\lambda a^2)$.
- $E_S = 2\mathcal{A}\gamma$, $\frac{dE_S}{d\mathcal{A}} = 2\gamma$.
- $G = -\frac{d\Pi}{d\mathcal{A}} = -\frac{1}{2t}\frac{d\Pi}{da} = \frac{\lambda a}{2E'}\sigma^2$.
- $\sigma_{cr} = \sqrt{\frac{E'\gamma}{\lambda a}} = \sqrt{\frac{2E'\gamma}{\pi a}}$.

## Double Cantilever Beam (DCB)



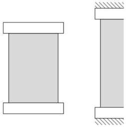

(Figure 4.14 in Gdoutos 2005)

- $u = CP = \frac{2a^3}{3EI}P$, $C = \frac{2a^3}{3EI}$.
- $U = \frac{Pu}{2} = \frac{CP^2}{2} = \frac{P^2}{3EI}a^3$,
  $W = Pu = CP^2 = \frac{2P^2}{3EI}a^3$,
  $\Pi = -\frac{P^2}{2}C = -\frac{P^2}{3EI}a^3$.
- $\mathcal{A} = aB$, $\partial_{\mathcal{A}} = \frac{1}{B}\partial_a$.
- $G = -\frac{d\Pi}{d\mathcal{A}} = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2a^2}{EIB} = \frac{12P^2a^2}{EB^2h^3}$

# 3.1. Griffith's Analysis and Energy Release Rate: Examples

Linear Elastic Fracture Mechanics

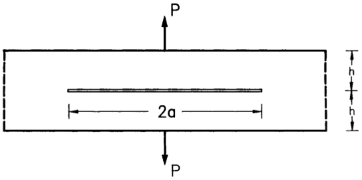
## Crack in Stre…



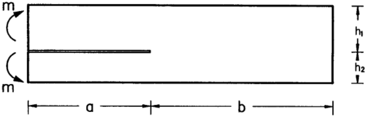
(Figure from sec…

- Crack: $\mathcal{A} = 2a$…
- $\Pi = U = \frac{\sigma^2 t}{2E'}($…
- $E_S = 2\mathcal{A}\gamma$, $\frac{dE}{d\mathcal{A}}$…
- $G = -\frac{d\Pi}{d\mathcal{A}} = -$…
- $\sigma_{cr} = \sqrt{\frac{E'\gamma}{\lambda a}} =$…

## Additional Cases to Consider

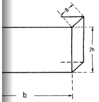



*(Figure 4.23 from Gdoutos (2005))*




*(Figure 4.20 from Gdoutos (2005))*

## …er Beam (DCB)




…Gdoutos 2005)

…, $C = \frac{2a^3}{3EI}$.

…$= \frac{P^2}{3EI}a^3$,

…$= \frac{2P^2}{3EI}a^3$,

…$\frac{P^2}{3EI}a^3$.

…$\partial_a$.

…$\frac{dC}{da} = \frac{P^2 a^2}{EIB} = \frac{12P^2a^2}{EB^2h^3}$

# 3.2. A Primer on 2D Elasticity

Linear Elastic Fracture Mechanics

- In 2D, the governing equations of elasticity (let us assume no body loads for simplicity) are written as,

$$\sigma_{x,x} + \tau_{xy,y} = 0, \quad \tau_{xy,x} + \sigma_{y,y} = 0.$$

- If we seek to obtain **solutions expressed directly in the stresses**, 2 equations won't cut it (we have 3 unique stresses $\sigma_x, \sigma_y, \tau_{xy}$). So we invoke strain compatibility, which is written as

$$\boxed{\varepsilon_{x,yy} + \varepsilon_{y,xx} = \gamma_{xy,xy}}$$

**Recall**: These are conditions that the strains must satisfy in order for them to have been generated by a continuously differentiable displacement field.

- This can be expressed in terms of the stresses if we invoke the **stress-strain constitutive relationships**.

# 3.2. A Primer on 2D Elasticity

Linear Elastic Fracture Mechanics

**Plane Stress**

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

**Compatibility**

$$\sigma_{x,yy} + \sigma_{y,xx} - \nu(\sigma_{x,xx} + \sigma_{y,yy}) = 2(1+\nu)\tau_{xy,xy}.$$

**Plane Strain**

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \frac{1+\nu}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

**Compatibility**

$$(1-\nu)(\sigma_{x,yy} + \sigma_{y,xx}) - \nu(\sigma_{x,xx} + \sigma_{y,yy}) = 2\tau_{xy,xy}.$$

- Making the substitution $\sigma_x = \phi_{,yy}$, $\sigma_y = \phi_{,xx}$, $\tau_{xy} = -\phi_{,xy}$, it is trivial to see that the equilibrium equations are satisfied automatically.
- In both the above cases, the compatibility equation comes out to be:

$$\phi_{,xxxx} + \phi_{,yyyy} + 2\phi_{,xxyy} = (\partial_{xx} + \partial_{yy})^2\phi = \nabla^4\phi = 0.$$

- Since the Laplacian when set to zero ($\nabla^2\phi = 0$) is referred to as the **harmonic equation** (recall complex analyticity), $\nabla^4\phi = 0$ is referred to as the **bi-harmonic equation**. $\phi$ is the **Airy Stress Function**.

# 3.3. Classical Solutions

Linear Elastic Fracture Mechanics

- Restricting ourselves to 2D problems, the governing equations may be written using the Airy's stress formulation as the biharmonic equation

$$\nabla^4 \phi = 0$$

- Let us look at this with cylindrical coordinates ($x = r\cos\theta$, $y = r\sin\theta$).

$$\underline{\nabla}\phi = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}\phi_{,r}\\ \frac{\phi_{,\theta}}{r}\end{bmatrix}, \quad \underline{\underline{\nabla u}} = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}u_{r,r} & \frac{u_{r,\theta}-u_\theta}{r}\\ u_{\theta,r} & \frac{u_{\theta,\theta}+u_r}{r}\end{bmatrix}\begin{bmatrix}\underline{e}_r\\ \underline{e}_\theta\end{bmatrix}$$

$$\underline{\underline{\nabla^2}}\phi = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}\phi_{,rr} & \partial_r(\frac{\phi_{,\theta}}{r})\\ \partial_r(\frac{\phi_{,\theta}}{r}) & \frac{\phi_{,r}}{r}+\frac{\phi_{,\theta\theta}}{r^2}\end{bmatrix}\begin{bmatrix}\underline{e}_r\\ \underline{e}_\theta\end{bmatrix}.$$

- The stresses are expressed (to satisfy equilibrium) as

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2}, \quad \sigma_{\theta\theta} = \phi_{,rr}, \quad \tau_{r\theta} = -\partial_r(\frac{\phi_{,\theta}}{r}).$$

# 3.3. Classical Solutions

Linear Elastic Fracture Mechanics

**General form of the Airy's Stress Function (Michell Solution, see Barber 2022, Ch. 8-9)**

$$\begin{aligned}
\phi =& a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\
& (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
& (a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r) \cos\theta \\
& (b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r) \sin\theta \\
& \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
& \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + b_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta.
\end{aligned}$$

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2}, \quad \sigma_{\theta\theta} = \phi_{,rr}, \quad \tau_{r\theta} = -\partial_r\left(\frac{\phi_{,\theta}}{r}\right).$$

# 3.3.1. The Michell Solution: Tabled Expressions

Classical Solutions

## Stress Components

| $\phi$ | $\sigma_{rr}$ | $\sigma_{r\theta}$ | $\sigma_{\theta\theta}$ |
|---|---|---|---|
| $r^2$ | $2$ | $0$ | $2$ |
| $r^2\ln(r)$ | $2\ln(r)+1$ | $0$ | $2\ln(r)+3$ |
| $\ln(r)$ | $1/r^2$ | $0$ | $-1/r^2$ |
| $\theta$ | $0$ | $1/r^2$ | $0$ |
| $r^3\cos\theta$ | $2r\cos\theta$ | $2r\sin\theta$ | $6r\cos\theta$ |
| $r\theta\sin\theta$ | $2\cos\theta/r$ | $0$ | $0$ |
| $r\ln(r)\cos\theta$ | $\cos\theta/r$ | $\sin\theta/r$ | $\cos\theta/r$ |
| $\cos\theta/r$ | $-2\cos\theta/r^3$ | $-2\sin\theta/r^3$ | $2\cos\theta/r^3$ |
| $r^3\sin\theta$ | $2r\sin\theta$ | $-2r\cos\theta$ | $6r\sin\theta$ |
| $r\theta\cos\theta$ | $-2\sin\theta/r$ | $0$ | $0$ |
| $r\ln(r)\sin\theta$ | $\sin\theta/r$ | $-\cos\theta/r$ | $\sin\theta/r$ |
| $\sin\theta/r$ | $-2\sin\theta/r^3$ | $2\cos\theta/r^3$ | $2\sin\theta/r^3$ |
| $r^{n+2}\cos n\theta$ | $-(n+1)(n-2)r^n\cos n\theta$ | $n(n+1)r^n\sin n\theta$ | $(n+1)(n+2)r^n\cos n\theta$ |
| $r^n\cos n\theta$ | $-n(n-1)r^{n-2}\cos n\theta$ | $n(n-1)r^{n-2}\sin n\theta$ | $n(n-1)r^{n-2}\cos n\theta$ |
| $r^{n+2}\sin n\theta$ | $-(n+1)(n-2)r^n\sin n\theta$ | $-n(n+1)r^n\cos n\theta$ | $(n+1)(n+2)r^n\sin n\theta$ |
| $r^n\sin n\theta$ | $-n(n-1)r^{n-2}\sin n\theta$ | $-n(n-1)r^{n-2}\cos n\theta$ | $n(n-1)r^{n-2}\sin n\theta$ |

*(Table 8.1 from Barber 2022)*

We set rigid body motion components to zero for the displacements

## Displacement Components

| $\phi$ | $2\mu u_r$ | $2\mu u_\theta$ |
|---|---|---|
| $r^2$ | $(\kappa-1)r$ | $0$ |
| $r^2\ln(r)$ | $(\kappa-1)r\ln(r)-r$ | $(\kappa+1)r\theta$ |
| $\ln(r)$ | $-1/r$ | $0$ |
| $\theta$ | $0$ | $-1/r$ |
| $r^3\cos\theta$ | $(\kappa-2)r^2\cos\theta$ | $(\kappa+2)r^2\sin\theta$ |
| $r\theta\sin\theta$ | $\frac{1}{2}\{(\kappa-1)\theta\sin\theta-\cos\theta$ $+(\kappa+1)\ln(r)\cos\theta\}$ | $\frac{1}{2}\{(\kappa-1)\theta\cos\theta-\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$ |
| $r\ln(r)\cos\theta$ | $\frac{1}{2}\{(\kappa+1)\theta\sin\theta-\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$ | $\frac{1}{2}\{(\kappa+1)\theta\cos\theta-\sin\theta$ $-(\kappa-1)\ln(r)\sin\theta\}$ |
| $\cos\theta/r$ | $\cos\theta/r^2$ | $\sin\theta/r^2$ |
| $r^3\sin\theta$ | $(\kappa-2)r^2\sin\theta$ | $-(\kappa+2)r^2\cos\theta$ |
| $r\theta\cos\theta$ | $\frac{1}{2}\{(\kappa-1)\theta\cos\theta+\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$ | $\frac{1}{2}\{-(\kappa-1)\theta\sin\theta-\cos\theta$ $-(\kappa+1)\ln(r)\cos\theta\}$ |
| $r\ln(r)\sin\theta$ | $\frac{1}{2}\{-(\kappa+1)\theta\cos\theta-\sin\theta$ $+(\kappa-1)\ln(r)\sin\theta\}$ | $\frac{1}{2}\{(\kappa+1)\theta\sin\theta+\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$ |
| $\sin\theta/r$ | $\sin\theta/r^2$ | $-\cos\theta/r^2$ |
| $r^{n+2}\cos n\theta$ | $(\kappa-n-1)r^{n+1}\cos n\theta$ | $(\kappa+n+1)r^{n+1}\sin n\theta$ |
| $r^n\cos n\theta$ | $-nr^{n-1}\cos n\theta$ | $nr^{n-1}\sin n\theta$ |
| $r^{n+2}\sin n\theta$ | $(\kappa-n-1)r^{n+1}\sin n\theta$ | $-(\kappa+n+1)r^{n+1}\cos n\theta$ |
| $r^n\sin n\theta$ | $-nr^{n-1}\sin n\theta$ | $-nr^{n-1}\cos n\theta$ |

*(Table 9.1 from Barber 2022)*

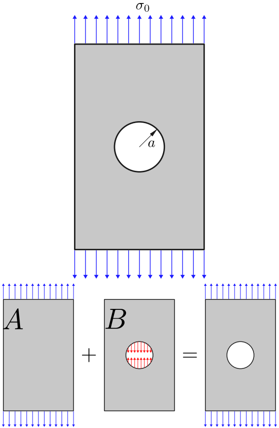
Plane Stress $\kappa = \frac{3-\nu}{1+\nu}$

Plane Strain $\kappa = 3-4\nu$

# 3.3.Plate with a Hole. Plate With a Hole Under Tension

Linear Elastic Fracture Mechanics

- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.



# 3.3.Plate with a Hole. Plate With a Hole Under Tension

Linear Elastic Fracture Mechanics

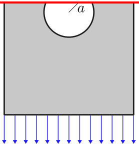
- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.

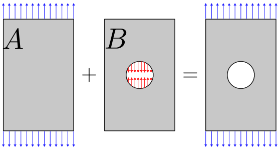
$\sigma_0$

**Displacement Field**

$$u_r = \frac{\sigma_0}{2}(\kappa - 1)r - \frac{\sigma_0}{2} r \cos 2\theta$$

$$u_\theta = \frac{\sigma_0}{2} r \sin 2\theta$$




**Problem A**

- The 2D stress field (cartesian) is

$$\underline{\underline{\sigma}}_{cart} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix}.$$

- Transforming to cylindrical coordinates,

$$\underline{\underline{\sigma}}_{cyl} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{bmatrix}$$
$$= \sigma_0 \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

The components can be written as

$$\sigma_{rr} = \sigma_0 \left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right), \quad \sigma_{r\theta} = \sigma_0 \frac{\sin 2\theta}{2},$$
$$\sigma_{\theta\theta} = \sigma_0 \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right).$$

# 3.3.Plate with a Hole. Plate With a Hole Under Tension

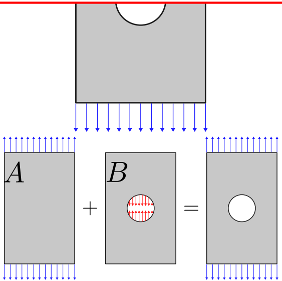


- Let us now try to use the above table for obtaining the stress distribution around a hole in a tensi

**Displacement Field**

$$u_r = \frac{\sigma_0 a^2}{2r}\left(1-(\kappa+1-(\frac{a}{r})^2)\cos 2\theta\right)$$

$$u_\theta = \frac{\sigma_0 a^2}{2r}\left(\kappa-1+(\frac{a}{r})^2\right)\sin 2\theta.$$




**Problem B**

- At $r=a$ we want

$$\sigma_{rr} = \sigma_0\left(\frac{1}{2}-\frac{\cos 2\theta}{2}\right), \quad \sigma_{r\theta} = \sigma_0\frac{\sin 2\theta}{2}.$$

  (no hoop component specified)
- As $r\to\infty$, we want $\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta} \to 0$ to match the far-field.
- Based on inspection (shown in class), we find the following Airy stress function to be a good starting point: $\phi = A\log r + B\theta + C\cos 2\theta + D\frac{\cos 2\theta}{r^2}$.
- Solving for $A, B, C, D$ based on the B.C.s we get,

$$\sigma_{rr} = -\frac{\sigma_0}{2}\left(\frac{a}{r}\right)^2 + 2\sigma_0\left(\frac{a}{r}\right)^2\left(1-\frac{3}{4}\left(\frac{a}{r}\right)^2\right)\cos 2\theta,$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(\frac{a}{r}\right)^2 + \frac{3\sigma_0}{4}\left(\frac{a}{r}\right)^4\cos 2\theta,$$

$$\sigma_{r\theta} = \sigma_0\left(\frac{a}{r}\right)^2\left(1-\frac{3}{2}\left(\frac{a}{r}\right)^2\right)\sin 2\theta$$

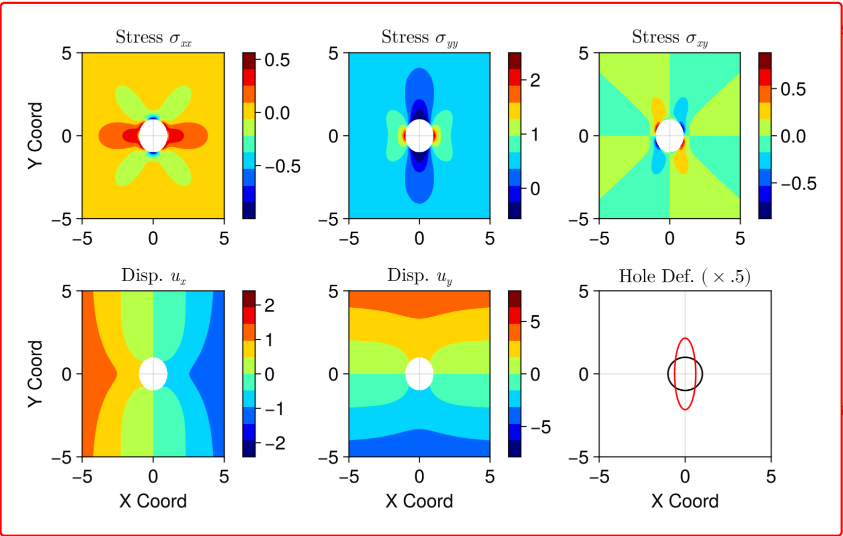
# 3.3.Plate with a Hole. Plate With a Hole Under Tension



- n

$u_r =$

$u_\theta =$

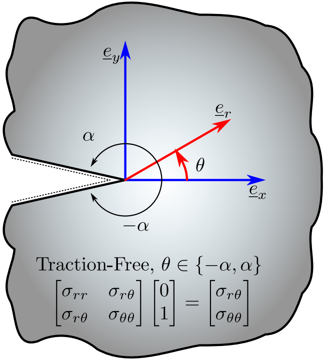



),

$\left(r\right) \quad \left( \quad 2 \left(r\right) \right)$

# 3.3.3. Notch Crack

Classical Solutions




- We seek an analytical solution for this problem setting **very close to the crack**.
- While we may intuitively expect stress to be singular at the crack tip, the strain energy has to be finite.
- Suppose $\sigma \sim \mathcal{O}(r^\lambda)$, $\varepsilon \sim \mathcal{O}(r^\lambda)$ necessarily.
  - So $\mathcal{U} = \int\int \frac{\sigma\varepsilon}{2} r dr d\theta \sim \mathcal{O}(r^{2\lambda+1})$.
  - For this to be finite, $2\lambda + 1 \geq 0 \implies \lambda \geq -\frac{1}{2}$.
- The only Airy stress functions that can show this are (refer sl. 18).

| $\phi$ | $\sigma_{rr}$ | $\sigma_{r\theta}$ | $\sigma_{\theta\theta}$ |
|---|---|---|---|
| $r^{n+2} \cos n\theta$ | $(..)r^n \cos n\theta$ | $(..)r^n \sin n\theta$ | $(..)r^n \cos n\theta$ |
| $r^n \cos n\theta$ | $(..)r^{n-2} \cos n\theta$ | $(..)r^{n-2} \sin n\theta$ | $(..)r^{n-2} \cos n\theta$ |
| $r^{n+2} \sin n\theta$ | $(..)r^n \sin n\theta$ | $(..)r^n \cos n\theta$ | $(..)r^n \sin n\theta$ |
| $r^n \sin n\theta$ | $(..)r^{n-2} \sin n\theta$ | $(..)r^{n-2} \cos n\theta$ | $(..)r^{n-2} \sin n\theta$ |

# 3.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

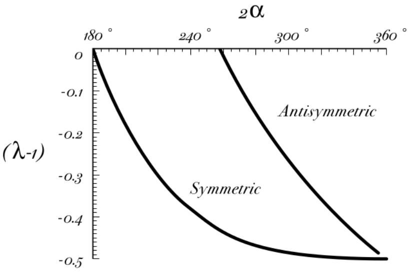
$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$

# 3.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$




*(Figure 11.7 from Barber 2022)*

# 3.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$

- Applying the boundary conditions (along with $\alpha = \pi$), we get a nonlinear eigenvalue problem that has the following solutions:

| $\lambda$ | Eigenfunction |
|---|---|
| $\frac{1}{2}$ | $A_2 = \frac{A_1}{3}$, $B_2 = -B_1$ |
| 1 | $A_2 = -A_1$, $B_2 = 0$ $(B_1 = 0)$ |
| $\frac{3}{2}$ | $A_2 = -\frac{A_1}{5}$, $B_2 = -B_1$ |
| $\vdots$ | |

- $\lambda = \frac{1}{2}$ corresponds to the near-field singular stress field, given by

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

# 3.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$

- Applying the boundary conditions (along with $\alpha = \pi$), we get a nonlinear eigenvalue problem that has the following solutions:

| $\lambda$ | Eigenfunction |
|---|---|

**Displacement Field**

$$2\mu u_r = K_I\sqrt{\frac{r}{2\pi}}\left((\kappa-\frac{1}{2})\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right) - K_{II}\sqrt{\frac{r}{2\pi}}\left((\kappa-\frac{1}{2})\sin\frac{\theta}{2} - \frac{3}{2}\sin\frac{3\theta}{2}\right)$$

$$2\mu u_\theta = K_I\sqrt{\frac{r}{2\pi}}\left(-(\kappa+\frac{1}{2})\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2}\right) - K_{II}\sqrt{\frac{r}{2\pi}}\left((\kappa+\frac{1}{2})\cos\frac{\theta}{2} - \frac{3}{2}\cos\frac{3\theta}{2}\right)$$

- $\lambda = \frac{1}{2}$ corresponds to the near-field singular stress field, given by

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

# 3.3.3. Singularity Close to Notch Crack

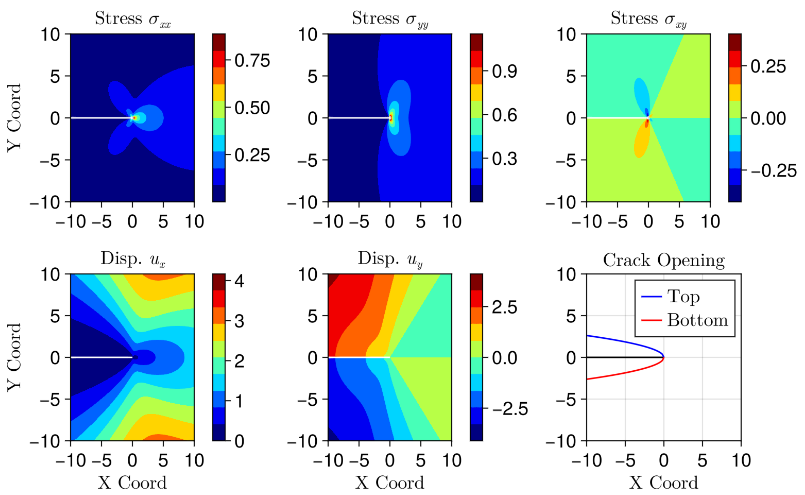
Classical Solutions

- ar

$2\mu u$

$2\mu u_\theta$

**Mode 1 Loading** (unit $K_I$)



# 3.3.3. Singularity Close to Notch Crack

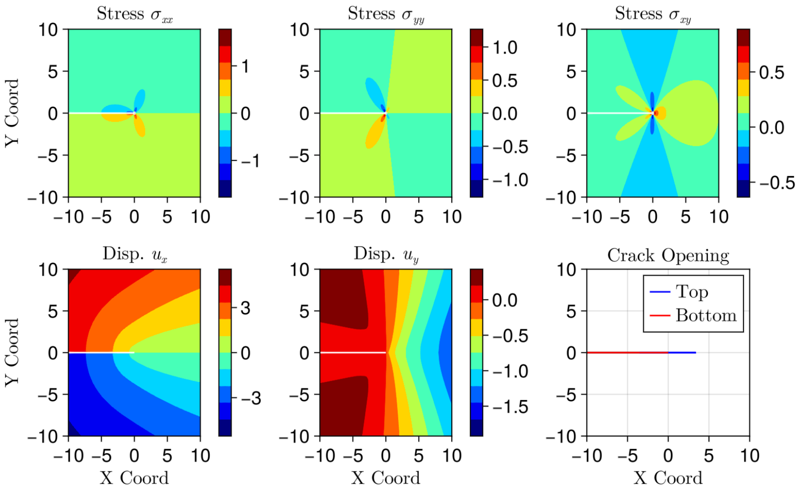
Classical Solutions

ar

$2\mu u$

$2\mu u_\theta$

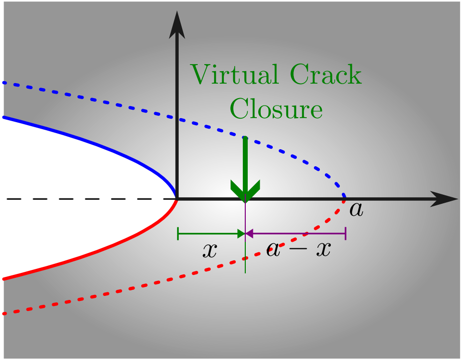
**Mode 2 Loading** (unit $K_{II}$)



# 3.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to “close” a crack.



# 3.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to "close" a crack.
- We observe that (all quantities in cylindrical):

$$@\,\theta = 0, \quad \underline{\underline{\sigma}} = \frac{K_I}{\sqrt{2\pi r}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa-1\\ 0\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ \kappa-1\end{bmatrix}$$

$$@\,\theta = \pi, \quad \underline{\underline{\sigma}} = \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix}, \quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ -(\kappa+1)\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa+1\\ 0\end{bmatrix}.$$

# 3.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to "close" a crack.
- We observe that (all quantities in cylindrical):

$$@\,\theta=0,\quad \underline{\underline{\sigma}} = \frac{K_I}{\sqrt{2\pi r}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa-1\\ 0\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ \kappa-1\end{bmatrix}$$

$$@\,\theta=\pi,\quad \underline{\underline{\sigma}} = \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix},\quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ -(\kappa+1)\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa+1\\ 0\end{bmatrix}.$$

- For virtual crack closure, the work done can be written as,

$$W(a) = 2\int_0^a \frac{1}{2}\left(\sigma_{\theta\theta}\Big|_{\theta=0}(-u_\theta)\Big|_{\theta=\pi} + \sigma_{r\theta}|_{\theta=0}(-u_r)|_{\theta=\pi}\right)dx$$

$$= \int_0^a \frac{K_I}{\sqrt{2\pi x}}K_I\sqrt{\frac{a-x}{2\pi}}\frac{\kappa+1}{2\mu} + \frac{K_{II}}{\sqrt{2\pi r}}K_{II}\sqrt{\frac{a-x}{2\pi}}\frac{\kappa+1}{2\mu}\,dx$$

$$= \frac{K_I^2+K_{II}^2}{2\pi}\frac{\kappa+1}{2\mu}\underbrace{\int_0^a\sqrt{\frac{a-x}{x}}\,dx}_{\frac{a\pi}{2}} = \frac{K_I^2+K_{II}^2}{8\mu^2}(\kappa+1)^2 a = \begin{cases}\frac{K_I^2+K_{II}^2}{E}a & \text{Plane Stress}\\ \frac{K_I^2+K_{II}^2}{E}(1-\nu^2)a & \text{Plane Strain}\end{cases}.$$

- The Griffith Energy Release Rate is the derivative $\lim_{a\to 0}\frac{1}{B}\frac{dW}{da}$, which evaluates as

$$G = \frac{1}{B}\begin{cases}\frac{K_I^2}{E}+\frac{K_{II}^2}{E} & \text{Plane Stress}\\ \frac{K_I^2}{E}(1-\nu^2)+\frac{K_{II}^2}{E}(1-\nu^2) & \text{Plane Strain}\end{cases}.$$

# 3.3.3. Stress Intensity Factor

Classical Solutions

- A crack is said to propagate when $G$ exceeds $G_{cr}$.
- Therefore, under "pure" mode 1 loading, the *Critical Stress Intensity Factor* ($K_{I,cr}$) is

$$K_{I,cr} = \begin{cases} \sqrt{BG_{cr}E} & \text{Plane Stress} \\ \sqrt{\frac{BG_{cr}E}{1-\nu^2}} & \text{Plane Strain} \end{cases}.$$

- This shows that for identical conditions, the Plane Stress case (thin plates) has **higher fracture toughness** than its plane stress counterpart (long prismatic structures).
- **But how do we relate $K_I, K_{II}$ with far-field applied stresses?**

# 3.3.3. Stress Intensity Factor

Classical Solutions

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$$K_{I,cr} = \begin{cases} \sqrt{BG_{cr}E} & \text{Plane Stress} \\ \sqrt{\frac{BG_{cr}E}{1-\nu^2}} & \text{Plane Strain} \end{cases}.$$

- This shows that for identical conditions, the Plane Stress case (thin plates) has **higher fracture toughness** than its plane stress counterpart (long prismatic structures).
- **But how do we relate $K_I, K_{II}$ with far-field applied stresses?** The answer is very closely tied in to the exact geometry, etc.

# 3.3.3. Griffith-Inglis Crack Revisited

Classical Solutions

- For the flat crack of length $2a$ (aka the Griffith-Inglis crack), the SIF is related to tensile stresses by

$$K_I = \sigma_0 \sqrt{\pi a}.$$

- Note that this is why we chose $\lambda = \frac{\pi}{2}$ in sl. 7. If we left it in, we'll have to satisfy (plane stress considered here):

$$\frac{4\lambda a}{E}\sigma_0^2 = \frac{2K_I^2}{E} = \frac{2\pi a}{E}\sigma_0^2.$$

# 3.4. Crack Propagation and the Paris Law

Linear Elastic Fracture Mechanics

- **Paris Law**: $\frac{da}{dN} = C(\Delta K)^m$.
- Usually $a_f$ is specified and we are interested in finding how many cycles until a crack of size $a_i$ grows to $a_f$. This is the "life" of the material.

*Values for common engineering materials, from Kumar 2009*

| Material | $C$ | $m$ |
|---:|:---:|:---:|
| Ferrite-Pearlite (S) | $6.8\times10^{-12}$ | 3.0 |
| Martensite (S) | $1.33\times10^{-10}$ | 2.25 |
| Austenite (S) | $5.5\times10^{-12}$ | 3.25 |
| Cast Iron (S) | $5.5\times10^{-12}$ | 3.25 |
| Al-Alloy | $1.1\times10^{-11}$ | 3.89 |

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