AS2070: Aerospace Structural Mechanics

Module 3: Introduction to Fatigue and Failure

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April 25, 2025

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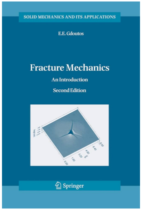
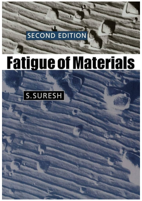
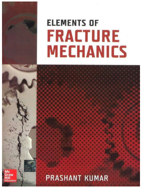
Also see https://www.fracturemechanics.org/



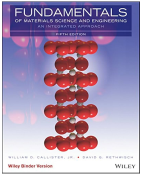
Chapters 1,4 in Gdoutos (2005)



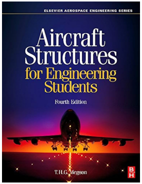
Chapters 1,7,9 in Suresh (1998)



Chapters 1-3 in Kumar (2009)



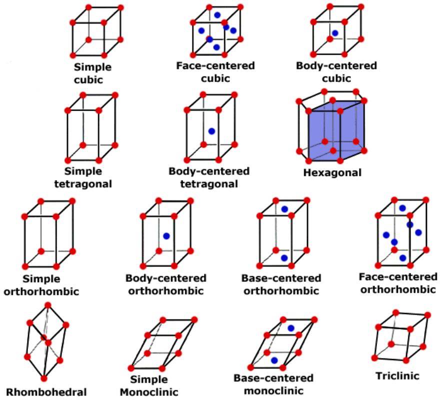
Chapter 3 in Jr and Rethwisch (2012)



Chapter 15 in Megson (2013)

1.1. Structure of Materials

Introduction



Types of crystal structures in metals Sparky (2013)

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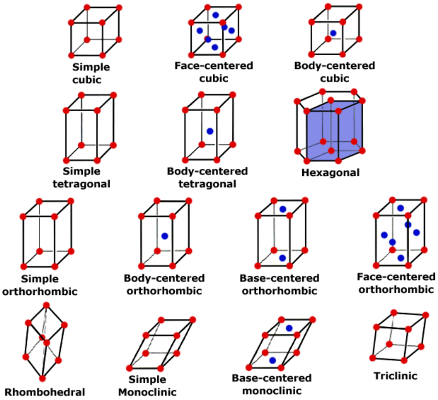

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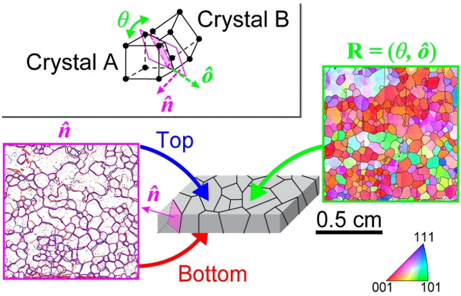

Crystal and Grain Structures New Technique Provides Detailed Views of Metals' Crystal Structure (2016). "Polycrystallinity"

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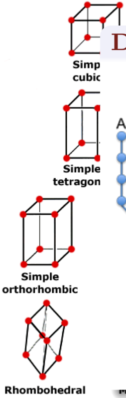


Types of crystal structures in metals Sparky (2013)

Dislocations and their "motion" under load

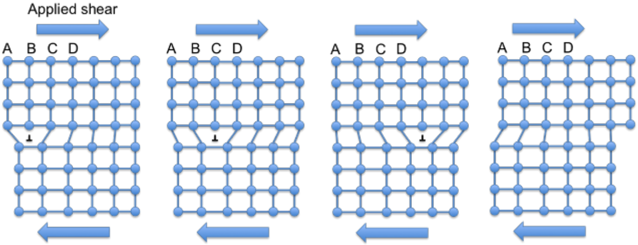
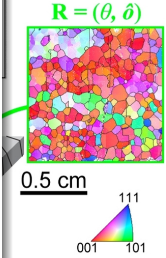


Figure from Sparky (2013)



New Technique Provides Detailed Views of Metals' Crystal Structure (2016). "Polycrystallinity"

1.2. Understanding the Stress-Strain Curve

Introduction

The Uniaxial Tensile Test

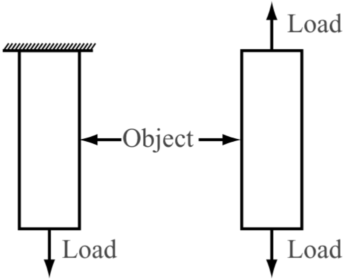


Figure from Rajendran 2011

1.2. Understanding the Stress-Strain Curve

Introduction

Terminology

1. Proportionality Limit;
2. Elastic Limit;
3. Yield Point;
4. Ultimate Strength;
5. Fracture Point;
6. Elongation at Failure;

Ductile Fracture

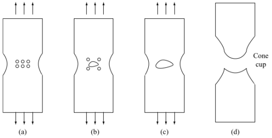


Figure from Rajendran 2011

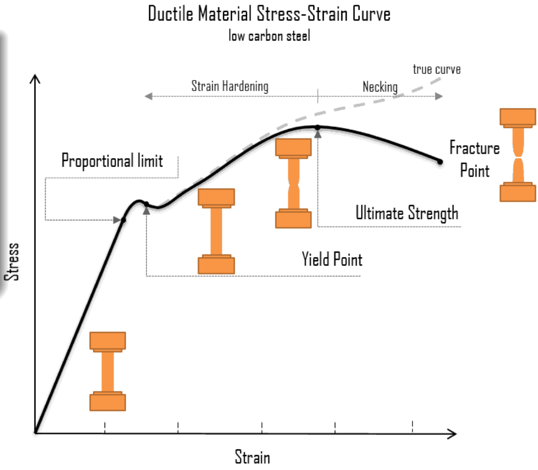


Figure from Connor 2020

1.3. Failure Mechanisms: Fracture

1. Introduction

"Griffith Theory" of brittle fracture

- Theoretical fracture stress $\sim \frac{E}{5} - \frac{E}{30}$ (steel $\sim \frac{E}{1000}$)
- Fracture occurs when $E_{strain} = E_{surface}$
- Crack propagates when $\frac{dE_{strain}}{dL} = \frac{dE_{surface}}{dL}$

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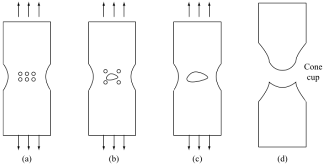

Ductile Fracture Rajendran 2011

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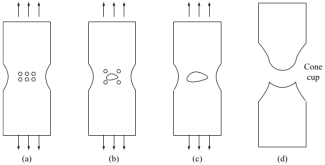


Ductile Fracture Rajendran 2011

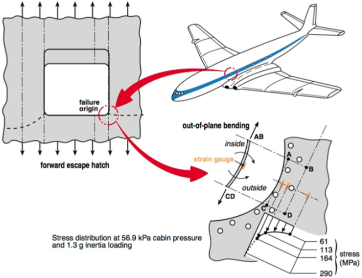
Sr. No	*Brittle Fracture*	*Ductile Fracture*
1.	It occurs with no or little plastic deformation.	It occurs with large plastic deformation.
2.	The rate of propagation of the crack is fast.	The rate of propagation of the crack is slow.
3.	It occurs suddenly without any warning.	It occurs slowly.
4.	The fractured surface is flat.	The fractured surface has rough contour and the shape is similar to cup and cone arrangement.
5.	The fractured surface appears shiny.	The fractured surface is dull when viewed with naked eye and the surface has dimpled appearance when viewed with scanning electron microscope.
6.	It occurs where micro crack is larger.	It occurs in localised region where the deformation is larger.

Ductile vs Brittle Fracture Rajendran 2011

1.3. Failure Mechanisms: Fatigue

1. Introduction

..over 90% of mechanical failures are caused because of metal fatigue *What Is Metal Fatigue?* 2021...



The De Havilland Comet *The deHavilland Comet Disaster 2019* [lecture]

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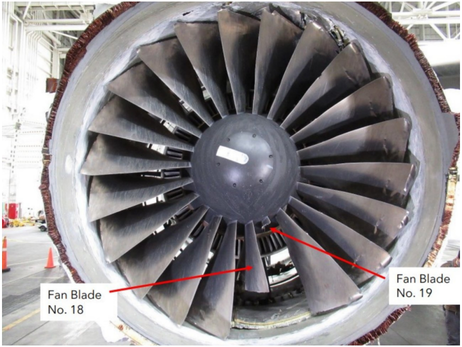
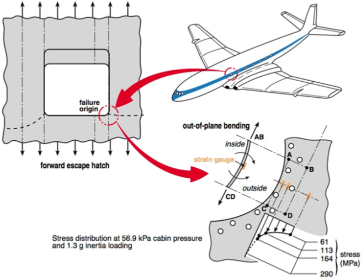


A more recent example (2021 United Airlines Boeing 777) **DCA21FA085Aspx**. *[video]*



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A more recent exam *Boeing 777)* **DCA**

Fatigue Crack Propagation: Beech Marks

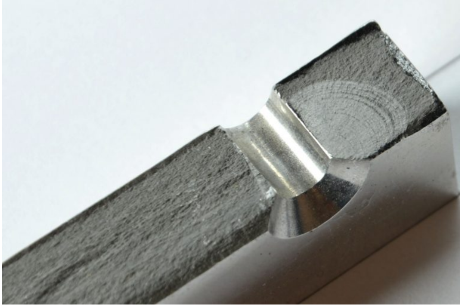
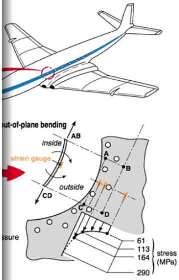


Figure from Fatigue Physics 2024



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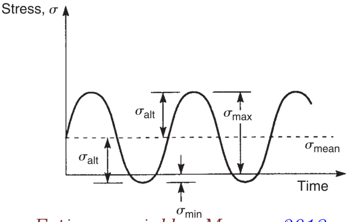
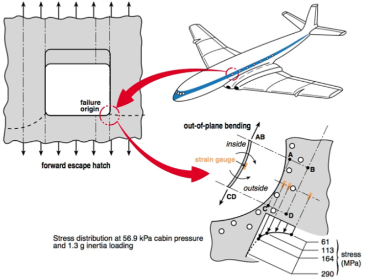


Fatigue variables Megson 2013



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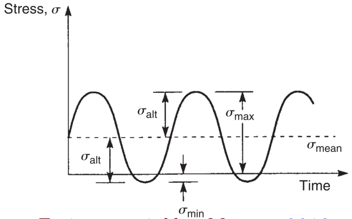

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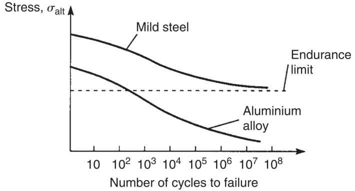
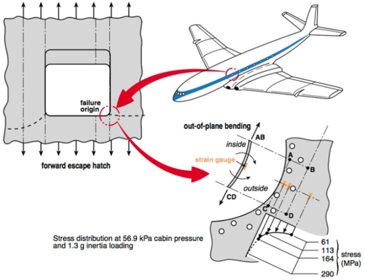


The S-n Diagram Megson 2013



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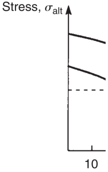
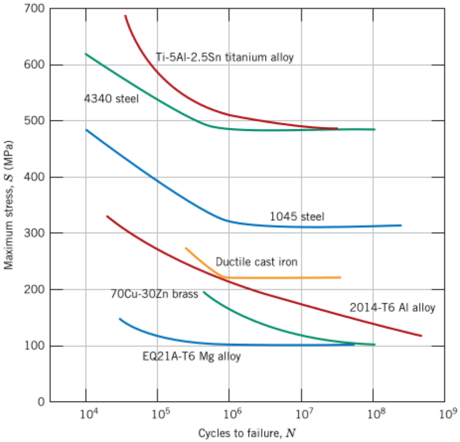
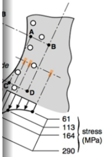
Fatigue



The S-n

S-N Curves for Common Metals (Jr and Rethwisch 2012)





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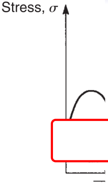
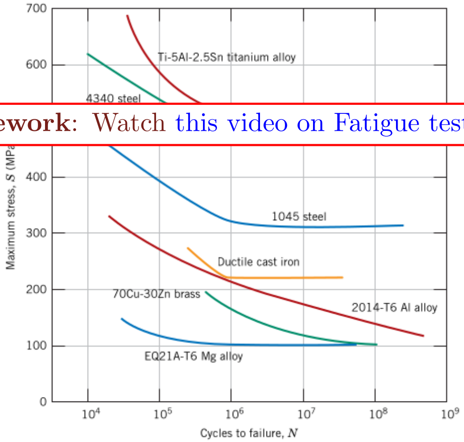
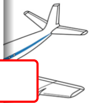
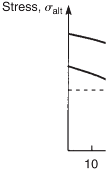






Homework: Watch this video on Fatigue testing.

Fatigue



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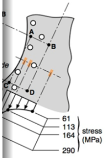

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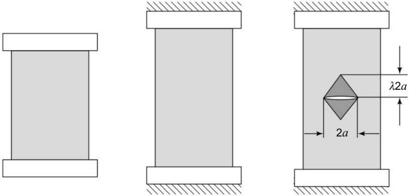
1.4. Energy Release Rate: Griffith's Analysis

Introduction



Simplistic picture of the introduction of a crack in a stretched specimen(Figure from sec 2.5 in Kumar 2009)

- Because of the crack, we assume $\sigma \approx 0$ in the triangles.
- Corresponding energy loss:

$$E_R = V_\Delta \times (\frac{\sigma^2}{2E}) = \frac{2a^2 \lambda t \sigma^2}{E}.$$

1.4. Energy Release Rate: Griffith's Analysis

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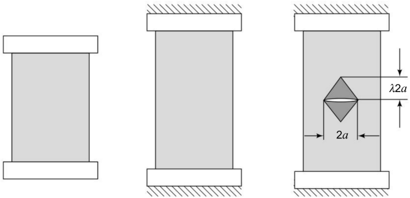

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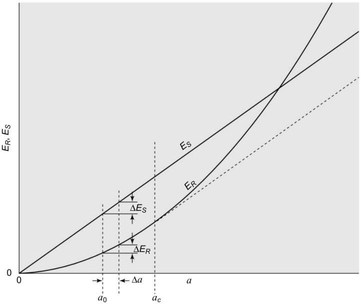
- For thin plates, $\lambda = \frac{\pi}{2}$. So,

$$E_R = \frac{\pi a^2 t\sigma^2}{E}.$$

- The "creation" of a surface takes energy. We write this as,

$$E_S = 2(2at)\gamma = 4at\gamma.$$



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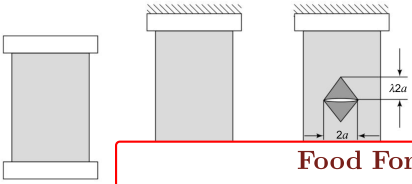

Simplistic pict
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Food For Thought

- What would a "safe size" of crack be, for a given loading condition? *Hint: Think incrementally*

- Becaus
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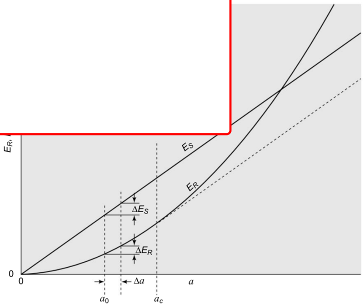

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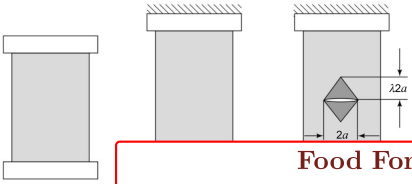

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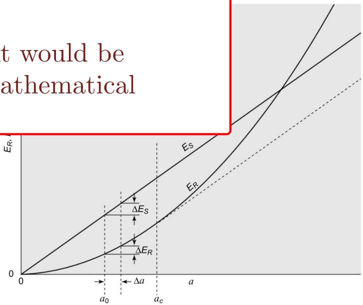


Food For Thought

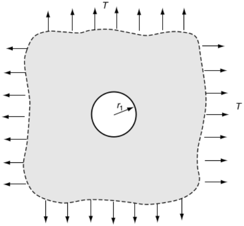
- What would a "safe size" of crack be, for a given loading condition? *Hint: Think incrementally*
- What type of an experiment would be necessary to confirm this mathematical framework?

1.5. Linear Elastic Fracture Mechanics

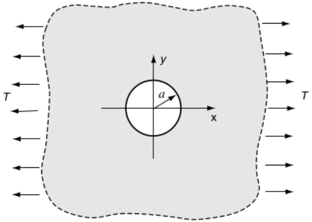
Introduction (Ref: Sec. 8.4.2 in Sadd 2009)

Consider the following two cases.
Question: Where will the point of peak stress occur? And which will have higher maximum stress?

Case 1



Case 2



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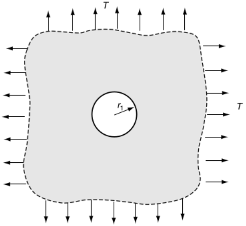

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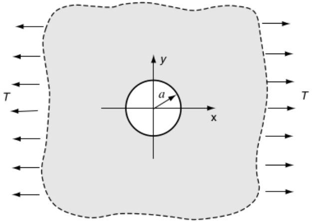

Analytical Solution

$\sigma_r = T(1 - \frac{r_1^2}{r^2})$, $\sigma_\theta = T(1 + \frac{r_1^2}{r^2})$

$\implies \boxed{\sigma_{\max} = 2T}$

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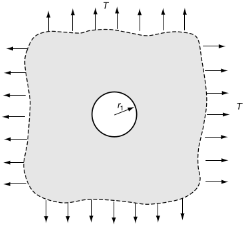

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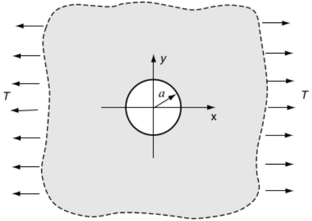

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Analytical Solution

$\sigma_r = T(1 - \frac{r_1^2}{r^2}) + (\cdot)\cos(2\theta)$, $\sigma_\theta = \dots$
$\implies \boxed{\sigma_{\max} = 3T}$

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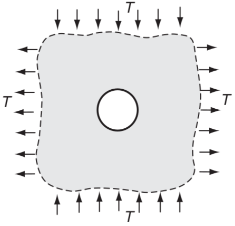
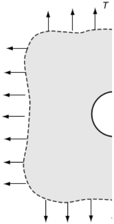
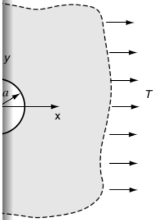
Case 3



$\boxed{\sigma_{\max} = 4T}$

Case 1





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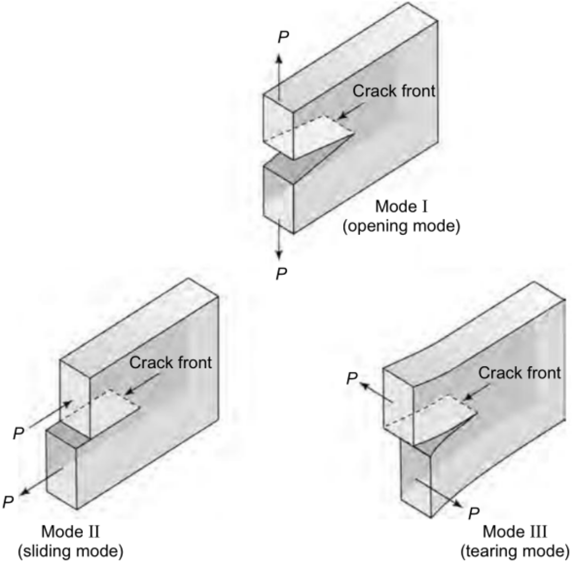
1.6. Modes of Fracture

Introduction

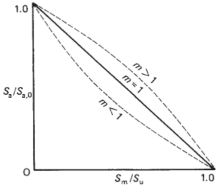


2. Introduction to Fatigue

Concepts

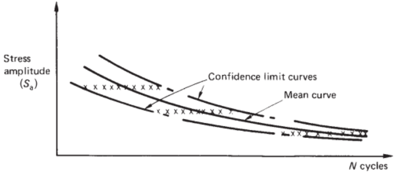
- Safe Life: RUL
- Fail-Safe: Redundancy

Tensile Stresses: The Goodman Diagram



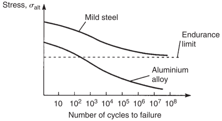
(Figure 15.2 from Megson 2013)

$$\frac{S_a}{S_{a,0}} = 1 - \left(\frac{S_m}{S_u}\right)^m$$



(Figure 15.1 from Megson 2013)

The S-N Curve



(Figure from Megson 2013)

$$\sigma_{alt} = \sigma_\infty \left(1 + \frac{C}{\sqrt{N}}\right), \quad N \propto \frac{1}{\sigma_{mean}}.$$

2.1. The deHavilland Comet

Introduction to Fatigue

No aircraft has contributed more to safety in the jet age than the Comet. The lessons it taught the world of aeronautics live in every jet airliner flying today. – D.D. Dempster, 1959, in The Tale of the Comet

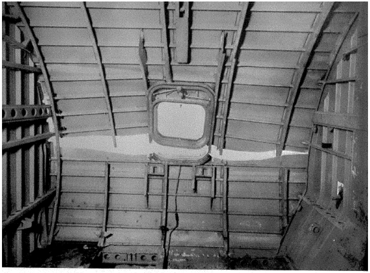


FIG. 7. VIEW FROM INSIDE OF FAILURE AT THE FORWARD ESCAPE HATCH ON THE PORT SIDE—COMET G-ALYU

(Figures from "De Havilland Comet" 2025)

2.1. The deHavilland Comet

Introduction to Fatigue

No aircraft has contributed more to safety in the jet age than the Comet. The lessons it taught the world of aeronautics live in every jet airliner flying

The Tale of the Comet

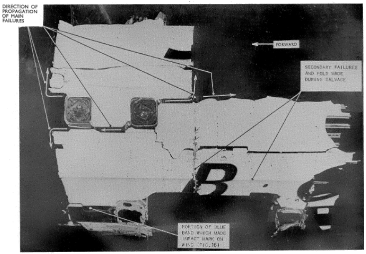
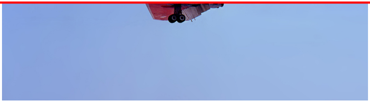


FIG. 12. PHOTOGRAPH OF WRECKAGE AROUND ADF AERIAL WINDOWS—G-ALYP.



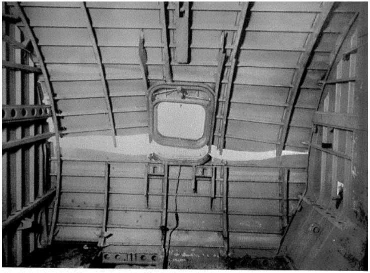


FIG. 7. VIEW FROM INSIDE OF FAILURE AT THE FORWARD ESCAPE HATCH ON THE PORT SIDE—COMET G-ALYU

(Figures from "De Havilland Comet" 2025)

2.1. The

Introduction to I

No air ... Comet.
The les ... r flying

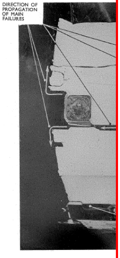
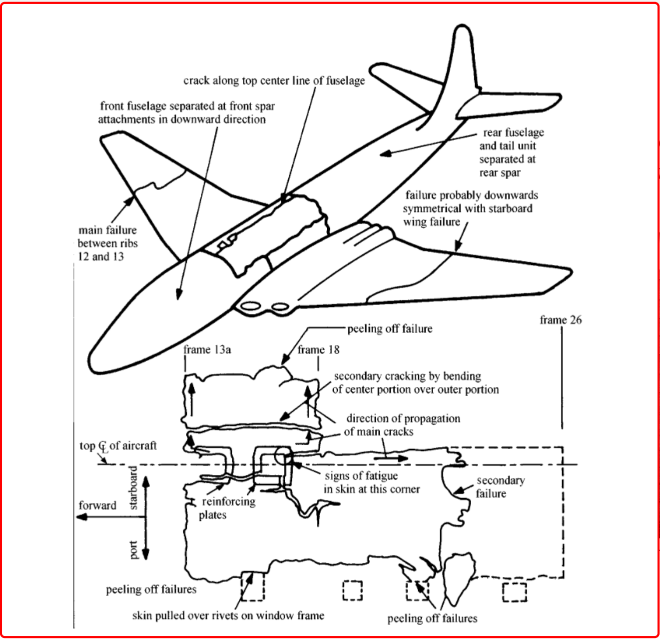


FIG. 12. PHOTO



TCH ON THE

t" 2025)

2.2. Miner's Rule

Introduction to Fatigue

- Suppose at an operation level of σ_m, σ_a, the fatigue life is N and the structure undergoes n cycles, Miner's rule posits that $\frac{n}{N}$ is the fraction of life that has been consumed.
- Suppose a structure undergoes multiple stress levels in its loading history, the total fraction of fatigue life that has been consumed is written as

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots.$$

- The structure is expected to fail when this sum becomes 1.0..

3.1. Griffith's Analysis and Energy Release Rate

Linear Elastic Fracture Mechanics

- The total energy of a loaded elastic body is written as

$$\Pi = \underbrace{U}_{\text{elastic}} - \underbrace{W}_{\text{external}}.$$

- **Griffith's principle**: The **energy lost due to the creation** of a cracked surface must be equal to the **energy required for the creation** of the cracked surface.
- Surface energy is usually expressed as $E_S = \mathcal{A}\gamma$.
- This is a general principle agnostic of the exact structure under consideration.

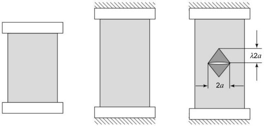
$$\boxed{G = -\frac{d\Pi}{d\mathcal{A}} = 2\gamma}.$$

(note: $2\mathcal{A}$ is the effective total "new" surface area that has been created)

3.1. Griffith's Analysis and Energy Release Rate: Examples

Linear Elastic Fracture Mechanics

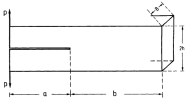
Crack in Stretched Specimen



(*Figure from sec 2.5 in Kumar 2009*)

- Crack: $\mathcal{A} = 2at$, $\partial_{\mathcal{A}} = \frac{1}{2t}\partial_a$
- $\Pi = U = \frac{\sigma^2 t}{2E'}(\mathcal{A}_{tot} - 4\lambda a^2)$.
- $E_S = 2\mathcal{A}\gamma$, $\frac{dE_S}{d\mathcal{A}} = 2\gamma$.
- $G = -\frac{d\Pi}{d\mathcal{A}} = -\frac{1}{2t}\frac{d\Pi}{da} = \frac{\lambda a}{2E'}\sigma^2$.
- $\sigma_{cr} = \sqrt{\frac{E'\gamma}{\lambda a}} = \sqrt{\frac{2E'\gamma}{\pi a}}$.

Double Cantilever Beam (DCB)



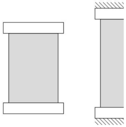
(*Figure 4.14 in Gdoutos 2005*)

- $u = CP = \frac{2a^3}{3EI}P$, $C = \frac{2a^3}{3EI}$.
- $U = \frac{Pu}{2} = \frac{CP^2}{2} = \frac{P^2}{3EI}a^3$,
  $W = Pu = CP^2 = \frac{2P^2}{3EI}a^3$,
  $\Pi = -\frac{P^2}{2}C = -\frac{P^2}{3EI}a^3$.
- $\mathcal{A} = aB$, $\partial_{\mathcal{A}} = \frac{1}{B}\partial_a$.
- $G = -\frac{d\Pi}{d\mathcal{A}} = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2a^2}{EIB} = \frac{12P^2a^2}{EB^2h^3}$

3.1. Griffith's Analysis and Energy Release Rate: Examples

Linear Elastic Fracture Mechanics

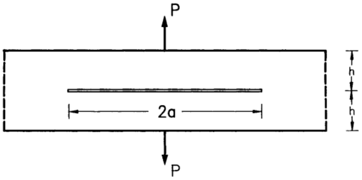
Crack in Stre



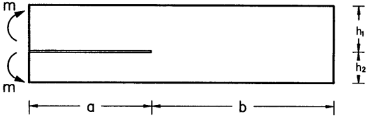
(Figure from sec

- Crack: $\mathcal{A} = 2a$
- $\Pi = U = \frac{\sigma^2 t}{2E'}($
- $E_S = 2\mathcal{A}\gamma,\ \frac{dE}{d\mathcal{A}}$
- $G = -\frac{d\Pi}{d\mathcal{A}} = -$
- $\sigma_{cr} = \sqrt{\frac{E'\gamma}{\lambda a}} =$

Additional Cases to Consider

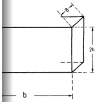


(Figure 4.23 from Gdoutos (2005))



(Figure 4.20 from Gdoutos (2005))

er Beam (DCB)



Gdoutos 2005)

$, C = \frac{2a^3}{3EI}.$

$= \frac{P^2}{3EI}a^3,$

$= \frac{2P^2}{3EI}a^3,$

$\frac{P^2}{3EI}a^3.$

$\partial_a.$

$\frac{dC}{da} = \frac{P^2a^2}{EIB} = \frac{12P^2a^2}{EB^2h^3}$

3.2. A Primer on 2D Elasticity

Linear Elastic Fracture Mechanics

- In 2D, the governing equations of elasticity (let us assume no body loads for simplicity) are written as,

$$\sigma_{x,x} + \tau_{xy,y} = 0, \quad \tau_{xy,x} + \sigma_{y,y} = 0.$$

- If we seek to obtain **solutions expressed directly in the stresses**, 2 equations won't cut it (we have 3 unique stresses $\sigma_x, \sigma_y, \tau_{xy}$). So we invoke strain compatibility, which is written as

$$\boxed{\varepsilon_{x,yy} + \varepsilon_{y,xx} = \gamma_{xy,xy}}$$

Recall: These are conditions that the strains must satisfy in order for them to have been generated by a continuously differentiable displacement field.

- This can be expressed in terms of the stresses if we invoke the **stress-strain constitutive relationships**.

3.2. A Primer on 2D Elasticity

Linear Elastic Fracture Mechanics

Plane Stress

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

Compatibility

$$\sigma_{x,yy} + \sigma_{y,xx} - \nu(\sigma_{x,xx} + \sigma_{y,yy}) = 2(1+\nu)\tau_{xy,xy}.$$

Plane Strain

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \frac{1+\nu}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

Compatibility

$$(1-\nu)(\sigma_{x,yy} + \sigma_{y,xx}) - \nu(\sigma_{x,xx} + \sigma_{y,yy}) = 2\tau_{xy,xy}.$$

- Making the substitution $\sigma_x = \phi_{,yy}$, $\sigma_y = \phi_{,xx}$, $\tau_{xy} = -\phi_{,xy}$, it is trivial to see that the equilibrium equations are satisfied automatically.
- In both the above cases, the compatibility equation comes out to be:

$$\phi_{,xxxx} + \phi_{,yyyy} + 2\phi_{,xxyy} = (\partial_{xx} + \partial_{yy})^2\phi = \nabla^4\phi = 0.$$

- Since the Laplacian when set to zero ($\nabla^2\phi = 0$) is referred to as the **harmonic equation** (recall complex analyticity), $\nabla^4\phi = 0$ is referred to as the **bi-harmonic equation**. ϕ is the **Airy Stress Function**.

3.3. Classical Solutions

Linear Elastic Fracture Mechanics

- Restricting ourselves to 2D problems, the governing equations may be written using the Airy's stress formulation as the biharmonic equation

$$\nabla^4 \phi = 0$$

- Let us look at this with cylindrical coordinates $(x = r\cos\theta,\ y = r\sin\theta)$.

$$\underline{\nabla}\phi = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}\phi_{,r}\\ \frac{\phi_{,\theta}}{r}\end{bmatrix}, \quad \underline{\underline{\nabla u}} = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}u_{r,r} & \frac{u_{r,\theta}-u_\theta}{r}\\ u_{\theta,r} & \frac{u_{\theta,\theta}+u_r}{r}\end{bmatrix}\begin{bmatrix}\underline{e}_r\\ \underline{e}_\theta\end{bmatrix}$$

$$\underline{\underline{\nabla^2}}\phi = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}\phi_{,rr} & \partial_r(\frac{\phi_{,\theta}}{r})\\ \partial_r(\frac{\phi_{,\theta}}{r}) & \frac{\phi_{,r}}{r}+\frac{\phi_{,\theta\theta}}{r^2}\end{bmatrix}\begin{bmatrix}\underline{e}_r\\ \underline{e}_\theta\end{bmatrix}.$$

- The stresses are expressed (to satisfy equilibrium) as

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2}, \quad \sigma_{\theta\theta} = \phi_{,rr}, \quad \tau_{r\theta} = -\partial_r(\frac{\phi_{,\theta}}{r}).$$

3.3. Classical Solutions

Linear Elastic Fracture Mechanics

General form of the Airy's Stress Function (Michell Solution, see Barber 2022, Ch. 8-9)

$$
\begin{aligned}
\phi =& a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\
&(a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
&\left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r\right) \cos\theta \\
&\left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r\right) \sin\theta \\
&\sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
&\sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + b_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta.
\end{aligned}
$$

$$
\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2}, \quad \sigma_{\theta\theta} = \phi_{,rr}, \quad \tau_{r\theta} = -\partial_r\left(\frac{\phi_{,\theta}}{r}\right).
$$

3.3.1. The Michell Solution: Tabled Expressions

Classical Solutions

Stress Components

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$r^2\ln(r)$	$2\ln(r)+1$	0	$2\ln(r)+3$
$\ln(r)$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^3\cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln(r)\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln(r)\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$n(n+1)r^n\sin n\theta$	$(n+1)(n+2)r^n\cos n\theta$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-n(n+1)r^n\cos n\theta$	$(n+1)(n+2)r^n\sin n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$

(Table 8.1 from Barber 2022)

We set rigid body motion components to zero for the displacements

Displacement Components

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$r^2\ln(r)$	$(\kappa-1)r\ln(r)-r$	$(\kappa+1)r\theta$
$\ln(r)$	$-1/r$	0
θ	0	$-1/r$
$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r\theta\sin\theta$	$\frac{1}{2}\{(\kappa-1)\theta\sin\theta-\cos\theta$ $+(\kappa+1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta-\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$
$r\ln(r)\cos\theta$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta-\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\cos\theta-\sin\theta$ $-(\kappa-1)\ln(r)\sin\theta\}$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa+2)r^2\cos\theta$
$r\theta\cos\theta$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta+\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{-(\kappa-1)\theta\sin\theta-\cos\theta$ $-(\kappa+1)\ln(r)\cos\theta\}$
$r\ln(r)\sin\theta$	$\frac{1}{2}\{-(\kappa+1)\theta\cos\theta-\sin\theta$ $+(\kappa-1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta+\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^{n+2}\cos n\theta$	$(\kappa-n-1)r^{n+1}\cos n\theta$	$(\kappa+n+1)r^{n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa-n-1)r^{n+1}\sin n\theta$	$-(\kappa+n+1)r^{n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$

(Table 9.1 from Barber 2022)

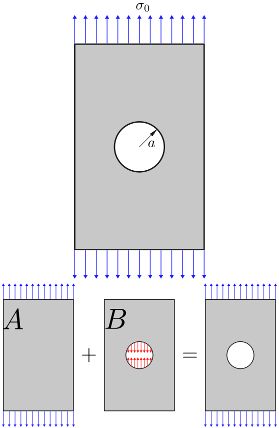
Plane Stress $\kappa = \frac{3-\nu}{1+\nu}$

Plane Strain $\kappa = 3 - 4\nu$

3.3.Plate with a Hole. Plate With a Hole Under Tension

Linear Elastic Fracture Mechanics

- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.



3.3.Plate with a Hole. Plate With a Hole Under Tension

Linear Elastic Fracture Mechanics

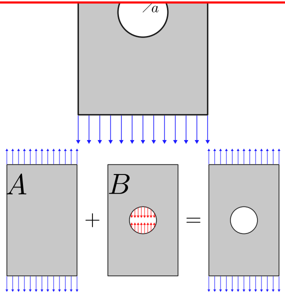
- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.

σ_0

Displacement Field

$$u_r = \frac{\sigma_0}{2}(\kappa - 1)r - \frac{\sigma_0}{2} r\cos 2\theta$$

$$u_\theta = \frac{\sigma_0}{2} r \sin 2\theta$$



Problem A

- The 2D stress field (cartesian) is

$$\underline{\underline{\sigma}}_{cart} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix}.$$

- Transforming to cylindrical coordinates,

$$\underline{\underline{\sigma}}_{cyl} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{bmatrix}$$

$$= \sigma_0 \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

The components can be written as

$$\sigma_{rr} = \sigma_0\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right), \quad \sigma_{r\theta} = \sigma_0 \frac{\sin 2\theta}{2},$$

$$\sigma_{\theta\theta} = \sigma_0\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right).$$

3.3.Plate with a Hole. Plate With a Hole Under Tension

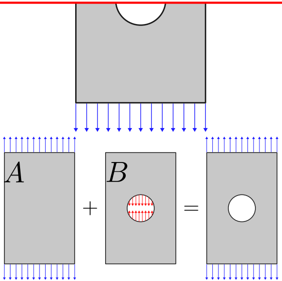
Linear Elastic Fracture Mechanics

- Let us now try to use the above table for obtaining the stress distribution around a hole in a tensi

Displacement Field

$$u_r = \frac{\sigma_0 a^2}{2r}\left(1-(\kappa+1-(\frac{a}{r})^2)\cos 2\theta\right)$$

$$u_\theta = \frac{\sigma_0 a^2}{2r}\left(\kappa-1+(\frac{a}{r})^2\right)\sin 2\theta.$$



Problem B

- At $r=a$ we want

$$\sigma_{rr} = \sigma_0\left(\frac{1}{2}-\frac{\cos 2\theta}{2}\right), \quad \sigma_{r\theta} = \sigma_0\frac{\sin 2\theta}{2}.$$

(no hoop component specified)

- As $r\to\infty$, we want $\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta} \to 0$ to match the far-field.
- Based on inspection (shown in class), we find the following Airy stress function to be a good starting point: $\phi = A\log r + B\theta + C\cos 2\theta + D\frac{\cos 2\theta}{r^2}$.
- Solving for A, B, C, D based on the B.C.s we get,

$$\sigma_{rr} = -\frac{\sigma_0}{2}\left(\frac{a}{r}\right)^2 + 2\sigma_0\left(\frac{a}{r}\right)^2\left(1-\frac{3}{4}\left(\frac{a}{r}\right)^2\right)\cos 2\theta,$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(\frac{a}{r}\right)^2 + \frac{3\sigma_0}{4}\left(\frac{a}{r}\right)^4\cos 2\theta,$$

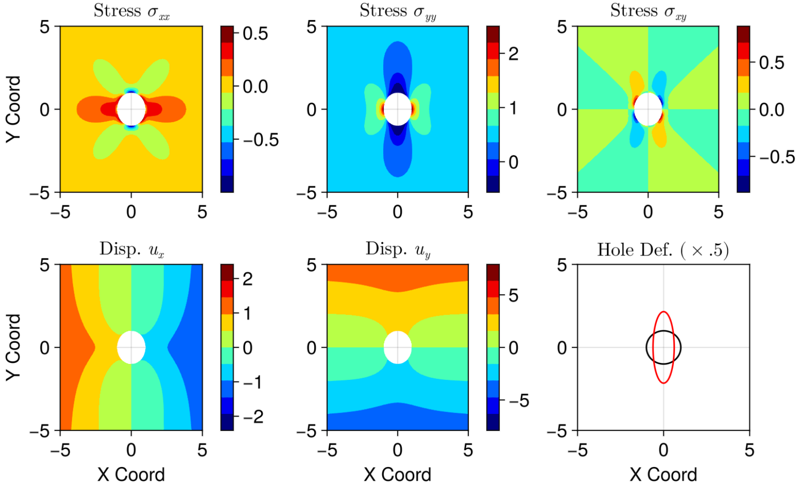
$$\sigma_{r\theta} = \sigma_0\left(\frac{a}{r}\right)^2\left(1-\frac{3}{2}\left(\frac{a}{r}\right)^2\right)\sin 2\theta$$

3.3.Plate with a Hole. Plate With a Hole Under Tension

- ... n

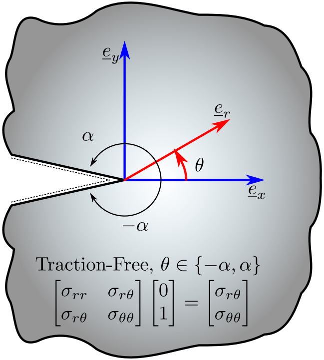
$u_r =$

$u_\theta =$



3.3.3. Notch Crack

Classical Solutions



- We seek an analytical solution for this problem setting **very close to the crack**.
- While we may intuitively expect stress to be singular at the crack tip, the strain energy has to be finite.
- Suppose $\sigma \sim \mathcal{O}(r^\lambda)$, $\varepsilon \sim \mathcal{O}(r^\lambda)$ necessarily.
  - So $\mathcal{U} = \int\int \frac{\sigma\varepsilon}{2} r dr d\theta \sim \mathcal{O}(r^{2\lambda+1})$.
  - For this to be finite, $2\lambda + 1 \geq 0 \implies \lambda \geq -\frac{1}{2}$.
- The only Airy stress functions that can show this are (refer sl. 18).

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
$r^{n+2}\cos n\theta$	$(..)r^n \cos n\theta$	$(..)r^n \sin n\theta$	$(..)r^n \cos n\theta$
$r^n \cos n\theta$	$(..)r^{n-2}\cos n\theta$	$(..)r^{n-2}\sin n\theta$	$(..)r^{n-2}\cos n\theta$
$r^{n+2}\sin n\theta$	$(..)r^n \sin n\theta$	$(..)r^n \cos n\theta$	$(..)r^n \sin n\theta$
$r^n \sin n\theta$	$(..)r^{n-2}\sin n\theta$	$(..)r^{n-2}\cos n\theta$	$(..)r^{n-2}\sin n\theta$

3.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1} \left(A_1 \cos((\lambda-1)\theta) + A_2 \cos((\lambda+1)\theta) + B_1 \sin((\lambda-1)\theta) + B_2 \sin((\lambda+1)\theta)\right)}.$$

3.3.3. Singularity Close to Notch Crack

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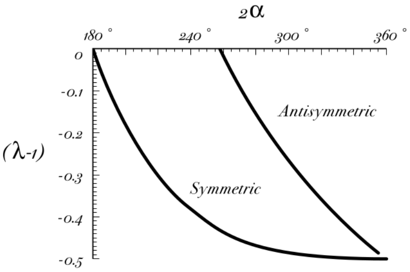


(Figure 11.7 from Barber 2022)

3.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$

- Applying the boundary conditions (along with $\alpha = \pi$), we get a nonlinear eigenvalue problem that has the following solutions:

λ	Eigenfunction
$\frac{1}{2}$	$A_2 = \frac{A_1}{3}$, $B_2 = -B_1$
1	$A_2 = -A_1$, $B_2 = 0$ $(B_1 = 0)$
$\frac{3}{2}$	$A_2 = -\frac{A_1}{5}$, $B_2 = -B_1$
$\vdots$	

- $\lambda = \frac{1}{2}$ corresponds to the near-field singular stress field, given by

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

3.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$

- Applying the boundary conditions (along with $\alpha = \pi$), we get a nonlinear eigenvalue problem that has the following solutions:

λ	Eigenfunction

Displacement Field

$$2\mu u_r = K_I\sqrt{\frac{r}{2\pi}}\left((\kappa-\frac{1}{2})\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right) - K_{II}\sqrt{\frac{r}{2\pi}}\left((\kappa-\frac{1}{2})\sin\frac{\theta}{2} - \frac{3}{2}\sin\frac{3\theta}{2}\right)$$

$$2\mu u_\theta = K_I\sqrt{\frac{r}{2\pi}}\left(-(\kappa+\frac{1}{2})\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2}\right) - K_{II}\sqrt{\frac{r}{2\pi}}\left((\kappa+\frac{1}{2})\cos\frac{\theta}{2} - \frac{3}{2}\cos\frac{3\theta}{2}\right)$$

- $\lambda = \frac{1}{2}$ corresponds to the near-field singular stress field, given by

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

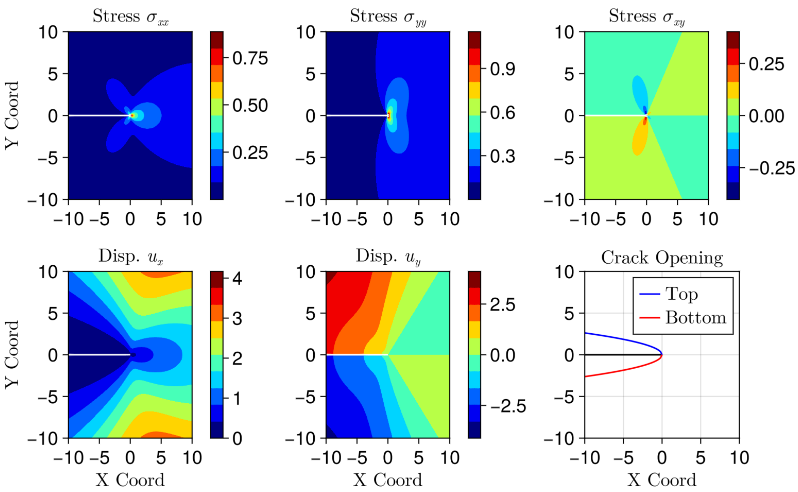
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

3.3.3. Singularity Close to Notch Crack

Classical Solutions

Mode 1 Loading (unit K_I)



3.3.3. Singularity Close to Notch Crack

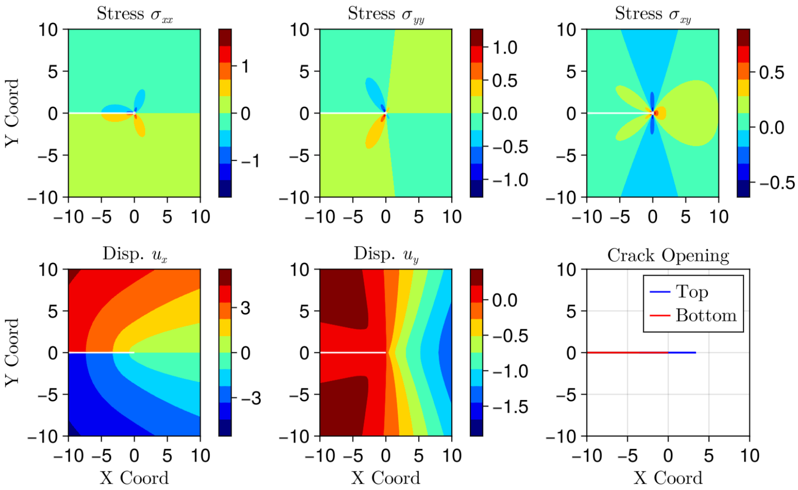
Classical Solutions

- ar

$2\mu u$

$2\mu u_\theta$

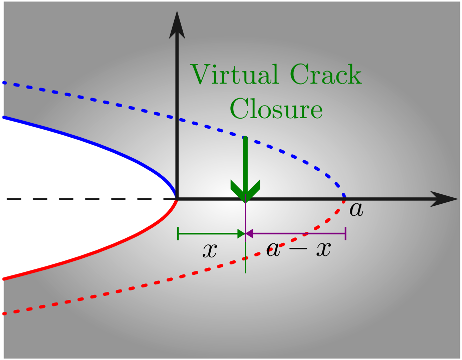
Mode 2 Loading (unit K_{II})



3.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to “close” a crack.



3.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to “close” a crack.
- We observe that (all quantities in cylindrical):

$$@\,\theta=0,\quad \underline{\underline{\sigma}}=\frac{K_I}{\sqrt{2\pi r}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad 2\mu\underline{u}=K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa-1\\ 0\end{bmatrix}-K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ \kappa-1\end{bmatrix}$$

$$@\,\theta=\pi,\quad \underline{\underline{\sigma}}=\frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix},\quad 2\mu\underline{u}=K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ -(\kappa+1)\end{bmatrix}-K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa+1\\ 0\end{bmatrix}.$$

3.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to "close" a crack.
- We observe that (all quantities in cylindrical):

$$@\,\theta=0,\quad \underline{\underline{\sigma}} = \frac{K_I}{\sqrt{2\pi r}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa-1\\ 0\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ \kappa-1\end{bmatrix}$$

$$@\,\theta=\pi,\quad \underline{\underline{\sigma}} = \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix},\quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ -(\kappa+1)\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa+1\\ 0\end{bmatrix}.$$

- For virtual crack closure, the work done can be written as,

$$\begin{aligned}W(a) &= 2\int_0^a \frac{1}{2}\left(\sigma_{\theta\theta}\Big|_{\theta=0}(-u_\theta)\Big|_{\theta=\pi} + \sigma_{r\theta}|_{\theta=0}(-u_r)|_{\theta=\pi}\right)dx\\ &= \int_0^a \frac{K_I}{\sqrt{2\pi x}}K_I\sqrt{\frac{a-x}{2\pi}}\frac{\kappa+1}{2\mu} + \frac{K_{II}}{\sqrt{2\pi r}}K_{II}\sqrt{\frac{a-x}{2\pi}}\frac{\kappa+1}{2\mu}\,dx\\ &= \frac{K_I^2+K_{II}^2}{2\pi}\frac{\kappa+1}{2\mu}\cancelto{\frac{a\pi}{2}}{\int_0^a\sqrt{\frac{a-x}{x}}\,dx} = \frac{K_I^2+K_{II}^2}{8\mu^2}(\kappa+1)^2 a = \begin{cases}\frac{K_I^2+K_{II}^2}{E}a & \text{Plane Stress}\\ \frac{K_I^2+K_{II}^2}{E}(1-\nu^2)a & \text{Plane Strain}\end{cases}.\end{aligned}$$

- The Griffith Energy Release Rate is the derivative $\lim_{a\to 0}\frac{1}{B}\frac{dW}{da}$, which evaluates as

$$G = \frac{1}{B}\begin{cases}\frac{K_I^2}{E}+\frac{K_{II}^2}{E} & \text{Plane Stress}\\ \frac{K_I^2}{E}(1-\nu^2)+\frac{K_{II}^2}{E}(1-\nu^2) & \text{Plane Strain}\end{cases}.$$

3.3.3. Stress Intensity Factor

Classical Solutions

- A crack is said to propagate when G exceeds G_{cr}.
- Therefore, under "pure" mode 1 loading, the *Critical Stress Intensity Factor* ($K_{I,cr}$) is

$$K_{I,cr} = \begin{cases} \sqrt{BG_{cr}E} & \text{Plane Stress} \\ \sqrt{\frac{BG_{cr}E}{1-\nu^2}} & \text{Plane Strain} \end{cases}.$$

- This shows that for identical conditions, the Plane Stress case (thin plates) has **higher fracture toughness** than its plane stress counterpart (long prismatic structures).
- **But how do we relate K_I, K_{II} with far-field applied stresses?**

3.3.3. Stress Intensity Factor

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- This shows that for identical conditions, the Plane Stress case (thin plates) has **higher fracture toughness** than its plane stress counterpart (long prismatic structures).
- **But how do we relate K_I, K_{II} with far-field applied stresses?** The answer is very closely tied in to the exact geometry, etc.

3.3.3. Griffith-Inglis Crack Revisited

Classical Solutions

- For the flat crack of length $2a$ (aka the Griffith-Inglis crack), the SIF is related to tensile stresses by

$$K_I = \sigma_0 \sqrt{\pi a}.$$

- Note that this is why we chose $\lambda = \frac{\pi}{2}$ in sl. 7. If we left it in, we'll have to satisfy (plane stress considered here):

$$\frac{4\lambda a}{E}\sigma_0^2 = \frac{2K_I^2}{E} = \frac{2\pi a}{E}\sigma_0^2.$$

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