Quiz 2 Solutions

1. $$\underset{\sim}{C} = \underset{\sim}{F}^T \cdot \underset{\sim}{F}$$

$$\frac{D\underset{\sim}{C}}{Dt} = \frac{D\underset{\sim}{F}^T}{Dt} \cdot \underset{\sim}{F} + \underset{\sim}{F}^T \cdot \frac{D\underset{\sim}{F}}{Dt}$$

$$= \left(\frac{D\underset{\sim}{F}}{Dt}\right)^T \cdot \underset{\sim}{F} + \quad ''$$

$$= (\underset{\sim}{L} \cdot \underset{\sim}{F})^T \cdot \underset{\sim}{F} + \underset{\sim}{F}^T \cdot (\underset{\sim}{L} \cdot \underset{\sim}{F})$$

$$= \underset{\sim}{F}^T \cdot (\underset{\sim}{L}^T + \underset{\sim}{L}) \cdot \underset{\sim}{F}$$

$$= \underset{\sim}{F}^T \cdot (2\underset{\sim}{D}) \cdot \underset{\sim}{F} \quad \blacksquare$$

2. We know:

$\dfrac{D\underset{\sim}{v}}{Dt} = \dfrac{\partial \underset{\sim}{v}}{\partial t} + (\text{grad}\, \underset{\sim}{v}) \cdot \underset{\sim}{v}$, so all we need show is:

$$(\text{grad}\, \underset{\sim}{v}) \cdot \underset{\sim}{v} = \frac{\text{grad}\,(\underset{\sim}{v} \cdot \underset{\sim}{v})}{2} + 2\underset{\sim}{W} \cdot \underset{\sim}{v}$$

$\Rightarrow$ To show:

$$\frac{\partial v_i}{\partial x_k} v_k = \frac{1}{2} \frac{\partial}{\partial x_i}(v_l v_l) + 2 W_{ij} v_j$$

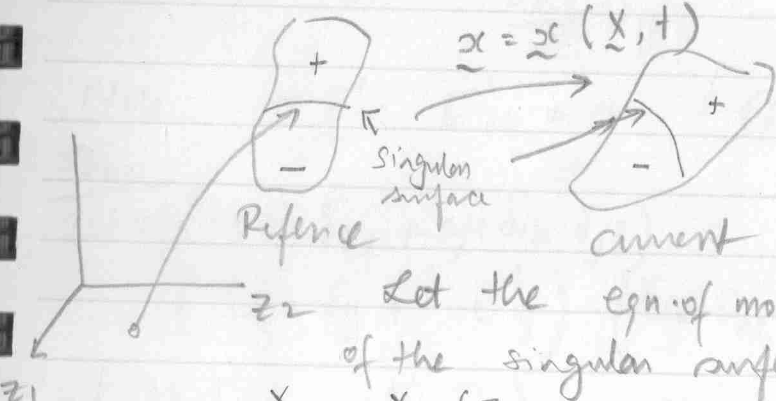
$$\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \longrightarrow \frac{L_{ij} - L_{ji}}{2}$$

To show:

$$\frac{\partial v_i}{\partial x_k} v_k = \frac{1}{\cancel{2}}\left(\cancel{\frac{\partial v_l}{\partial x_i} v_l}\right) \cdot 2 +$$

$$\left(\frac{\partial v_i}{\partial x_j} - \cancel{\frac{\partial v_j}{\partial x_i}}\right) v_j \quad \text{— true!!}$$

3. Recall that particle motion is diffent from that of the singular surface



Let the eqn. of motion of the singular surface be $\underset{\sim}{X} = \underset{\sim}{X}(z_1, z_2, t)$ in the reference config.

Then, in the current config, the sing. surface moves as: $\underset{\sim}{x} = \underset{\sim}{x}\left(\underset{\sim}{X}(z_1, z_2, t), t\right)$.

We must have

$$\left.\frac{\partial \underset{\sim}{x}}{\partial \zeta_1}\right|_+ = \left.\frac{\partial \underset{\sim}{x}}{\partial \zeta_1}\right|_- \quad \& \quad \left.\frac{\partial \underset{\sim}{x}}{\partial \zeta_2}\right|_+ = \left.\frac{\partial \underset{\sim}{x}}{\partial \zeta_2}\right|_- \quad \text{across the singular surface}$$

$$\left.\frac{\partial \underset{\sim}{x}}{\partial \zeta_1}\right|_+ = \left.\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}}\right|_+ \cdot \frac{\partial \underset{\sim}{X}}{\partial \zeta_1} = \left.\underset{\sim}{F}\right|_+ \cdot \underset{\sim}{l}_1$$

a vector tangent to the singular surface in the current config.

$\parallel^{rly}$ $\left.\frac{\partial \underset{\sim}{x}}{\partial \zeta_1}\right|_- = \left.\underset{\sim}{F}\right|_- \cdot \underset{\sim}{l}_1$.

and $\parallel^{rly}$, $\left.\underset{\sim}{F}\right|_+ \cdot \underset{\sim}{l}_2 = \left.\underset{\sim}{F}\right|_- \cdot \underset{\sim}{l}_2$.

Combining,

$$\left.\underset{\sim}{F}\right|_+ \cdot (\alpha \underset{\sim}{l}_1 + \beta \underset{\sim}{l}_2) = \left.\underset{\sim}{F}\right|_- \cdot (\alpha \underset{\sim}{l}_1 + \beta \underset{\sim}{l}_2)$$

call this $\underset{\sim}{l}$ to get the result!

4. Given: $\mu > 0$, $\lambda + \frac{2\mu}{3} > 0$.

To show:

$$2W = (\lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}) E_{ij} E_{kl} > 0$$

Break $\underset{\sim}{E}$ into hydrostatic & deviatoric parts.

$$\underset{\sim}{E} = \underset{\sim}{E}' + \frac{tr(\underset{\sim}{E})}{3} \underset{\sim}{I}, \text{ or}$$

$$E_{ij} = E_{ij}' + \frac{E_{kk}}{3} \delta_{ij}.$$

Note: $tr\, \underset{\sim}{E}' = E_{kk}' = 0$. — ★

Then:

$$2W = (\lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}) \cdot \left(\frac{E_{mm}}{3} \delta_{ij} + E_{ij}' \right) \cdot \left(\frac{E_{nn}}{3} \delta_{kl} + E_{kl}' \right)$$

$$= (\lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}) \cdot \left(\frac{E_{mm} E_{nn}}{9} \delta_{ij} \delta_{kl} + \frac{E_{mm}}{3} \delta_{ij} E_{kl}' + \frac{E_{nn}}{3} E_{ij}' \delta_{kl} + E_{ij}' E_{kl}' \right)$$

recall (*)

$$\overset{\downarrow}{=} \lambda \frac{E_{mm} E_{nn}}{\cancel{9}} \cdot \cancel{\delta_{ii}}^{\,3} \cancel{\delta_{kk}}^{\,3} +$$

$$\lambda \frac{E_{mm}}{\cancel{3}} \cancel{\delta_{ii}}^{\,3} \cancelto{0}{E'_{kk}} + \quad \text{— by } \star$$

$$\lambda \frac{E_{nn}}{3} \cancelto{0\,(\star)}{E'_{ii}} \cdot \delta_{kk} + \lambda \cancelto{0}{E'_{ii}} \cancelto{0}{E'_{kk}} +$$

$$2\mu \frac{E_{mm} E_{nn}}{\cancel{9}\;3} \cdot \cancel{\delta_{ll}}^{\,3} +$$

$$2\mu \frac{E_{mm}}{3} \cdot \cancelto{0}{E'_{kk}} + \text{(another 0 term)}$$

$$+ 2\mu E'_{jk} E'_{jk}$$

$$= \left(\lambda + \frac{2\mu}{3}\right)\left(E_{11} + E_{22} + E_{33}\right)^2 +$$

$$2\mu\left(E_{11}^2 + E_{22}^2 + E_{33}^2 + 2E_{12}^2 + 2E_{13}^2 + 2E_{23}^2\right).$$

sum of squares, $\therefore \geq 0$

If $\lambda + \frac{2\mu}{3}, \mu > 0$, the alone is surely ≥ 0.