Ans 1)

L.H.S.

$\nabla \cdot \phi \underline{V}$

$$\Rightarrow \frac{\partial}{\partial x_i}(\phi v_j e_j) e_i = \frac{\partial}{\partial x_i}(\phi v_j)\delta_{ij}$$

$$= \phi \frac{\partial}{\partial x_i} v_j \delta_{ij} + v_j \frac{\partial \phi}{\partial x_i}\delta_{ij}$$

$$= \underbrace{\phi \frac{\partial v_i}{\partial x_i}}_{\text{div } V} + v_i \underbrace{\frac{\partial \phi}{\partial x_i}}_{\text{grad } \phi}$$

$$= \phi(\nabla \cdot \underline{v}) + \underline{v} \cdot \nabla \phi$$

$= \underline{R.H.S}$

2) If $\underset{\sim}{Q}$ is an orthonormal.

$$\underset{\sim}{Q}\,\underset{\sim}{Q}^T = \underset{\sim}{I} \quad \text{or} \quad \underset{\sim}{Q}^T = \underset{\sim}{Q}^{-1}$$

Given

$$\underset{\sim}{Q} \cdot \underline{e} = \underline{e}$$

i.e, $Q_{ij} \cdot e_j = e_i$

$Q_{ji} Q_{ij} e_j = Q_{ji} e_i$ (Multiply by transpose)

$\cancel{E e_j}\ I e_j = Q_{ji} e_i$

$\Rightarrow \quad \underline{e} = \underset{\sim}{Q}^T \underline{e}$

$\underline{e}$ is parallel to axis of rotation.
Since rotation of any vector around rotation axis result in same vector.

3).

Given

$$\underline{v}(\underline{x}, t) = v_1(x_2)\underline{e}_1$$

Show

$$\frac{Dv_i}{Dt}(x_j, t) = \frac{\partial v_i}{\partial t}(x_j, t)$$

~~In gen~~

$$\because \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t}(x_j, t) + v_j \frac{\partial v_i}{\partial x_j}(x_j, t)$$

check the term $v_j \frac{\partial v_i}{\partial x_j}(x_j, t)$

for $i = 1$,

$$\Rightarrow v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \qquad \left[\because \begin{array}{l} v_3 = v_2 = 0 \\ v_1 = v_1(x_2) \end{array}\right]$$

$$= v_1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0$$

llly for $i = 2, 3$.

$$v_j \frac{\partial v_i}{\partial x_j}(x_j, t) \equiv 0$$

$$\Rightarrow \frac{Dv_i}{Dt}(x_j, t) = \frac{\partial v_i}{\partial t}(x_j, t)$$

4)

$$x_1 = X_1 e^{t^2}$$

$$x_2 = X_2 e^{t}$$

$$x_3 = X_3 .$$

$$\left[\begin{array}{l} X_R \rightarrow \text{Reference co-ordinate} \\ x_i \rightarrow \text{spatial co-ordinate} \end{array}\right.$$

(~~spatial~~)

$$\Rightarrow.\quad x_i = x_i(X_R, t).$$

a)

$\because$ velocity $\rightarrow \underline{v}(\underline{X}, t)$

label of particle.

label does not change with time.

$$\underline{v}(\underline{X}, t) = \frac{\partial u}{\partial t}(\underline{X}, t) = \frac{\partial x}{\partial t}(\underline{X}, t) - \cancel{\frac{\partial X}{\partial t}}$$

$$\therefore\quad v_i(X_R, t) = \frac{\partial x_i}{\partial t}(X_R, t)$$

$$v_1 = \frac{\partial}{\partial t}(X_1 e^{t^2})$$

$$= 2t X_1 e^{t^2}$$

$$v_2 = \frac{\partial}{\partial t}(X_2 e^{t}) = X_2 e^{t}$$

$$v_3 = \frac{\partial}{\partial t}(X_3) = 0.$$

~~In~~ velocity components in spatial form

$$\boxed{\begin{aligned} v_1 &= 2t X_1 e^{t^2} = 2t \frac{x_1}{e^{t^2}} \times e^{t^2} = 2t x_1 \\ v_2 &= x_2 \\ v_3 &= 0 \end{aligned}}$$

b) $F_{iR}(X_s, t) = \dfrac{\partial x_i}{\partial X_R}(X_s, t)$

Deformation gradient.

$$\underset{\sim}{F} = \begin{pmatrix} ct^2 & 0 & 0 \\ 0 & ct & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) Streamlins.

$$dx_i = v_i(\underbrace{x_j}_{(X_R, t)}, t)\,dt$$

$\Rightarrow$ $\dfrac{dx_1}{dt} = v_1(x_j, t) = 2tx_1$

$\therefore \dfrac{dx_1}{x_1} = 2t\,dt$

$\Rightarrow$ $\ln x_1 = t^2 + \ln c_1$

~~or~~ $\therefore x_1 = X_1 e^{t^2}$

III^{ly}. $v_2 = \dfrac{dx_2}{dt}$

$\Rightarrow x_2 = X_2 e^t$

and $x_3 = X_3$ $\left[\because \dfrac{dx_3}{dt} = 0\right]$