AS3020: Aerospace Structures

Module 7: Elastic Stability

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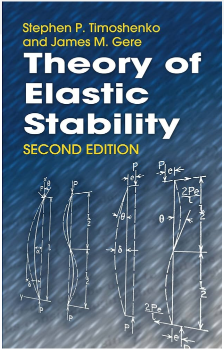
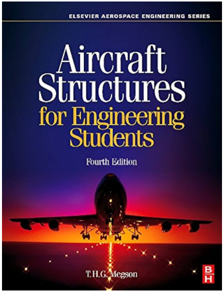


Chapter 9 in Timoshenko and Gere [1]. Good reference in general.

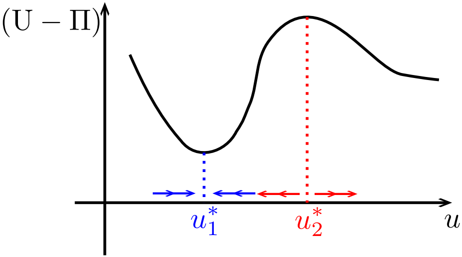


Chapters 7-9 in Megson [2]

1. Introduction

- The key intuition for elastic stability comes from analyzing the quantity $\text{U} - \Pi$ around its extrema.
  - Maxima in $\text{U} - \Pi$ correspond to **unstable solutions**;
  - Minima in $\text{U} - \Pi$ correspond to **stable solutions**.

- Investigating the second derivative ("Hessian") of the quantity allows for efficient classification;
- In 1D ($u \in \mathbb{R}$), the sign of $\frac{\partial(U-\Pi)}{\partial u^2}$ is sufficient for this;
- In higher dimensions, we obtain an eigenvalue problem.



1.1. Column Buckling

Introduction

- We already derived the governing equations for a beam under uniform axial stress $\frac{P}{A}$. When this is compressive, the governing equation can be written as

$$\boxed{EIv'''' + Pv'' = 0}.$$

- We showed in class that this can be used to recover Euler's Critical Loads,

$$P_n = n^2 \frac{\pi^2 EI}{\ell^2}, \quad v(X_1) = V \sin\left(n \frac{\pi X_1}{\ell}\right).$$

- We solved a **Sturm-Liouville Problem** to obtain these.

2. Plates

- We will now derive the governing equations of thin plates with the **Kirchhoff-Love Plate Theory**, which is the simplest generalization of **Euler-Bernoulli Beam Theory**.

Euler-Bernoulli Beams

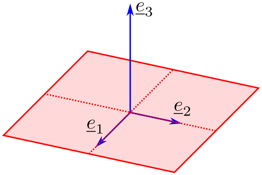
- Sections *move* rigidly;
- Plane sections remain perpendicular to the centroidal axis.

KL Plates

- Line elements along thickness *move* rigidly;
- Line elements remain perpendicular to the mid-plane.

- The above assumptions lead to the zeroing out of certain strains in the formulation that leads to a simplified kinematic description. For plates this is,

$$u_1 = -X_3 w_{,1}$$
$$u_2 = -X_3 w_{,2}$$
$$u_3 = w,$$



where w is a function of X_1, X_2.

2. Plates

Variational Approach for Derivation

- Using the kinematic description we write out the strains (linear and nonlinear) as

$$\begin{aligned} E_{11} &= u_{1,1} + \frac{1}{2}(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2) \\ &= -X_3 w_{,11} + \frac{1}{2}\left(X_3^2 w_{,11}^2 + X_3^2 w_{,12}^2 + w_{,1}^2\right) \\ E_{22} &= u_{2,2} + \frac{1}{2}(u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2) \\ &= -X_3 w_{,22} + \frac{1}{2}\left(X_3^2 w_{,12}^2 + X_3^2 w_{,22}^2 + w_{,2}^2\right) \\ \gamma_{12} &= u_{1,2} + u_{2,1} + (u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}) \\ &= -2X_3 w_{,12} + \left(X_3^2 w_{,11} w_{,12} + X_3^2 w_{,12} w_{,22} + w_{,1} w_{,2}\right), \end{aligned}$$

  where the nonlinear (quadratic) terms are highlighted in blue.
- Just like in the case of the beam, we **retain only the quadratic terms** for the internal energy.

2. Plates

Bending Strain Energy under Plane Stress

- We have to first write down the stresses before the energy can be expressed. Under **plane stress** assumptions we get,

$$\begin{bmatrix} E_{11} \\ E_{12} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\implies \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{12} \\ \gamma_{12} \end{bmatrix}$$

- The bending energy (up to $\mathcal{O}(v^2)$) is

$$\mathrm{U}_b = \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{1}{2} \left(\sigma_{11} E_{11} + \sigma_{22} E_{22} + \sigma_{12} \gamma_{12}\right) dX_3$$

$$= \frac{1}{2} \underbrace{\frac{Et^3}{12(1-\nu^2)}}_{D} \left(w_{,11}^2 + w_{,22}^2 + 2(1-\nu) w_{,12}^2 + 2 w_{,11} w_{,22}\right)$$

2. Plates

Work Done by Axial Stresses

- We consider axial loads P_1, P_2, P_{12} as shown. The work done by these is contributed by the quadratic strains

$$\begin{aligned}\mathrm{U}_c =&\frac{P_1}{24}\left(t^2(w_{,11}^2+w_{,12}^2)+12w_{,1}^2\right)+\frac{P_2}{24}\left(t^2(w_{,12}^2+w_{,22}^2)+12w_{,2}^2\right)\\ &+\frac{P_{12}}{12}\left(t^2 w_{,12}(w_{,11}+w_{,22})+12w_{,1}w_{,2}\right).\end{aligned}$$

- We will ignore the t^2 terms in the above to give,

$$\mathrm{U}_c = \frac{1}{2}\left(P_1 w_{,1}^2 + P_2 w_{,2}^2 + 2P_{12} w_{,1} w_{,2}\right).$$

Other Loads

When there is also a distributed transverse load f acting, the load work done is given by

$$\Pi = \int_{\mathcal{D}} f w dX_1 dX_2$$

2.1. Principle of Virtual Work

Plates

- The total work done by the system is written as,

$$\mathcal{L} = U_b + U_c - \Pi = \frac{D}{2}\left(w_{,11}^2 + w_{,22}^2 + 2(1-\nu)w_{,12}^2 + 2w_{,11}w_{,22}\right) + \frac{1}{2}\left(P_1 w_{,1}^2 + P_2 w_{,2}^2 + 2P_{12}w_{,1}w_{,2}\right) - fw$$

- The Euler-Lagrange Equations are written as:

$$\frac{d^2}{dX_1^2}\frac{\partial \mathcal{L}}{\partial w_{,11}} + \frac{d^2}{dX_2^2}\frac{\partial \mathcal{L}}{\partial w_{,22}} + \frac{d^2}{dX_1 dX_2}\frac{\partial \mathcal{L}}{\partial w_{,12}} - \frac{d}{dX_1}\frac{\partial \mathcal{L}}{\partial w_{,1}} - \frac{d}{dX_2}\frac{\partial \mathcal{L}}{\partial w_{,2}} + \frac{\partial \mathcal{L}}{\partial w} = 0.$$

- This leads to,

$$\boxed{\underbrace{\frac{Et^3}{12(1-\nu^2)}}_{D}\left(w_{,1111} + w_{,2222} + 2w_{,1122}\right) - \left(P_1 w_{,11} + P_2 w_{,22} + 2P_{12}w_{,12}\right) - f = 0}$$

2.1. Principle of Virtual Work

Plates

- The general plate equation can be interpreted in two ways just as before.

$$D\left(w_{,1111} + w_{,2222} + 2w_{,1122}\right) - \left(P_1 w_{,11} + P_2 w_{,22} + 2P_{12} w_{,12}\right) - f = 0$$

Membranes

- When the quantity D is very small, the system is approximated well as

$$(P_1 w_{,11} + P_2 w_{,22} + 2P_{12} w_{,12}) + f = 0$$

- For the isotropic case shear-free case ($P_1 = P_2 = P$, $P_{12} = 0$) we have,

$$P\nabla^2 w + f = 0$$

Plate Buckling

- For the $f = 0$ case undergoing compressive loading ($P_1 \to -P_1$, $P_2 \to -P_2$ $P_{12} \to -P_{12}$), the governing equation is

$$D\nabla^4 w + (P_1 w_{,11} + P_2 w_{,22} + 2P_{12} w_{,12}) = 0.$$

- This is a slightly more complicated Sturm-Liouville type problem than the one encountered with column buckling.

2.1. Principle of Virtual Work

Plates

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$$D\left(w_{,1111} + w_{,2222} + 2w_{,1122}\right) - \left(P_1 w_{,11} + P_2 w_{,22} + 2P_{12} w_{,12}\right) - f = 0$$

Note that it is also possible to express the governing equations in terms of moments and section-normal shear forces (like we did with E.B.T.). But we will not pursue this here.

Mem

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Buckling

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2.2. Classical Solutions

Plates

- One of the simplest case to consider is a plate with **simply supported edges** ($w = 0$ on $\partial\mathcal{D}$). The governing equations (for zero loading) is

$$D\nabla^4 w - f = 0, \quad (X_1, X_2) \in \mathcal{D}, \qquad w = 0, \quad (X_1, X_2) \in \partial\mathcal{D}.$$

($\partial\mathcal{D}$ is the closure of the open set $\mathcal{D}$).

- For a rectangular plate (sides $a_1 \times a_2$ such that $X_1 \in [0, a_1]$, $X_2 \in [0, a_2]$), a popular approach is to use a **Fourier Decomposition** of the form

$$w(X_1, X_2) = \sum_{n_1, n_2} A_{n_1 n_2} \sin\left(n_1 \frac{\pi}{a_1} X_1\right) \sin\left(n_2 \frac{\pi}{a_2} X_2\right).$$

- Note that the coefficients $A_{n_1 n_2}$ may be retrieved by the integral,

$$A_{n_1 n_2} = \frac{4}{a_1 a_2} \int_0^{a_1} \int_0^{a_2} w(X_1, X_2) \sin\left(n_1 \frac{\pi}{a_1} X_1\right) \sin\left(n_2 \frac{\pi}{a_2} X_2\right) dX_1 dX_2.$$

2.2. Classical Solutions

Plates

- Using this ansatz, the equilibrium equation now reads,

$$\underbrace{\sum_{n_1,n_2} D\pi^4\left(\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}\right)^2 A_{n_1n_2}\sin\left(n_1\frac{\pi}{a_1}X_1\right)\sin\left(n_2\frac{\pi}{a_2}X_2\right)}_{D\nabla^4 w} = f.$$

- Expressing the Fourier coefficients of the load f as $F_{n_1n_2}$ we can write,

$$A_{n_1n_2} = \frac{1}{D\pi^4}\left(\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}\right)^{-2} F_{n_1n_2}.$$

- This means that excitation along the function $\sin(n_1\frac{\pi}{a_1}X_1)\sin(n_2\frac{\pi}{a_2}X_2)$ will result in **deformation in the same shape**.
- For an arbitrary deformation, this leads to a **series representation** of the deformation shape.

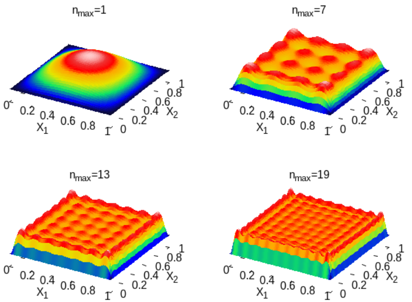
2.2. Classical Solutions: Uniform Loading

Plates

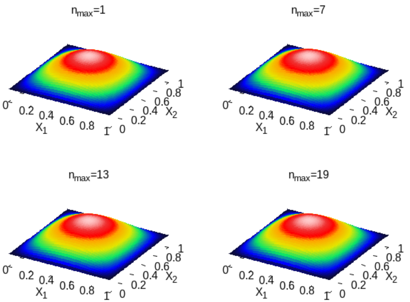
- For the case of uniform loading ($f(X_1, X_2) = 1$), it can be shown that

$$F_{n_1 n_2} = \begin{cases} \frac{16}{\pi^2 n_1 n_2} & n_1, n_2 \text{ both odd} \\ 0 & \text{otherwise} \end{cases}.$$

Force Decomposition



Deflection Decomposition



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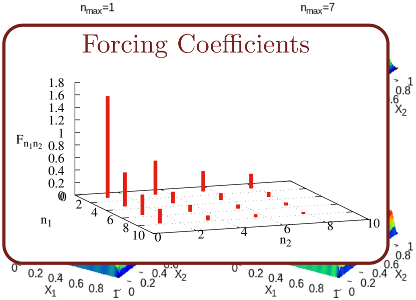

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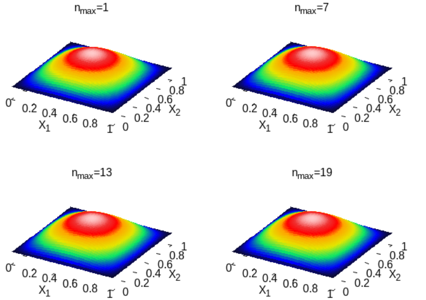

2.3. Buckling of Plates

- We will consider buckling of plates also under the same conditions (simply supported ends). Let us set $P_{12} = 0$ here (since it introduces cosine terms also).
- The governing equations become

$$D\nabla^4 w - (P_1 w_{,11} + P_2 w_{,22}) =$$
$$\sum_{n_1, n_2} \left(D\pi^4 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2 - \pi^2 \left(\frac{n_1^2}{a_1^2} P_1 + \frac{n_2^2}{a_2^2} P_2 \right) \right) A_{n_1 n_2} \sin\left(n_1 \frac{\pi}{a_1} X_1 \right) \sin\left(n_2 \frac{\pi}{a_2} X_2 \right) = 0.$$

Non-trivial $A_{n_1 n_2}$ for $P_2 = 0$

$$P_1 = \pi^2 D \frac{a_1^2}{n_1^2} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2.$$

The critical load (lowest P_1) corresponds to $n_2 = 1$:

$$P_1^* = \frac{\pi^2 D}{a_2^2} \left(\frac{n_1 a_2}{a_1} + \frac{a_1}{n_1 a_2} \right)^2.$$

Non-trivial $A_{n_1 n_2}$ for $P_1 = 0$

$$P_2 = \pi^2 D \frac{a_2^2}{n_2^2} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2.$$

The critical load (lowest P_2) corresponds to $n_1 = 1$:

$$P_2^* = \frac{\pi^2 D}{a_1^2} \left(\frac{n_2 a_1}{a_2} + \frac{a_2}{n_2 a_1} \right)^2.$$

2.3. Buckling of Plates

Plates

- We will co ditions (simply su ntroduces cosine term
- The govern

$D\nabla^4 w - (P_1$

$\sum_{n_1, n_2} \left(D\pi^4 \right.$ $\sin\left(n_2 \frac{\pi}{a_2} X_2\right) = 0.$

Denoting $k = \left(\frac{n_1 a_1}{a_2} + \frac{a_2}{n_1 a_1}\right)^2$ s.t. $P_1^* = k\frac{\pi^2 D}{a_2^2}$

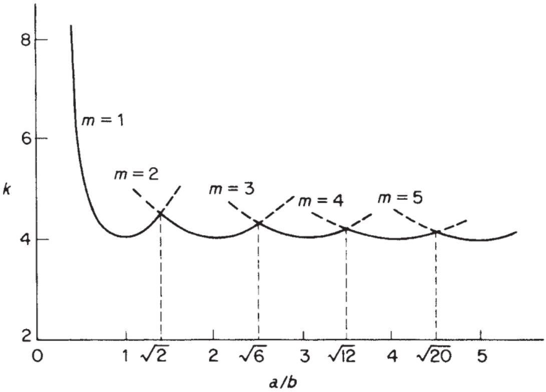

Figure from [2], $a = a_1$, $b = a_2$.

Non-trivia **for $P_1 = 0$**

$P_1 = \pi^2 D$ $+ \frac{n_2^2}{a_2^2}\Big)^2.$

The critical load P_2) corresponds to $n_2 = 1$:

$$P_1^* = \frac{\pi^2 D}{a_2^2}\left(\frac{n_1 a_2}{a_1} + \frac{a_1}{n_1 a_2}\right)^2.$$

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2.3. Buckling of Plates

Plates

- We will co[…] (simply su[…] cosine term[…]
- The govern[…]

$D\nabla^4 w - (P_1 […]$

$\sum_{n_1,n_2} \left(D\pi^4 \left([…] \sin\left(n_2 \frac{\pi}{a_2} X_2\right) = 0.$

Denoting $k = \left(\frac{n_1 a_1}{a_2} + \frac{[…]}{n[…]}\right.$

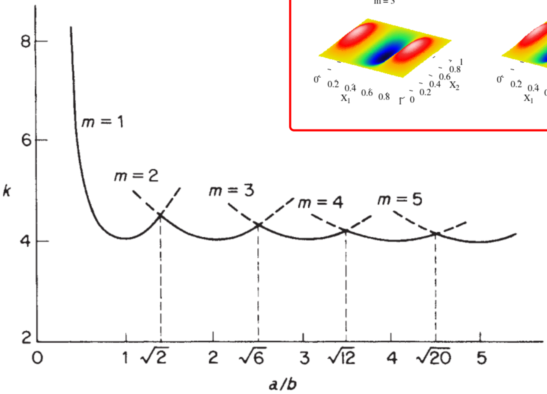
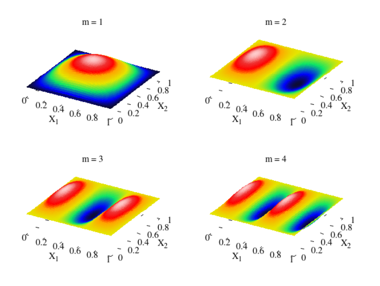


Figure from [2], $a = a_1$, $b = a_2$.



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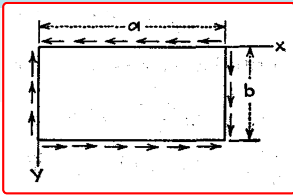
2.3. Buckling of Plates: Shear Buckling

Plates



- Under pure shear, the governing equations is

$$D\nabla^4 w - P_{12} 2 w_{,12} = 0.$$

- Using the same ansatz (simply supported boundaries) we have,

$$\sum_{n_1,n_2} \left[D\pi^4 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2 A_{n_1 n_2} \mathcal{S}_{n_1} \mathcal{S}_{n_2} - P_{12} 2\pi^2 \frac{n_1 n_2}{a_1 a_2} \mathcal{C}_{n_1} \mathcal{C}_{n_2} \right] = 0.$$

- Note that $\int_0^a \mathcal{S}_n S_m dx = \frac{a}{2}\delta_{nm}$ and

$$\int_0^a \mathcal{S}_n \mathcal{C}_m dx = \begin{cases} 0 & n \pm m \text{ is even,} \\ \frac{2a}{\pi} \frac{n}{n^2 - m^2} & n \pm m \text{ is odd} \end{cases}.$$

- Multiplying the above equation by $\mathcal{S}_{m_1} \mathcal{S}_{m_2}$ and integrating over $(0, a_1) \times (0, a_2)$ we get

$$D \frac{\pi^4}{a_1^2 a_2^2} \left(\frac{m_1^2 a_2}{a_1} + \frac{m_2^2 a_1}{a_2} \right)^2 A_{m_1 m_2} - P_{12} \frac{32}{a_1 a_2} \sum_{n_1 n_2} \frac{n_1 n_2 m_1 m_2}{(n_1^2 - m_1^2)(n_2^2 - m_2^2)} A_{n_1 n_2} = 0.$$

2.3. Buckling of Plates: Shear Buckling

Plates

- We can move around the terms a little bit and setting the aspect ratio $\beta = \frac{a_2}{a_1}$ we get

$$\underbrace{\frac{\pi^4 D}{32 a_2^2}}_{\alpha} \left(m_1^2\beta^2 + m_2^2\right)^2 A_{m_1 m_2} - P_{12}\beta \sum_{n_1 n_2} \frac{n_1 n_2 m_1 m_2}{(n_1^2 - m_1^2)(n_2^2 - m_2^2)} A_{n_1 n_2} = 0.$$

- If we truncate the n_1 and n_2 to finite N_1 and N_2, this represents a **Generalized Eigenvalue Problem** (GEVP). Restricting ourselves to $N_1, N_2 = 2$ we have,

$$\left(\alpha \begin{bmatrix} (\beta^2+1)^2 & 0 & 0 & 0 \\ 0 & (\beta^2+4)^2 & 0 & 0 \\ 0 & 0 & (4\beta^2+1)^2 & 0 \\ 0 & 0 & 0 & (4\beta^2+4)^2 \end{bmatrix} - P_{12}\frac{4\beta}{9} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- This is quite convenient since we can analytically estimate its eigen solutions. (I use maxima)

2.3. Buckling of Plates: Shear Buckling

- The eigenpairs are evaluated as

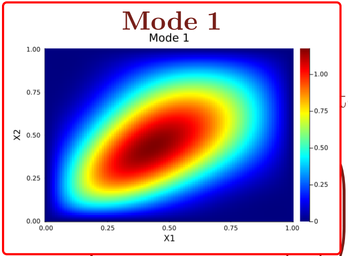
$$\left(\pm 9\alpha\frac{(\beta^2+1)^2}{\beta}, \begin{bmatrix}1\\0\\0\\\frac{1}{4}\end{bmatrix}\right), \qquad \left(\pm 9\alpha\frac{(\beta^2+4)(4\beta^2+1)}{4\beta}, \begin{bmatrix}0\\1\\\pm\frac{\beta^2+4}{4\beta^2+1}\\0\end{bmatrix}\right)$$

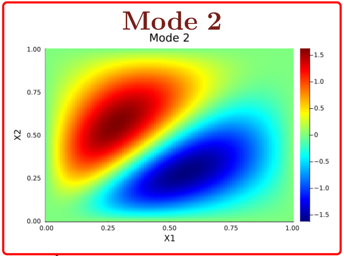
- Substituting for $\alpha = \frac{\pi^4 D}{32 a_2^2}$ we have,

$$P_{12}^* = \pm\frac{\pi^2 D}{a_2^2}\frac{9\pi^2}{32}\frac{(\beta^2+1)^2}{\beta}, \quad \pm\frac{\pi^2 D}{a_2^2}\frac{9\pi^2}{128}\frac{(\beta^2+4)(4\beta^2+1)}{\beta}$$

2.3. Buckling of Plates: Shear Buckling

Plates





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$$\ldots, \quad \left(\ldots \begin{bmatrix} 0 \\ 1 \\ \pm\frac{\beta^2+4}{4\beta^2+1} \\ 0 \end{bmatrix} \right)$$

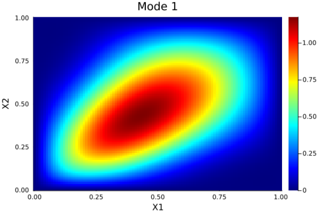
- Substituting for $\alpha = \frac{\pi^4 D}{32 a_2^2}$ we have,

$$P_{12}^* = \pm \frac{\pi^2 D}{a_2^2} \frac{9\pi^2}{32} \frac{(\beta^2+1)^2}{\beta}, \quad \pm \frac{\pi^2 D}{a_2^2} \frac{9\pi^2}{128} \frac{(\beta^2+4)(4\beta^2+1)}{\beta}$$

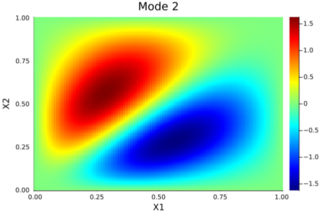
2.3. Buckling of Plates: Shear Buckling

Plates

Mode 1



Mode 2



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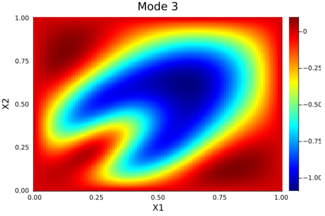
$$\begin{bmatrix} 0 \\ 1 \\ \pm\frac{\beta^2+4}{4\beta^2+1} \\ 0 \end{bmatrix}$$

- Substitutin

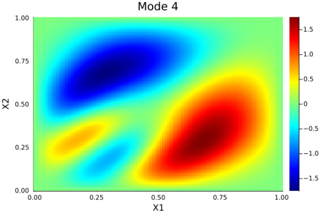
Mode 3



P^*_{12}

$\frac{\tau^2 D}{a_2^2}$

Mode 4

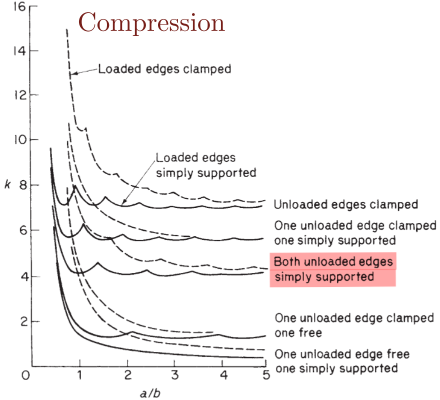


2.3. Buckling of Plates

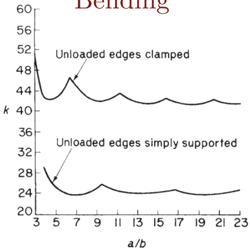
Plates

- In general, the boundary conditions as well as loads may be completely different, warranting different analysis to be done to estimate the buckling coefficient k.

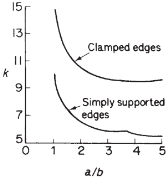
Compression



Bending



Shear



NOTICE: k approx. constant for $a/b > 3$.

Figures from [2].

2.3. Buckling of Plates

Plates

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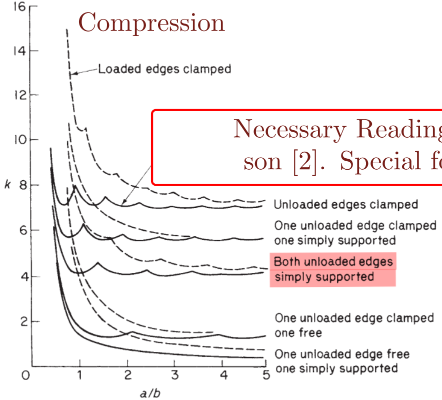
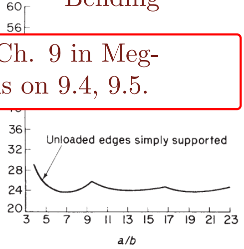
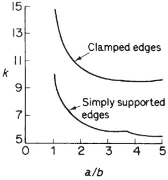
Compression



Bending



Shear



Necessary Reading: Ch. 9 in Megson [2]. Special focus on 9.4, 9.5.

NOTICE: k approx. constant for $a/b > 3$.

Figures from [2].

References I

[1] S. P. Timoshenko and J. M. Gere. Theory of Elastic Stability, Courier Corporation, June 2009. ISBN: 978-0-486-47207-2 (cit. on p. 2).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on pp. 2, 16–18, 24, 25).