AS3020: Aerospace Structures

Module 7: Elastic Stability

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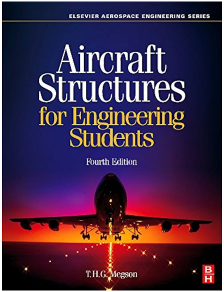
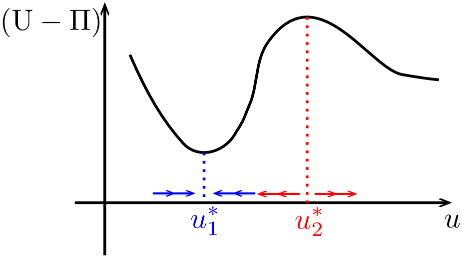


Chapters 7-9 in Megson [1]

1. Introduction

- The key intuition for elastic stability comes from analyzing the quantity $\mathrm{U} - \Pi$ around its extrema.
  - Maxima in $\mathrm{U} - \Pi$ correspond to **unstable solutions**;
  - Minima in $\mathrm{U} - \Pi$ correspond to **stable solutions**.

- Investigating the second derivative ("Hessian") of the quantity allows for efficient classification;
- In 1D ($u \in \mathbb{R}$), the sign of $\frac{\partial(U - \Pi)}{\partial u^2}$ is sufficient for this;
- In higher dimensions, we obtain an eigenvalue problem.



1.1. Column Buckling

Introduction

- We already derived the governing equations for a beam under uniform axial stress $\frac{P}{A}$. When this is compressive, the governing equation can be written as

$$\boxed{EIv'''' + Pv'' = 0}.$$

- We showed in class that this can be used to recover Euler's Critical Loads,

$$P_n = n^2 \frac{\pi^2 EI}{\ell^2}, \quad v(X_1) = V \sin\left(n \frac{\pi X_1}{\ell}\right).$$

- We solved a **Sturm-Liouville Problem** to obtain these.

2. Plates

- We will now derive the governing equations of thin plates with the **Kirchhoff-Love Plate Theory**, which is the simplest generalization of **Euler-Bernoulli Beam Theory**.

Euler-Bernoulli Beams

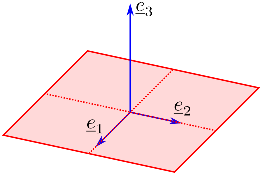
- Sections *move* rigidly;
- Plane sections remain perpendicular to the centroidal axis.

KL Plates

- Line elements along thickness *move* rigidly;
- Line elements remain perpendicular to the mid-plane.

- The above assumptions lead to the zeroing out of certain strains in the formulation that leads to a simplified kinematic description. For plates this is,

$$u_1 = -X_3 w_{,1}$$
$$u_2 = -X_3 w_{,2}$$
$$u_3 = w,$$



where w is a function of X_1, X_2.

2. Plates

Variational Approach for Derivation

- Using the kinematic description we write out the strains (linear and nonlinear) as

$$
\begin{aligned}
E_{11} &= u_{1,1} + \frac{1}{2}(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2) \\
&= -X_3 w_{,11} + \frac{1}{2}\left(X_3^2 w_{,11}^2 + X_3^2 w_{,12}^2 + w_{,1}^2\right) \\
E_{22} &= u_{2,2} + \frac{1}{2}(u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2) \\
&= -X_3 w_{,22} + \frac{1}{2}\left(X_3^2 w_{,12}^2 + X_3^2 w_{,22}^2 + w_{,2}^2\right) \\
\gamma_{12} &= u_{1,2} + u_{2,1} + (u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}) \\
&= -2X_3 w_{,12} + \left(X_3^2 w_{,11}w_{,12} + X_3^2 w_{,12}w_{,22} + w_{,1}w_{,2}\right),
\end{aligned}
$$

  where the nonlinear (quadratic) terms are highlighted in blue.
- Just like in the case of the beam, we **retain only the quadratic terms** for the internal energy.

2. Plates

Bending Strain Energy under Plane Stress

- We have to first write down the stresses before the energy can be expressed. Under **plane stress** assumptions we get,

$$\begin{bmatrix} E_{11} \\ E_{12} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\implies \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{12} \\ \gamma_{12} \end{bmatrix}$$

- The bending energy (up to $\mathcal{O}(v^2)$) is

$$\begin{aligned} \mathrm{U}_b &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{1}{2} \left(\sigma_{11} E_{11} + \sigma_{22} E_{22} + \sigma_{12} \gamma_{12}\right) dX_3 \\ &= \frac{1}{2} \underbrace{\frac{Et^3}{12(1-\nu^2)}}_{D} \left(w_{,11}^2 + w_{,22}^2 + 2(1-\nu) w_{,12}^2 + 2 w_{,11} w_{,22}\right) \end{aligned}$$

2. Plates

Work Done by Axial Stresses

- We consider axial loads P_1, P_2, P_{12} as shown. The work done by these is contributed by the quadratic strains

$$\begin{aligned}\mathrm{U}_c =& \frac{P_1}{24}\left(t^2(w_{,11}^2 + w_{,12}^2) + 12w_{,1}^2\right) + \frac{P_2}{24}\left(t^2(w_{,12}^2 + w_{,22}^2) + 12w_{,2}^2\right)\\ &+ \frac{P_{12}}{12}\left(t^2 w_{,12}(w_{,11} + w_{,22}) + 12w_{,1}w_{,2}\right).\end{aligned}$$

- We will ignore the t^2 terms in the above to give,

$$\mathrm{U}_c = \frac{1}{2}\left(P_1 w_{,1}^2 + P_2 w_{,2}^2 + 2P_{12}w_{,1}w_{,2}\right).$$

Other Loads

When there is also a distributed transverse load f acting, the load work done is given by

$$\Pi = \int_{\mathcal{D}} f w dX_1 dX_2$$

2.1. Principle of Virtual Work

Plates

- The total work done by the system is written as,

$$\mathcal{L} = U_b + U_c - \Pi = \frac{D}{2}\left(w_{,11}^2 + w_{,22}^2 + 2(1-\nu)w_{,12}^2 + 2w_{,11}w_{,22}\right) + \frac{1}{2}\left(P_1 w_{,1}^2 + P_2 w_{,2}^2 + 2P_{12}w_{,1}w_{,2}\right) - fw$$

- The Euler-Lagrange Equations are written as:

$$\frac{d^2}{dX_1^2}\frac{\partial\mathcal{L}}{\partial w_{,11}} + \frac{d^2}{dX_2^2}\frac{\partial\mathcal{L}}{\partial w_{,22}} + \frac{d^2}{dX_1 dX_2}\frac{\partial\mathcal{L}}{\partial w_{,12}} - \frac{d}{dX_1}\frac{\partial\mathcal{L}}{\partial w_{,1}} - \frac{d}{dX_2}\frac{\partial\mathcal{L}}{\partial w_{,2}} + \frac{\partial\mathcal{L}}{\partial w} = 0.$$

- This leads to,

$$\boxed{\underbrace{\frac{Et^3}{12(1-\nu^2)}}_{D}\left(w_{,1111} + w_{,2222} + 2w_{,1122}\right) - \left(P_1 w_{,11} + P_2 w_{,22} + 2P_{12}w_{,12}\right) - f = 0}$$

2.1. Principle of Virtual Work

Plates

- The general plate equation can be interpreted in two ways just as before.

$$D\left(w_{,1111}+w_{,2222}+2w_{,1122}\right)-\left(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12}\right)-f=0$$

Membranes

- When the quantity D is very small, the system is approximated well as

$$(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12})+f=0$$

- For the isotropic case shear-free case ($P_1=P_2=P$, $P_{12}=0$) we have,

$$P\nabla^2 w+f=0$$

Plate Buckling

- For the $f=0$ case undergoing compressive loading ($P_1\to -P_1$, $P_2\to -P_2$ $P_{12}\to -P_{12}$), the governing equation is

$$D\nabla^4 w+(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12})=0.$$

- This is a slightly more complicated Sturm-Liouville type problem than the one encountered with column buckling.

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- This is a slightly more complicated Sturm-Liouville type problem than the one encountered with column buckling.

Note that it is also possible to express the governing equations in terms of moments and section-normal shear forces (like we did with E.B.T.). But we will not pursue this here.

2.2. Classical Solutions

- One of the simplest case to consider is a plate with **simply supported edges** ($w = 0$ on $\partial\mathcal{D}$). The governing equations (for zero loading) is

$$D\nabla^4 w - f = 0, \quad (X_1, X_2) \in \mathcal{D}, \qquad w = 0, \quad (X_1, X_2) \in \partial\mathcal{D}.$$

  ($\partial\mathcal{D}$ is the closure of the open set $\mathcal{D}$).

- For a rectangular plate (sides $a_1 \times a_2$ such that $X_1 \in [0, a_1]$, $X_2 \in [0, a_2]$), a popular approach is to use a **Fourier Decomposition** of the form

$$w(X_1, X_2) = \sum_{n_1, n_2} A_{n_1 n_2} \sin\left(n_1 \frac{\pi}{a_1} X_1\right) \sin\left(n_2 \frac{\pi}{a_2} X_2\right).$$

- Note that the coefficients $A_{n_1 n_2}$ may be retrieved by the integral,

$$A_{n_1 n_2} = \frac{4}{a_1 a_2} \int_0^{a_1}\int_0^{a_2} w(X_1, X_2) \sin\left(n_1 \frac{\pi}{a_1} X_1\right) \sin\left(n_2 \frac{\pi}{a_2} X_2\right) dX_1 dX_2.$$

2.2. Classical Solutions

Plates

- Using this ansatz, the equilibrium equation now reads,

$$\underbrace{\sum_{n_1,n_2} D\pi^4 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2}\right)^2 A_{n_1 n_2} \sin\left(n_1 \frac{\pi}{a_1} X_1\right) \sin\left(n_2 \frac{\pi}{a_2} X_2\right)}_{D\nabla^4 w} = f.$$

- Expressing the Fourier coefficients of the load f as $F_{n_1 n_2}$ we can write,

$$A_{n_1 n_2} = \frac{1}{D\pi^4} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2}\right)^{-2} F_{n_1 n_2}.$$

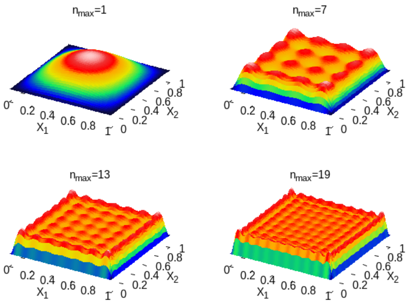
- This means that excitation along the function $\sin(n_1 \frac{\pi}{a_1} X_1) \sin(n_2 \frac{\pi}{a_2} X_2)$ will result in **deformation in the same shape**.
- For an arbitrary deformation, this leads to a **series representation** of the deformation shape.

2.2. Classical Solutions: Uniform Loading

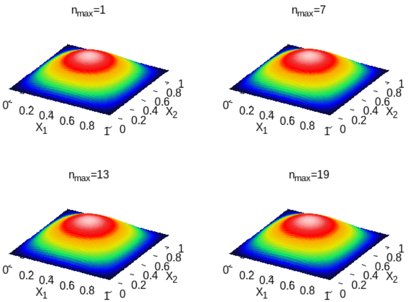
- For the case of uniform loading ($f(X_1, X_2) = 1$), it can be shown that

$$F_{n_1 n_2} = \begin{cases} \dfrac{16}{\pi^2 n_1 n_2} & n_1, n_2 \text{ both odd} \\ 0 & \text{otherwise} \end{cases}.$$

Force Decomposition



Deflection Decomposition



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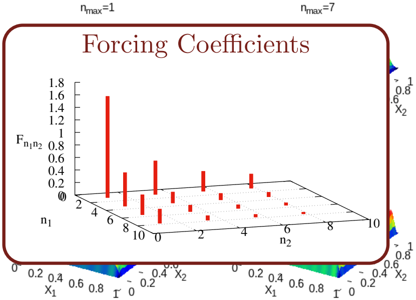

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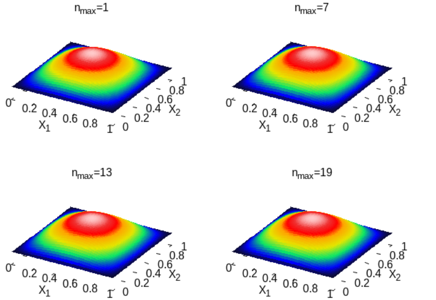

2.3. Buckling of Plates

- We will consider buckling of plates also under the same conditions (simply supported ends). Let us set $P_{12} = 0$ here (since it introduces cosine terms also).
- The governing equations become

$$D\nabla^4 w - (P_1 w_{,11} + P_2 w_{,22}) =$$

$$\sum_{n_1, n_2} \left(D\pi^4 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2 - \pi^2 \left(\frac{n_1^2}{a_1^2} P_1 + \frac{n_2^2}{a_2^2} P_2 \right) \right) A_{n_1 n_2} \sin\left(n_1 \frac{\pi}{a_1} X_1 \right) \sin\left(n_2 \frac{\pi}{a_2} X_2 \right) = 0.$$

Non-trivial $A_{n_1 n_2}$ for $P_2 = 0$

$$P_1 = \pi^2 D \frac{a_1^2}{n_1^2} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2.$$

The critical load (lowest P_1) corresponds to $n_2 = 1$:

$$P_1^* = \frac{\pi^2 D}{a_2^2} \left(\frac{n_1 a_2}{a_1} + \frac{a_1}{n_1 a_2} \right)^2.$$

Non-trivial $A_{n_1 n_2}$ for $P_1 = 0$

$$P_2 = \pi^2 D \frac{a_2^2}{n_2^2} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2.$$

The critical load (lowest P_2) corresponds to $n_1 = 1$:

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2.3. Buckling of Plates

Plates

- We will co… ditions (simply su… ntroduces cosine term…
- The govern…

$D\nabla^4 w - (P_1 \ldots$

$\sum_{n_1,n_2}\left(D\pi^4 \ldots \right. \ldots \sin\left(n_2\frac{\pi}{a_2}X_2\right) = 0.$

Denoting $k = \left(\frac{n_1 a_1}{a_2} + \frac{a_2}{n_1 a_1}\right)^2$ s.t. $P_1^* = k\frac{\pi^2 D}{a_2^2}$

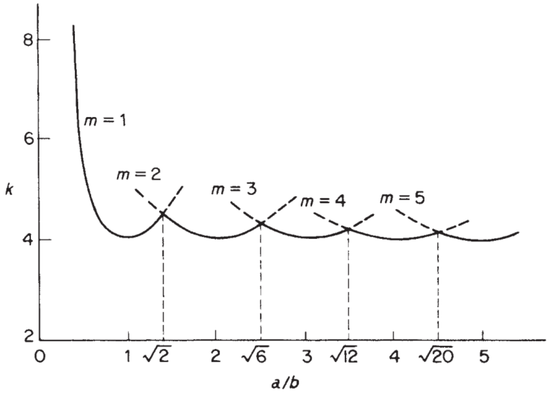

Figure from [1], $a = a_1$, $b = a_2$.

Non-trivia… for $P_1 = 0$

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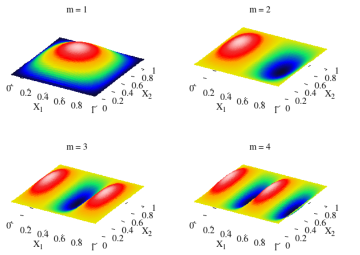
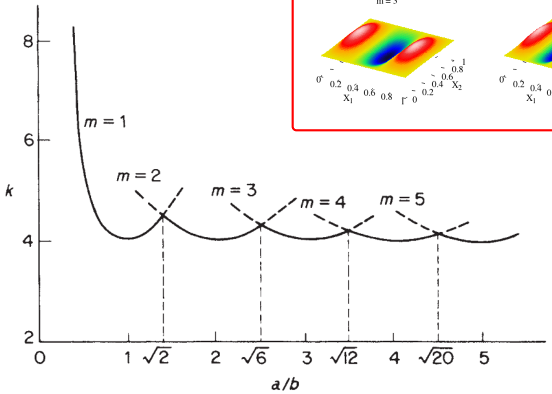
Denoting $k = \left(\frac{n_1 a_1}{a_2} + \frac{\ldots}{n \ldots}\right.$





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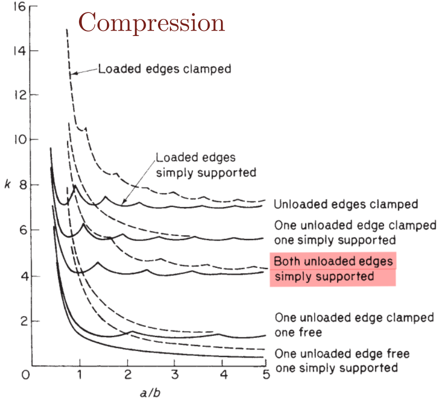
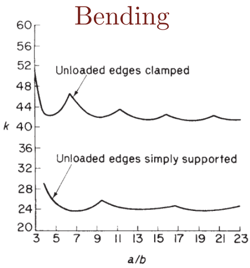
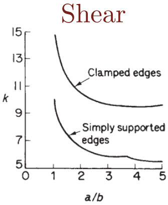
2.3. Buckling of Plates

Plates

- In general, the boundary conditions as well as loads may be completely different, warranting different analysis to be done to estimate the buckling coefficient k.







NOTICE: k approx. constant for $a/b > 3$.

Figures from [1].

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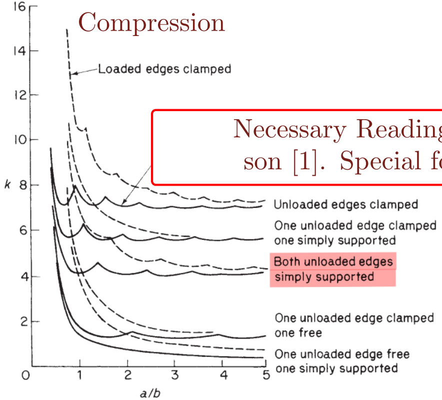

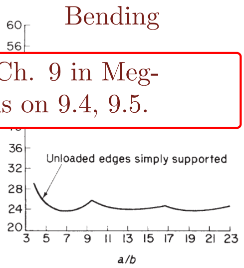

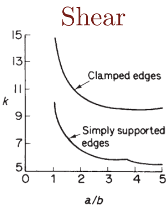

Necessary Reading: Ch. 9 in Megson [1]. Special focus on 9.4, 9.5.

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Figures from [1].

References I

[1] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on pp. 2, 16–20).