AS3020: Aerospace Structures

Module 7: Elastic Stability

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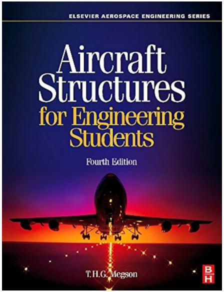
Table of Contents

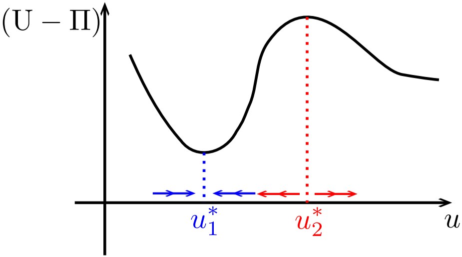


Chapters 7-9 in Megson [1]

1. Introduction

- The key intuition for elastic stability comes from analyzing the quantity $\mathrm{U} - \Pi$ around its extrema.
  - Maxima in $\mathrm{U} - \Pi$ correspond to **unstable solutions**;
  - Minima in $\mathrm{U} - \Pi$ correspond to **stable solutions**.

- Investigating the second derivative ("Hessian") of the quantity allows for efficient classification;
- In 1D ($u \in \mathbb{R}$), the sign of $\frac{\partial(U-\Pi)}{\partial u^2}$ is sufficient for this;
- In higher dimensions, we obtain an eigenvalue problem.



1.1. Column Buckling

Introduction

- We already derived the governing equations for a beam under uniform axial stress $\frac{P}{A}$. When this is compressive, the governing equation can be written as

$$\boxed{EIv'''' + Pv'' = 0}.$$

- We showed in class that this can be used to recover Euler's Critical Loads,

$$P_n = n^2 \frac{\pi^2 EI}{\ell^2}, \quad v(X_1) = V \sin\left(n\frac{\pi X_1}{\ell}\right).$$

- We solved a **Sturm-Liouville Problem** to obtain these.

2. Plates

- We will now derive the governing equations of thin plates with the **Kirchhoff-Love Plate Theory**, which is the simplest generalization of **Euler-Bernoulli Beam Theory**.

Euler-Bernoulli Beams

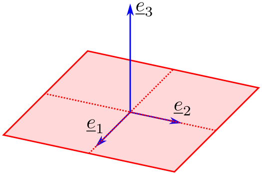
- Sections *move* rigidly;
- Plane sections remain perpendicular to the centroidal axis.

KL Plates

- Line elements along thickness *move* rigidly;
- Line elements remain perpendicular to the mid-plane.

- The above assumptions lead to the zeroing out of certain strains in the formulation that leads to a simplified kinematic description. For plates this is,

$$u_1 = -X_3 w_{,1}$$
$$u_2 = -X_3 w_{,2}$$
$$u_3 = w,$$



where w is a function of X_1, X_2.

2. Plates

Variational Approach for Derivation

- Using the kinematic description we write out the strains (linear and nonlinear) as

$$
\begin{aligned}
E_{11} &= u_{1,1} + \frac{1}{2}(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2) \\
&= -X_3 w_{,11} + \frac{1}{2}\left(X_3^2 w_{,11}^2 + X_3^2 w_{,12}^2 + w_{,1}^2\right) \\
E_{22} &= u_{2,2} + \frac{1}{2}(u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2) \\
&= -X_3 w_{,22} + \frac{1}{2}\left(X_3^2 w_{,12}^2 + X_3^2 w_{,22}^2 + w_{,2}^2\right) \\
\gamma_{12} &= u_{1,2} + u_{2,1} + (u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}) \\
&= -2X_3 w_{,12} + \left(X_3^2 w_{,11}w_{,12} + X_3^2 w_{,12}w_{,22} + w_{,1}w_{,2}\right),
\end{aligned}
$$

  where the nonlinear (quadratic) terms are highlighted in blue.
- Just like in the case of the beam, we **retain only the quadratic terms** for the internal energy.

2. Plates

Bending Strain Energy under Plane Stress

- We have to first write down the stresses before the energy can be expressed. Under **plane stress** assumptions we get,

$$\begin{bmatrix} E_{11} \\ E_{12} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\implies \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{12} \\ \gamma_{12} \end{bmatrix}$$

- The bending energy (up to $\mathcal{O}(v^2)$) is

$$\mathrm{U}_b = \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{1}{2} \left(\sigma_{11} E_{11} + \sigma_{22} E_{22} + \sigma_{12}\gamma_{12}\right) dX_3$$

$$= \frac{1}{2} \underbrace{\frac{Et^3}{12(1-\nu^2)}}_{D} \left(w_{,11}^2 + w_{,22}^2 + 2(1-\nu) w_{,12}^2 + 2 w_{,11} w_{,22}\right)$$

2. Plates

Work Done by Axial Stresses

- We consider axial loads P_1, P_2, P_{12} as shown. The work done by these is contributed by the quadratic strains

$$\begin{aligned} \mathrm{U}_c =& \frac{P_1}{24}\left(t^2(w_{,11}^2 + w_{,12}^2) + 12w_{,1}^2\right) + \frac{P_2}{24}\left(t^2(w_{,12}^2 + w_{,22}^2) + 12w_{,2}^2\right) \\ &+ \frac{P_{12}}{12}\left(t^2 w_{,12}(w_{,11} + w_{,22}) + 12w_{,1}w_{,2}\right). \end{aligned}$$

- We will ignore the t^2 terms in the above to give,

$$\mathrm{U}_c = \frac{1}{2}\left(P_1 w_{,1}^2 + P_2 w_{,2}^2 + 2P_{12}w_{,1}w_{,2}\right).$$

Other Loads

When there is also a distributed transverse load f acting, the load work done is given by

$$\Pi = \int_{\mathcal{D}} f w dX_1 dX_2$$

2.1. Principle of Virtual Work

Plates

- The total work done by the system is written as,

$$\mathcal{L} = U_b + U_c - \Pi = \frac{D}{2}\left(w_{,11}^2 + w_{,22}^2 + 2(1-\nu)w_{,12}^2 + 2w_{,11}w_{,22}\right) + \frac{1}{2}\left(P_1 w_{,1}^2 + P_2 w_{,2}^2 + 2P_{12}w_{,1}w_{,2}\right) - fw$$

- The Euler-Lagrange Equations are written as:

$$\frac{d^2}{dX_1^2}\frac{\partial \mathcal{L}}{\partial w_{,11}} + \frac{d^2}{dX_2^2}\frac{\partial \mathcal{L}}{\partial w_{,22}} + \frac{d^2}{dX_1 dX_2}\frac{\partial \mathcal{L}}{\partial w_{,12}} - \frac{d}{dX_1}\frac{\partial \mathcal{L}}{\partial w_{,1}} - \frac{d}{dX_2}\frac{\partial \mathcal{L}}{\partial w_{,2}} + \frac{\partial \mathcal{L}}{\partial w} = 0.$$

- This leads to,

$$\boxed{\underbrace{\frac{Et^3}{12(1-\nu^2)}}_{D}\left(w_{,1111} + w_{,2222} + 2w_{,1122}\right) - \left(P_1 w_{,11} + P_2 w_{,22} + 2P_{12}w_{,12}\right) - f = 0}$$

2.1. Principle of Virtual Work

Plates

- The general plate equation can be interpreted in two ways just as before.

$$D\left(w_{,1111}+w_{,2222}+2w_{,1122}\right)-\left(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12}\right)-f=0$$

Membranes

- When the quantity D is very small, the system is approximated well as

$$(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12})+f=0$$

- For the isotropic case shear-free case ($P_1=P_2=P$, $P_{12}=0$) we have,

$$P\nabla^2 w+f$$

Plate Buckling

- For the $f=0$ case undergoing compressive loading ($P_1\rightarrow -P_1$, $P_2\rightarrow -P_2$ $P_{12}\rightarrow -P_{12}$), the governing equation is

$$D\nabla^4 w+(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12})=0.$$

- This is a slightly more complicated Sturm-Liouville type problem than the one encountered with column buckling.

References I

[1] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).