AS3020: Aerospace Structures

Module 7: Elastic Stability

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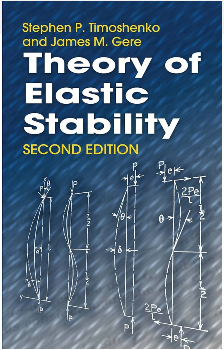
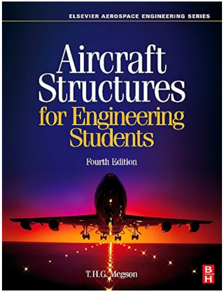


Chapter 9 in Timoshenko and Gere [1]. Good reference in general.

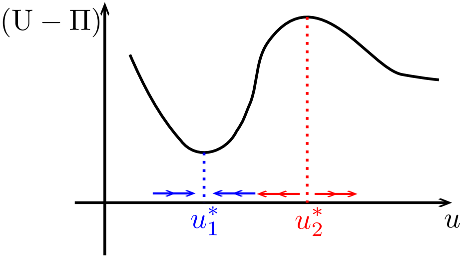


Chapters 7-9 in Megson [2]

1. Introduction

- The key intuition for elastic stability comes from analyzing the quantity $\mathrm{U}-\Pi$ around its extrema.
  - Maxima in $\mathrm{U}-\Pi$ correspond to **unstable solutions**;
  - Minima in $\mathrm{U}-\Pi$ correspond to **stable solutions**.

- Investigating the second derivative ("Hessian") of the quantity allows for efficient classification;
- In 1D ($u \in \mathbb{R}$), the sign of $\frac{\partial(U-\Pi)}{\partial u^2}$ is sufficient for this;
- In higher dimensions, we obtain an eigenvalue problem.



1.1. Column Buckling

Introduction

- We already derived the governing equations for a beam under uniform axial stress $\frac{P}{A}$. When this is compressive, the governing equation can be written as

$$\boxed{EIv'''' + Pv'' = 0}.$$

- We showed in class that this can be used to recover Euler's Critical Loads,

$$P_n = n^2 \frac{\pi^2 EI}{\ell^2}, \quad v(X_1) = V \sin\left(n\frac{\pi X_1}{\ell}\right).$$

  (above expressions are for a simply supported beam)
- We solved a **Sturm-Liouville Problem** to obtain these.

2. Plates

- We will now derive the governing equations of thin plates with the **Kirchhoff-Love Plate Theory**, which is the simplest generalization of **Euler-Bernoulli Beam Theory**.

Euler-Bernoulli Beams

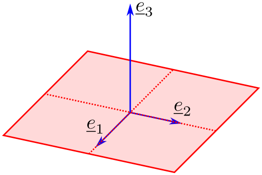
- Sections *move* rigidly;
- Plane sections remain perpendicular to the centroidal axis.

KL Plates

- Line elements along thickness *move* rigidly;
- Line elements remain perpendicular to the mid-plane.

- The above assumptions lead to the zeroing out of certain strains in the formulation that leads to a simplified kinematic description. For plates this is,

$$u_1 = -X_3 w_{,1}$$
$$u_2 = -X_3 w_{,2}$$
$$u_3 = w,$$



where w is a function of X_1, X_2.

2. Plates

Variational Approach for Derivation

- Using the kinematic description we write out the strains (linear and nonlinear) as

$$
\begin{aligned}
E_{11} &= u_{1,1} + \frac{1}{2}(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2) \\
&= -X_3 w_{,11} + \frac{1}{2}\left(X_3^2 w_{,11}^2 + X_3^2 w_{,12}^2 + w_{,1}^2\right) \\
E_{22} &= u_{2,2} + \frac{1}{2}(u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2) \\
&= -X_3 w_{,22} + \frac{1}{2}\left(X_3^2 w_{,12}^2 + X_3^2 w_{,22}^2 + w_{,2}^2\right) \\
\gamma_{12} &= u_{1,2} + u_{2,1} + (u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}) \\
&= -2X_3 w_{,12} + \left(X_3^2 w_{,11}w_{,12} + X_3^2 w_{,12}w_{,22} + w_{,1}w_{,2}\right),
\end{aligned}
$$

  where the nonlinear (quadratic) terms are highlighted in blue.
- Just like in the case of the beam, we **retain only the quadratic terms** for the internal energy.

2. Plates

Bending Strain Energy under Plane Stress

- We have to first write down the stresses before the energy can be expressed. Under **plane stress** assumptions we get,

$$\begin{bmatrix} E_{11} \\ E_{12} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\implies \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{12} \\ \gamma_{12} \end{bmatrix}$$

- The bending energy (up to $\mathcal{O}(v^2)$) is

$$\mathrm{U}_b = \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{1}{2} \left(\sigma_{11} E_{11} + \sigma_{22} E_{22} + \sigma_{12} \gamma_{12}\right) dX_3$$

$$= \frac{1}{2} \underbrace{\frac{Et^3}{12(1-\nu^2)}}_{D} \left(w_{,11}^2 + w_{,22}^2 + 2(1-\nu) w_{,12}^2 + 2 w_{,11} w_{,22}\right)$$

2. Plates

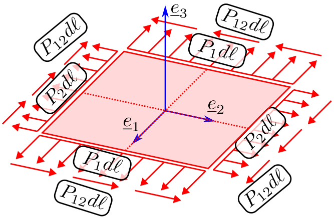
Work Done by Axial Stresses

- We consider axial loads P_1, P_2, P_{12} as shown. The work done by these is contributed by the quadratic strains

$$\begin{aligned} \mathrm{U}_c =& \frac{P_1}{24}\left(t^2(w_{,11}^2 + w_{,12}^2) + 12w_{,1}^2\right) + \frac{P_2}{24}\left(t^2(w_{,12}^2 + w_{,22}^2) + 12w_{,2}^2\right) \\ &+ \frac{P_{12}}{12}\left(t^2 w_{,12}(w_{,11} + w_{,22}) + 12w_{,1}w_{,2}\right). \end{aligned}$$

- We will ignore the t^2 terms in the above to give,

$$\mathrm{U}_c = \frac{1}{2}\left(P_1 w_{,1}^2 + P_2 w_{,2}^2 + 2P_{12}w_{,1}w_{,2}\right).$$



Other Loads

When there is also a distributed transverse load f acting, the load work done is given by

$$\Pi = \int_{\mathcal{D}} f w dX_1 dX_2$$

2.1. Principle of Virtual Work

Plates

- The total work done by the system is written as,

$$\mathcal{L} = U_b + U_c - \Pi = \frac{D}{2}\left(w_{,11}^2 + w_{,22}^2 + 2(1-\nu)w_{,12}^2 + 2w_{,11}w_{,22}\right) + \frac{1}{2}\left(P_1 w_{,1}^2 + P_2 w_{,2}^2 + 2P_{12}w_{,1}w_{,2}\right) - fw$$

- The Euler-Lagrange Equations are written as:

$$\frac{d^2}{dX_1^2}\frac{\partial \mathcal{L}}{\partial w_{,11}} + \frac{d^2}{dX_2^2}\frac{\partial \mathcal{L}}{\partial w_{,22}} + \frac{d^2}{dX_1 dX_2}\frac{\partial \mathcal{L}}{\partial w_{,12}} - \frac{d}{dX_1}\frac{\partial \mathcal{L}}{\partial w_{,1}} - \frac{d}{dX_2}\frac{\partial \mathcal{L}}{\partial w_{,2}} + \frac{\partial \mathcal{L}}{\partial w} = 0.$$

- This leads to,

$$\boxed{\underbrace{\frac{Et^3}{12(1-\nu^2)}}_{D}\left(w_{,1111} + w_{,2222} + 2w_{,1122}\right) - \left(P_1 w_{,11} + P_2 w_{,22} + 2P_{12}w_{,12}\right) - f = 0}$$

2.1. Principle of Virtual Work

Plates

- The general plate equation can be interpreted in two ways just as before.

$$D\left(w_{,1111}+w_{,2222}+2w_{,1122}\right)-\left(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12}\right)-f=0$$

Membranes

- When the quantity D is very small, the system is approximated well as

$$(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12})+f=0$$

- For the isotropic case shear-free case ($P_1=P_2=P$, $P_{12}=0$) we have,

$$P\nabla^2 w+f=0$$

Plate Buckling

- For the $f=0$ case undergoing compressive loading ($P_1\to -P_1$, $P_2\to -P_2$ $P_{12}\to -P_{12}$), the governing equation is

$$D\nabla^4 w+(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12})=0.$$

- This is a slightly more complicated Sturm-Liouville type problem than the one encountered with column buckling.

2.1. Principle of Virtual Work

Plates

- The general plate equation can be interpreted in two ways just as before.

$$D\left(w_{,1111}+w_{,2222}+2w_{,1122}\right)-\left(P_1 w_{,11}+P_2 w_{,22}+2P_{12}w_{,12}\right)-f=0$$

Mem

- When the qua
  the system is a

  $(P_1 w_{,11} + P_2 w$

- For the isotropic case shear-free case ($P_1 = P_2 = P$, $P_{12} = 0$) we have,

$$P\nabla^2 w + f = 0$$

Note that it is also possible to express the governing equations in terms of moments and section-normal shear forces (like we did with E.B.T.). But we will not pursue this here.

Plate Buckling

- ase undergoing
  ding ($P_1 \to -P_1$,
  $\to -P_{12}$), the
  tion is

  $D\nabla^4 w + (P_1 w_{,11} + P_2 w_{,22} + 2P_{12} w_{,12}) = 0.$

- This is a slightly more complicated Sturm-Liouville type problem than the one encountered with column buckling.

2.2. Classical Solutions

Plates

- One of the simplest case to consider is a plate with **simply supported edges** ($w=0$ on $\partial\mathcal{D}$). The governing equations (for zero loading) is

$$D\nabla^4 w - f = 0, \quad (X_1, X_2) \in \mathcal{D}, \qquad w = 0, \quad (X_1, X_2) \in \partial\mathcal{D}.$$

  ($\partial\mathcal{D}$ is the closure of the open set $\mathcal{D}$).

- For a rectangular plate (sides $a_1 \times a_2$ such that $X_1 \in [0, a_1]$, $X_2 \in [0, a_2]$), a popular approach is to use a **Fourier Decomposition** of the form

$$w(X_1, X_2) = \sum_{n_1, n_2} A_{n_1 n_2} \sin\left(n_1 \frac{\pi}{a_1} X_1\right) \sin\left(n_2 \frac{\pi}{a_2} X_2\right).$$

- Note that the coefficients $A_{n_1 n_2}$ may be retrieved by the integral,

$$A_{n_1 n_2} = \frac{4}{a_1 a_2} \int_0^{a_1} \int_0^{a_2} w(X_1, X_2) \sin\left(n_1 \frac{\pi}{a_1} X_1\right) \sin\left(n_2 \frac{\pi}{a_2} X_2\right) dX_1 dX_2.$$

2.2. Classical Solutions

Plates

- Using this ansatz, the equilibrium equation now reads,

$$\underbrace{\sum_{n_1,n_2} D\pi^4 \left(\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}\right)^2 A_{n_1n_2} \sin\left(n_1\frac{\pi}{a_1}X_1\right) \sin\left(n_2\frac{\pi}{a_2}X_2\right)}_{D\nabla^4 w} = f.$$

- Expressing the Fourier coefficients of the load f as $F_{n_1n_2}$ we can write,

$$A_{n_1n_2} = \frac{1}{D\pi^4}\left(\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}\right)^{-2} F_{n_1n_2}.$$

- This means that excitation along the function $\sin(n_1\frac{\pi}{a_1}X_1)\sin(n_2\frac{\pi}{a_2}X_2)$ will result in **deformation in the same shape**.
- For an arbitrary deformation, this leads to a **series representation** of the deformation shape.

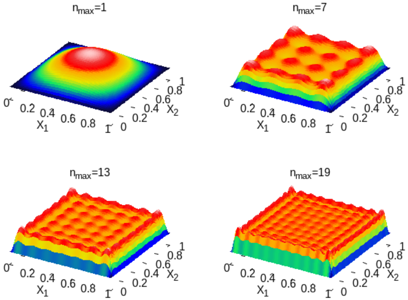
2.2. Classical Solutions: Uniform Loading

Plates

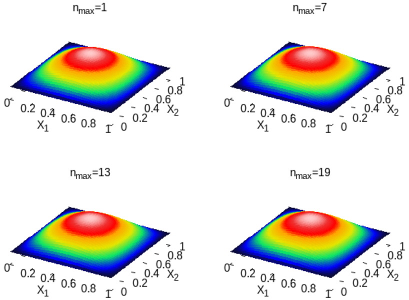
- For the case of uniform loading ($f(X_1, X_2) = 1$), it can be shown that

$$F_{n_1 n_2} = \begin{cases} \dfrac{16}{\pi^2 n_1 n_2} & n_1, n_2 \text{ both odd} \\ 0 & \text{otherwise} \end{cases}.$$

Force Decomposition



Deflection Decomposition



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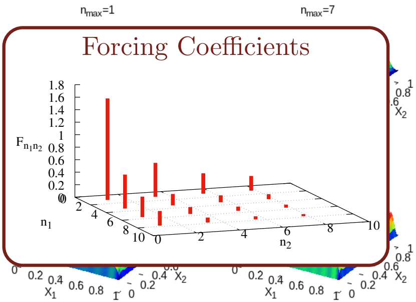
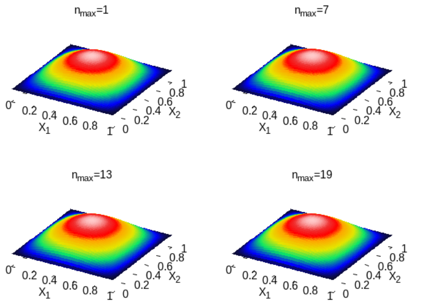


Deflection Decomposition



2.3. Buckling of Plates

Plates

- We will consider buckling of plates also under the same conditions (simply supported ends). Let us set $P_{12} = 0$ here (since it introduces cosine terms also).
- The governing equations become

$$D\nabla^4 w - (P_1 w_{,11} + P_2 w_{,22}) =$$

$$\sum_{n_1,n_2} \left(D\pi^4 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2 - \pi^2 \left(\frac{n_1^2}{a_1^2} P_1 + \frac{n_2^2}{a_2^2} P_2 \right) \right) A_{n_1 n_2} \sin\left(n_1 \frac{\pi}{a_1} X_1 \right) \sin\left(n_2 \frac{\pi}{a_2} X_2 \right) = 0.$$

Non-trivial $A_{n_1 n_2}$ for $P_2 = 0$

$$P_1 = \pi^2 D \frac{a_1^2}{n_1^2} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2.$$

The critical load (lowest P_1) corresponds to $n_2 = 1$:

$$P_1^* = \frac{\pi^2 D}{a_2^2} \left(\frac{n_1 a_2}{a_1} + \frac{a_1}{n_1 a_2} \right)^2.$$

Non-trivial $A_{n_1 n_2}$ for $P_1 = 0$

$$P_2 = \pi^2 D \frac{a_2^2}{n_2^2} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} \right)^2.$$

The critical load (lowest P_2) corresponds to $n_1 = 1$:

$$P_2^* = \frac{\pi^2 D}{a_1^2} \left(\frac{n_2 a_1}{a_2} + \frac{a_2}{n_2 a_1} \right)^2.$$

2.3. Buckling of Plates

Plates

- We will co… …ditions (simply su… …ntroduces cosine term…
- The gover…

$D\nabla^4 w - (P_1 \ldots$

$$\sum_{n_1,n_2} \left(D\pi^4 \left(\ldots \right. \right. \ldots \sin\left(n_2 \frac{\pi}{a_2} X_2\right) = 0.$$

Denoting $k = \left(\frac{n_1 a_1}{a_2} + \frac{a_2}{n_1 a_1}\right)^2$ s.t. $P_1^* = k \frac{\pi^2 D}{a_2^2}$

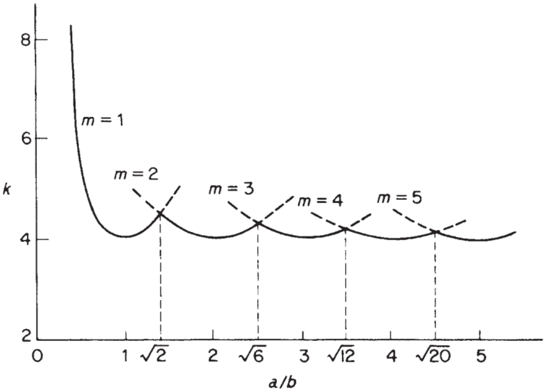


Figure from [2], $a = a_1$, $b = a_2$.

Non-trivia… …for $P_1 = 0$

$P_1 = \pi^2 D \ldots + \frac{n_2^2}{a_2^2}\Big)^2.$

The critical load … P_2) corresponds to $n_2 = 1$:

$$P_1^* = \frac{\pi^2 D}{a_2^2}\left(\frac{n_1 a_2}{a_1} + \frac{a_1}{n_1 a_2}\right)^2. \qquad P_2^* = \frac{\pi^2 D}{a_1^2}\left(\frac{n_2 a_1}{a_2} + \frac{a_2}{n_2 a_1}\right)^2.$$

2.3. Buckling of Plates

Plates

- We will co[...]
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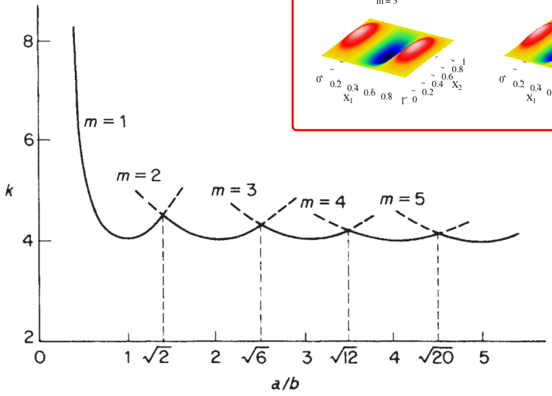
  cosine term[...]
- The govern[...]

$D\nabla^4 w - (P_1 \ldots$

$\sum_{n_1, n_2} \left(D\pi^4 \left(\ldots \right. \right. \ldots \sin\left(n_2 \frac{\pi}{a_2} X_2\right) = 0.$

Denoting $k = \left(\frac{n_1 a_1}{a_2} + \ldots \right.$



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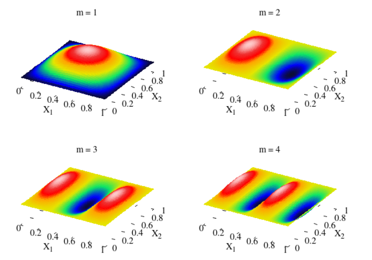

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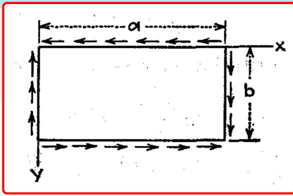
$$P_1^* = \frac{\pi^2 D}{a_2^2} \left(\frac{n_1 a_2}{a_1} + \frac{a_1}{n_1 a_2} \right)^2. \qquad P_2^* = \frac{\pi^2 D}{a_1^2} \left(\frac{n_2 a_1}{a_2} + \frac{a_2}{n_2 a_1} \right)^2.$$

2.3. Buckling of Plates: Shear Buckling

Plates



- Under pure shear, the governing equations is
$$D\nabla^4 w - P_{12} 2 w_{,12} = 0.$$
- Using the same ansatz (simply supported boundaries) we have,
$$\sum_{n_1,n_2} \left[D\pi^4 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2}\right)^2 A_{n_1 n_2} \mathcal{S}_{n_1} \mathcal{S}_{n_2} - P_{12} 2\pi^2 \frac{n_1 n_2}{a_1 a_2} \mathcal{C}_{n_1} \mathcal{C}_{n_2} \right] = 0.$$
- Note that $\int_0^a \mathcal{S}_n S_m dx = \frac{a}{2}\delta_{nm}$ and
$$\int_0^a \mathcal{S}_n \mathcal{C}_m dx = \begin{cases} 0 & n \pm m \text{ is even,} \\ \frac{2a}{\pi}\frac{n}{n^2-m^2} & n \pm m \text{ is odd} \end{cases}.$$
- Multiplying the above equation by $\mathcal{S}_{m_1}\mathcal{S}_{m_2}$ and integrating over $(0, a_1) \times (0, a_2)$ we get

$$D\frac{\pi^4}{a_1^2 a_2^2}\left(\frac{m_1^2 a_2}{a_1} + \frac{m_2^2 a_1}{a_2}\right)^2 A_{m_1 m_2} - P_{12}\frac{32}{a_1 a_2}\sum_{n_1 n_2} \frac{n_1 n_2 m_1 m_2}{(n_1^2 - m_1^2)(n_2^2 - m_2^2)} A_{n_1 n_2} = 0.$$

2.3. Buckling of Plates: Shear Buckling

- We can move around the terms a little bit and setting the aspect ratio $\beta = \frac{a_2}{a_1}$ we get

$$\underbrace{\frac{\pi^4 D}{32 a_2^2}}_{\alpha} \left(m_1^2\beta^2 + m_2^2\right)^2 A_{m_1 m_2} - P_{12}\beta \sum_{n_1 n_2} \frac{n_1 n_2 m_1 m_2}{(n_1^2 - m_1^2)(n_2^2 - m_2^2)} A_{n_1 n_2} = 0.$$

- If we truncate the n_1 and n_2 to finite N_1 and N_2, this represents a **Generalized Eigenvalue Problem** (GEVP). Restricting ourselves to $N_1, N_2 = 2$ we have,

$$\left(\alpha \begin{bmatrix} (\beta^2+1)^2 & 0 & 0 & 0 \\ 0 & (\beta^2+4)^2 & 0 & 0 \\ 0 & 0 & (4\beta^2+1)^2 & 0 \\ 0 & 0 & 0 & (4\beta^2+4)^2 \end{bmatrix} - P_{12}\frac{4\beta}{9} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- This is quite convenient since we can analytically estimate its eigen solutions. (I use maxima)

2.3. Buckling of Plates: Shear Buckling

Plates

- The eigenpairs are evaluated as

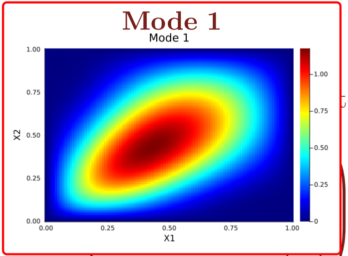
$$\left(\pm 9\alpha\frac{(\beta^2+1)^2}{\beta}, \begin{bmatrix}1\\0\\0\\\frac{1}{4}\end{bmatrix}\right), \qquad \left(\pm 9\alpha\frac{(\beta^2+4)(4\beta^2+1)}{4\beta}, \begin{bmatrix}0\\1\\\pm\frac{\beta^2+4}{4\beta^2+1}\\0\end{bmatrix}\right)$$

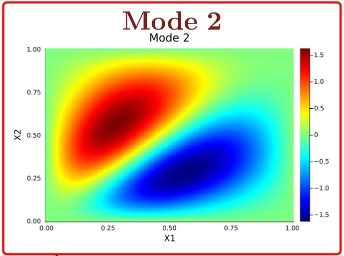
- Substituting for $\alpha = \frac{\pi^4 D}{32 a_2^2}$ we have,

$$P_{12}^* = \pm\frac{\pi^2 D}{a_2^2}\frac{9\pi^2}{32}\frac{(\beta^2+1)^2}{\beta}, \quad \pm\frac{\pi^2 D}{a_2^2}\frac{9\pi^2}{128}\frac{(\beta^2+4)(4\beta^2+1)}{\beta}$$

2.3. Buckling of Plates: Shear Buckling

Plates





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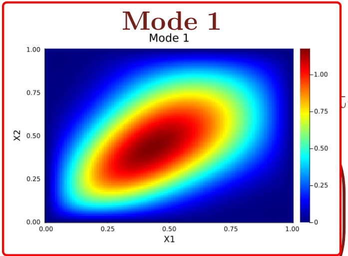
$$\left(\ldots \frac{}{4} \right), \quad \left(\ldots \begin{bmatrix} 0 \\ 1 \\ \pm\frac{\beta^2+4}{4\beta^2+1} \\ 0 \end{bmatrix} \right)$$

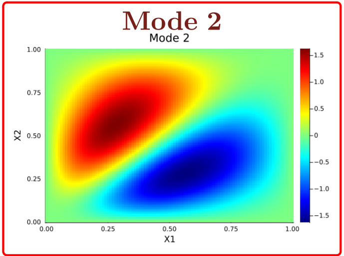
- Substituting for $\alpha = \frac{\pi^4 D}{32 a_2^2}$ we have,

$$P_{12}^* = \pm \frac{\pi^2 D}{a_2^2} \frac{9\pi^2}{32} \frac{(\beta^2+1)^2}{\beta}, \quad \pm \frac{\pi^2 D}{a_2^2} \frac{9\pi^2}{128} \frac{(\beta^2+4)(4\beta^2+1)}{\beta}$$

2.3. Buckling of Plates: Shear Buckling

Plates





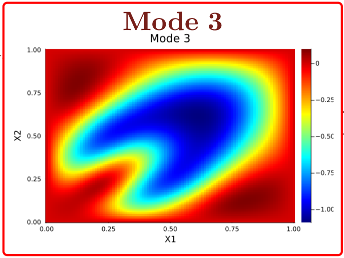
ted

,

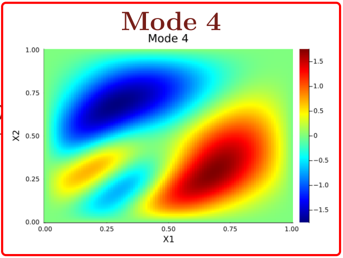
$$\begin{bmatrix} 0 \\ 1 \\ \pm\frac{\beta^2+4}{4\beta^2+1} \\ 0 \end{bmatrix}$$

- Substitutin

P^*_{12}



$\frac{\tau^2 D}{a_2^2}$

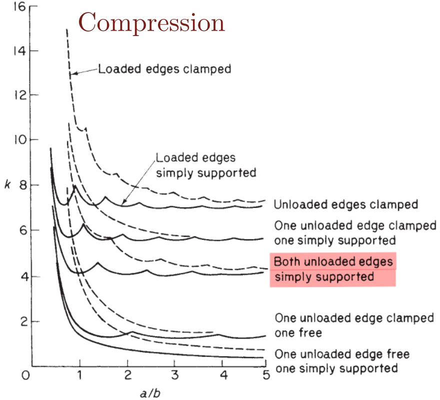


2.3. Buckling of Plates

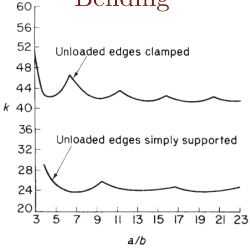
Plates

- In general, the boundary conditions as well as loads may be completely different, warranting different analysis to be done to estimate the buckling coefficient k.

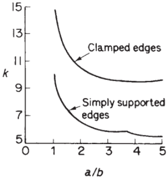
Compression



Bending



Shear



NOTICE: k approx. constant for $a/b > 3$.

Figures from [2].

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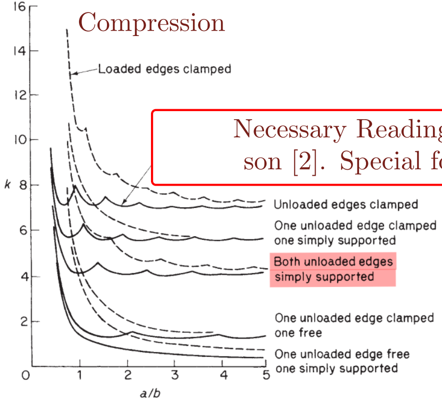
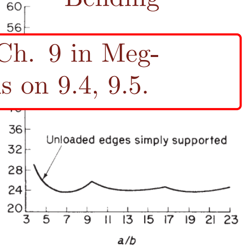
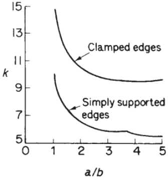


Necessary Reading: Ch. 9 in Megson [2]. Special focus on 9.4, 9.5.

Bending



Shear



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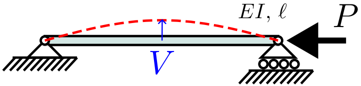
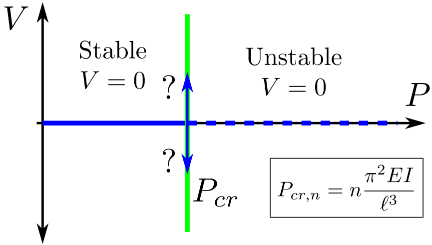
Figures from [2].

3. Basic Post Buckling Analysis

- We have so far only looked at **buckling/pre-buckling analysis**.



But what about the exact amplitudes post buckling?



- The study of postbuckling requires the consideration of **nonlinear strain energy contributions**.

3. Basic Post Buckling Analysis

- Under the kinematic description $u_1 = -X_2 v'$, and $u_2 = v$, the **von Karman strain** E_{11} is,

$$E_{11} = u_{1,1} + \frac{u_{2,1}^2}{2} = -X_2 v'' + \frac{(v')^2}{2}.$$

- The corresponding total strain energy is

$$\text{U} = \int_0^{\ell} \left(\frac{E_y I_{33}}{2} (v'')^2 - \frac{P}{2} (v')^2 + \frac{E_y A}{8} (v')^4 \right) dX_1$$

NOTE: Positive P is compressive here.

Von Karman Beam Equations

Applying the Euler-Lagrange equations directly here, we get:

$$E_y I_{33} v'''' - \frac{3E_y A}{2} (v')^2 v'' + P v'' = 0.$$

This is the starting point for the **von Karman beam theory** which allows the study for nominally finite amplitude deformations of beams.

3. Basic Post Buckling Analysis

- For P values slightly above P_{cr}, the deflection may be written as

$$v(X_1) = V\sin\left(n\frac{\pi}{\ell}X_1\right).$$

- Choosing $n=1$, substituting this into the strain energy expression yields,

$$\begin{aligned}\mathrm{U} &= \int_0^\ell \left[\frac{\pi^4 E_y I_{33} V^2}{2\ell^4}\sin^2\left(\frac{\pi X_1}{\ell}\right) - \frac{\pi^2 P V^2}{2\ell^2}\cos^2\left(\frac{\pi X_1}{\ell}\right) + \frac{\pi^4 E_y A V^4}{8\ell^4}\cos^4\left(\frac{\pi X_1}{\ell}\right)\right] dX_1 \\ &= \frac{\pi^4 E_y I_{33} V^2}{2\ell^4}\frac{\ell}{2} - \frac{\pi^2 P V^2}{2\ell^2}\frac{\ell}{2} + \frac{\pi^4 E_y A V^4}{8\ell^4}\frac{3\ell}{8} = \frac{\pi^2}{4\ell}\left(P_{cr}-P\right)V^2 + \frac{3\pi^2 A}{64 I_{33}\ell}P_{cr}V^4 \\ &= \boxed{\frac{\pi^2}{4\ell}\left[(P_{cr}-P)V^2 + \frac{3A}{16 I_{33}}P_{cr}V^4\right]}.\end{aligned}$$

- Stationarizing this with respect to variations in V (setting $\delta\mathrm{U} = \frac{\partial \mathrm{U}}{\partial V}\delta V = 0$ for all δV), we obtain

$$\boxed{(P_{cr}-P)V + \frac{3A}{8I_{33}}P_{cr}V^3 = 0} \implies V^* = 0, \pm\sqrt{\frac{8I_{33}}{3A}\left(\frac{P}{P_{cr}}-1\right)}.$$

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Real only for
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3. Basic Post Buckling Analysis

- So equilibrium is achieved for either

$$V_0^2 = 0, \qquad \text{or} \qquad V_{1,2}^2 = \frac{8I_{33}}{3A}\left(\frac{P}{P_{cr}} - 1\right).$$

- Looking for the second order optimality conditions we have,

$$\mathrm{U} = \frac{\pi^2}{4\ell}\left[(P_{cr} - P)V^2 + \frac{3A}{16I_{33}}V^4\right].$$

$$\frac{d\mathrm{U}}{dV} = \frac{\pi^2}{2\ell}\left[(P_{cr} - P)V + \frac{3A}{8I_{33}}V^3\right]$$

$$\frac{d^2\mathrm{U}}{dV^2} = \frac{\pi^2}{2\ell}\left[(P_{cr} - P) + \frac{9A}{8I_{33}}V^2\right]$$

- Substituting the equilibrium solutions we have,

$V = V_0 = 0$

$$\frac{d^2U}{dV^2} = -\frac{\pi^2}{2\ell}(P - P_{cr})$$

$V = V_{1,2}$

$$\frac{d^2U}{dV^2} = \frac{\pi^2}{\ell}(P - P_{cr}).$$

3. Basic Post Buckling Analysis

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$d^2\text{U}$ … π^2 … $(P_{cr} -$ … $9A$ … $]$

- S … olutions

$P < P_{cr}$ **Stable**

$P > P_{cr}$ **Unstable**

$P < P_{cr}$ **Non-Real**

$P > P_{cr}$ **Real, Stable**

$$V = V_0 = 0$$

$$\frac{d^2U}{dV^2} = -\frac{\pi^2}{2\ell}(P - P_{cr})$$

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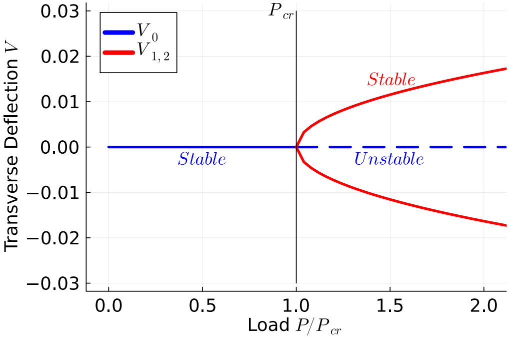
3.1. The Bifurcation Diagram

Basic Post Buckling Analysis

- The above analysis allows us to sketch the **bifurcation diagram**. This type of bifurcation is often termed the **Pitchfork bifurcation** (for obvious reasons).



- Unlike linearized stability analysis, the nonlinear analysis allows us to study the force-deflection curve of the system post buckling also.

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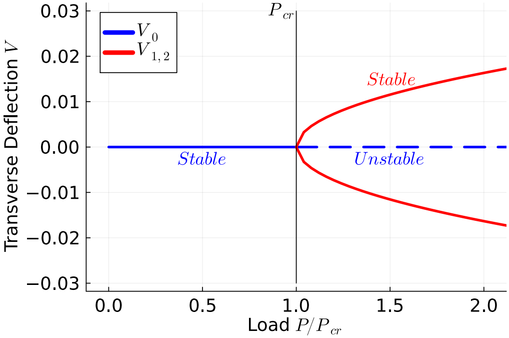

- Unlike linearized stability analysis, the nonlinear analysis allows us to study the force-deflection curve of the system post buckling also.

But what about axial deformations??

3.2. Axial Deflections

Basic Post Buckling Analysis

- For studying the axial deflections also, we modify the kinematics to allow these, such that

$$u_1 = u - X_2 v, \qquad u_2 = v \implies \boxed{E_{11} = u' - X_2 v' + \frac{(v')^2}{2}}.$$

- Using this, the strain energy density becomes

$$\mathrm{U} = \frac{E_y I_{33}}{2}(v'')^2 + \frac{E_y A}{8}(v')^4 - \frac{P}{2}(v')^2 + \frac{E_y A}{2}(u')^2 + \frac{E_y A}{2}u'(v')^2 - (-P u_T),$$

with the new terms highlighted in blue. (u_T is tip axial displacement)

- The equation governing axial deflection u cdis

$$E_y A u'' + E_y A v' v'' = 0, \quad u = 0, @\, X_1 = 0, \quad u = u_T, @\, X_1 = \ell.$$

- Substituting $v = V \sin\left(\frac{\pi X_1}{\ell}\right)$ and applying the boundary conditions leads to,

$$\boxed{u = -\frac{\pi V^2}{8\ell}\sin\left(\frac{2\pi X_1}{\ell}\right) + u_T \frac{X_1}{\ell}}.$$

3.2. Axial Deflections

Basic Post Buckling Analysis

- Resubstituting the above and extremizing in u_T yields,

$$\boxed{u_T = -\frac{P\ell}{E_y A} - \frac{\ell}{4E_y I_{33}} P_{cr} V^2}.$$

- On the "main branch", $V = 0$ so we have $\boxed{u_T = -\frac{\ell}{E_y A} P}$.
- On the bifurcated branch, $V^2 = \frac{8I_{33}}{3A}(\frac{P}{P_{cr}} - 1)$. So,

$$\boxed{u_T = -\frac{5}{3}\frac{\ell}{E_y A} P + \frac{2\ell}{3E_y A} P_{cr}}.$$

- In simpler terms, the axial stiffness before and after bifurcation are:

Before bifurcation $(P < P_{cr})$

$$\frac{du_T}{dP} = \frac{E_y A}{\ell}$$

After Bifurcation $P > P_{cr}$

$$\frac{du_T}{dP} = \frac{3}{5}\frac{E_y A}{\ell}$$

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- In simpler terms, the axial stiffness before and

Loss in stiffness

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After Bifurcation $P > P_{cr}$

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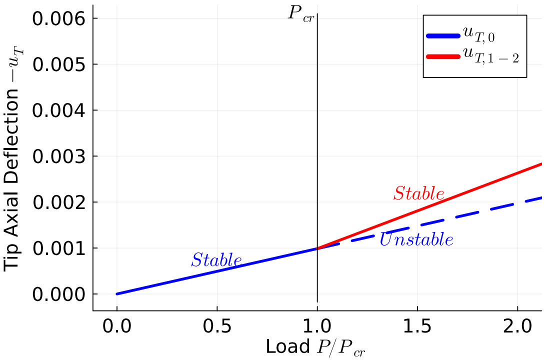
3.2. Axial Deflections

Basic Post Buckling Analysis

- We can now sketch the bifurcation diagram of the axial component also in terms of u_T:

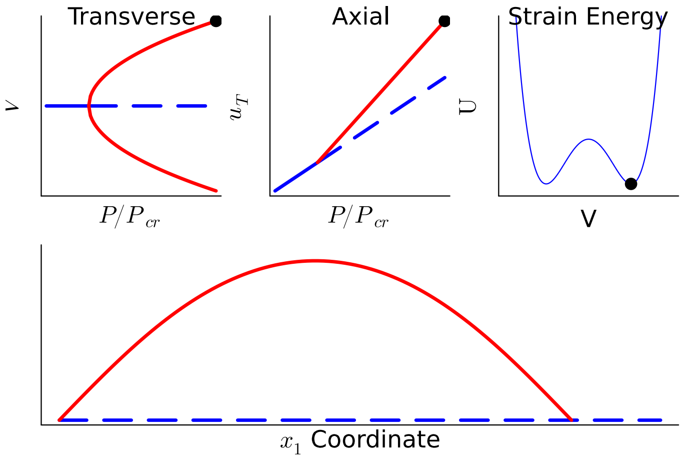


- The axial deformation field is written as

$$u(X_1) = -\frac{\pi}{8\ell}V^2\left[\sin\left(\frac{2\pi}{\ell}X_1\right) + \frac{2\pi}{\ell}X_1\right] - \frac{P}{E_yA}X_1.$$

3.2. Axial Deflections

Basic Post Buckling Analysis



Summary

References I

[1] S. P. Timoshenko and J. M. Gere. Theory of Elastic Stability, Courier Corporation, June 2009. ISBN: 978-0-486-47207-2 (cit. on p. 2).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on pp. 2, 16–18, 24, 25).