AS3020: Aerospace Structures

Module 6: Introduction to Variational Mechanics

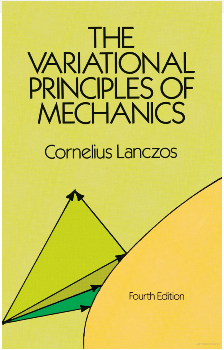
Instructor: Nidish Narayanaa Balaji

Dept. of Aerospace Engg., IIT-Madras, Chennai

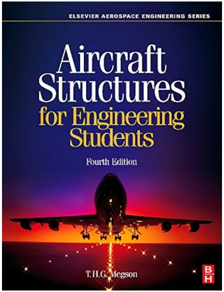
November 2, 2024

Table of Contents

1. The Principle of Virtual Work
   - The Elastic Solid
   - Interpretation
2. Beam Bending: A Motivating Problem Setting
   - Calculus of Variations
   - Variational Derivation of Euler Bernoulli Beam Theory
3. Tutorial Example
4. Bending of Axially Loaded Beams
   - Cables and Beams
   - Column Buckling



Very good book to read Lanczos [1]



Chapters 4, 5 in Megson [2]

1. The Principle of Virtual Work

- The idea behind the equilibrium principle is that the sum of all forces acting on a body is zero:
$$\int_\Omega forces = 0.$$
- In module 3, we used this in conjunction with the Cauchy Stress Principle to obtain general governing equations for an elastic solid:
$$\sigma_{ij,j} + f_i = 0, \quad \text{on } \Omega, \quad + \qquad b.c.s \quad \text{on } \partial\Omega.$$
- The work done by any force f_i on a system as it goes from $u_i^{(0)}$ to $u_i^{(0)} + \delta u_i$ is written as $f_i \delta u_i$.
- The displacement field δu_i denotes a “virtual displacement”, which is a mathematical idealization such that
  - it is small enough so as not to introduce changes in the force field;
  - it is compatible with any constraints that exist (B.C.s, for instance).

1. The Principle of Virtual Work

- The idea behind the equilibrium principle is that the sum of all forces acting on a body is zero:
$$\int_\Omega forces = 0.$$
- In module 3, we used this in conjunction with the Cauchy Stress Principle to obtain general governing equations for an elastic solid:
$$\sigma_{ij,j} + f_i = 0, \quad \text{on } \Omega, \quad + \qquad b.c.s \quad \text{on } \partial\Omega.$$
- The work done by any force f_i on a system as it goes from $u_i^{(0)}$ to $u_i^{(0)} + \delta u_i$ is written as $f_i \delta u_i$.
- The displacement field δu_i denotes a "virtual displacement", which is a mathematical idealization such that
  - it is small enough so as not to introduce changes in the force field;
  - it is compatible with any constraints that exist (B.C.s, for instance).

Under the stated assumptions, $\delta(\cdot)$ can be treated as a differential operator and we call it the **variational operator**.

1. The Principle of Virtual Work

- The work done by a virtual displacement field is termed as the *virtual work*. For a system under a certain force-field, the *virtual work* is a property of the system since so further deformation needs to be done..

Principle of Virtual Work

The virtual work of a system at equilibrium is zero.

- The *principle of virtual work* is merely a restatement of the principle of equilibrium, but it sometimes provides a more convenient analytical framework.
- Variational Mechanics is sometimes also referred to as analytical mechanics.

1.1. The Elastic Solid

The Principle of Virtual Work

- For the elastic solid, the principle of virtual work may be mathematically expressed as

$$\int_\Omega \underbrace{\sigma_{ij,j}\delta u_i}_{(\sigma_{ij}\delta u_i)_{,j}-\sigma_{ij}\delta u_{i,j}} + \int_\Omega f_i \delta u_i = 0.$$

- Applying Gauss divergence to the first term in the above we get,

$$\int_{\partial\Omega} \underbrace{\sigma_{ij} n_j}_{\text{surface traction } t_i} \delta u_i - \int_\Omega \sigma_{ij}\delta u_{i,j} + \int_\Omega f_i \delta u_i = 0.$$

- Due to stress tensor symmetry, the following equality holds:

$$\sigma_{ij}\delta u_{i,j} = \sigma_{ij}\delta \underbrace{\left(\frac{u_{i,j}+u_{j,i}}{2}\right)}_{E_{ij}} = \sigma_{ij}\delta E_{ij}.$$

- So the principle of virtual work reads,

$$\int_\Omega \sigma_{ij}\delta E_{ij} = \int_\Omega f_i \delta u_i + \int_{\partial\Omega} t_i \delta u_i$$

1.1. The Elastic Solid

The Principle of Virtual Work

- For the elastic solid, the principle of virtual work may be mathematically expressed as
$$\int_\Omega \underbrace{\sigma_{ij,j}\delta u_i}_{(\sigma_{ij}\delta u_i)_{,j}-\sigma_{ij}\delta u_{i,j}} + \int_\Omega f_i \delta u_i = 0.$$
- Applying Gauss divergence to the first term in the above we get,
$$\int_{\partial\Omega} \underbrace{\sigma_{ij} n_j}_{\text{surface traction } t_i} \delta u_i - \int_\Omega \sigma_{ij}\delta u_{i,j} + \int_\Omega f_i \delta u_i = 0.$$
- Due to stress tensor symmetry, the following equality holds:
$$\sigma_{ij}\delta u_{i,j} = \sigma_{ij}\delta \underbrace{\left(\frac{u_{i,j}+u_{j,i}}{2}\right)}_{E_{ij}} = \sigma_{ij}\delta E_{ij}.$$
- So the principle of virtual work reads,

$$\boxed{\text{"internal" contributions}} \longrightarrow \int_\Omega \sigma_{ij}\delta E_{ij} = \int_\Omega f_i \delta u_i + \int_{\partial\Omega} t_i \delta u_i \longleftarrow \boxed{\text{"external" contributions}}$$

1.1. The Elastic Solid

The Principle of Virtual Work

- The formula above is valid in the general case but further simplifications are possible for the non-dissipative solid. Here we know that a strain energy density $\mathcal{U}$ exists such that

$$\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial E_{ij}}, \quad \text{and} \int_\Omega \mathcal{U} = \mathrm{U}.$$

- Substituting this for the internal components we get,

$$\int_\Omega \sigma_{ij} \delta E_{ij} = \int_\Omega \frac{\partial \mathcal{U}}{\partial E_{ij}} \delta E_{ij} = \int_\Omega \delta \mathcal{U} = \delta \mathrm{U}.$$

- Denoting the external contributions by Π such that

$$\delta \Pi = \int_\Omega f_i \delta u_i + \int_{\partial\Omega} t_i \delta u_i,$$

  the principle of virtual work can be simply written as,

$$\boxed{\delta(\mathrm{U} - \Pi) = 0}.$$

- Note that while the strain energy U is **fully described by the system**, Π is loading-state dependent.

1.2. Interpretation

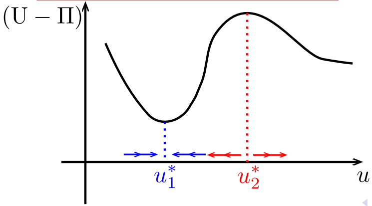
The Principle of Virtual Work

- $\delta(\mathrm{U}-\Pi)=0$ can be stated as
  For a system in equilibrium, the variation of total work done is zero.
- Intuitively, this means that
  the equilibrium deformation field of a system extremizes the quantity $\mathrm{U}-\Pi$:
  Out of all possible deformation fields that satisfy the constraints, the system deforms in a way that ***extremizes*** *the quantity* $\mathrm{U}-\Pi$.

Stability

- Suppose u^* is the equilibrium field, the "surplus energy" $\mathrm{U}-\Pi$ in its neighborhood governs its stability.
  - Consider the 1D example here with two extremal (equilibrium) points: u_1^* and u_2^*.
  - The local behavior of the function $\mathrm{U}-\Pi$ governs the stability of the equilibria.

$$\boxed{\text{Sign of } \frac{\partial(\mathrm{U}-\Pi)}{\partial u} = \text{- Sign of } force}$$



2. Beam Bending: A Motivating Problem Setting

- The kinematic and stress descriptions of a symmetric slender beam on the $(\underline{e}_1, \underline{e}_2)$ plane is:

$$u_1 = -X_2 v', \quad u_2 = v, \quad E_{11} = -X_2 E_y v'', \qquad \sigma_{11} = E_y E_{11}.$$

- The strain energy density in this case is

$$\mathcal{U} = \frac{E_y}{2} X_2^2 \left(v''\right)^2.$$

- Integrating this over the section $\mathcal{S}$ we get the linear density

$$d\mathrm{U} = \frac{E_y I_{33}}{2} \left(v''\right)^2.$$

- Considering transverse body force (per unit length) f and some point force F_P, the external energy is given as,

$$\Pi = \int_0^\ell f v + F_P v(X_P).$$

- Combining the two and integrating over the length we have,

$$\mathrm{U} - \Pi = F_P v(X_P) + \int_0^\ell \frac{E_y I_{33}}{2} \left(v''\right)^2 - f v.$$

2.1. Calculus of Variations

Beam Bending: A Motivating Problem Setting

- Ignoring the $F_P v(X_P)$ term for the moment, the principle of virtual work is given as,

$$\delta(\mathrm{U}-\Pi) = \delta\left(\int_0^\ell \frac{E_y I_{33}}{2}\left(v''\right)^2 - fv\right) = 0.$$

- This is a variational equation of the form,

$$\boxed{\delta\left(\int_{\mathcal{D}} \mathcal{L}(v, v', v'', \dots)\right) = 0}.$$

- Since δv are small quantities, we can apply Taylor's expansion on:

$$\int_{\mathcal{D}} \frac{\partial \mathcal{L}}{\partial v}\delta v + \frac{\partial \mathcal{L}}{\partial v'}\delta v' + \frac{\partial \mathcal{L}}{\partial v''}\delta v'' + \dots = 0.$$

- Applying integration by parts and observing that the variations $\delta v^{(n)}$ vanish at the boundaries, this simplifies as,

$$\int_{\mathcal{D}} \left(\frac{\partial \mathcal{L}}{\partial v} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial v'} + \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial v''} + \dots\right)\delta v = 0.$$

2.1. Calculus of Variations

Beam Bending: A Motivating Problem Setting

- Since the integral condition needs to be satisfied for all kinds of variations δu, the term within the parens must be zero:

$$\frac{\partial \mathcal{L}}{\partial v} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial v'} + \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial v''} + \cdots = 0.$$

- This is known as *Euler-Lagrange Equations* and forms the basis of variational mechanics. It is the *functional analog* of function extremization.

Regular Calculus

- Find $x \in \mathbb{R}$ such that $f(x) : \mathbb{R} \to \mathbb{R}$ is **extremized**.
- First order optimiality condition: $\frac{df}{dx} = 0$.
- Second order optimality: $\frac{d^2 f}{dx^2} :> 0(\text{min}), < 0(\text{max})$.

Calculus of Variations

- Find $v(x) : \mathbb{R} \to \mathbb{R}$ to extremize **functional** $J = \int \mathcal{L} dx$.
- First order optimality condition: $\frac{\partial \mathcal{L}}{\partial v} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial v'} + \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial v''} + \cdots = 0$.
- Second order optimality: `not so trivial`.

2.2. Variational Derivation of Euler Bernoulli Beam Theory

Beam Bending: A Motivating Problem Setting

- Returning to the slender beam, we have as the Lagrangian,

$$\mathcal{L} = \frac{E_y I_{33}}{2}(v'')^2 - fv.$$

- Applying the Euler-Lagrange Equations we obtain,

$$\underbrace{-f}_{\frac{\partial \mathcal{L}}{\partial v}} + \frac{d^2}{dX_1^2}\underbrace{\left(E_y I_{33} v''\right)}_{\frac{\partial \mathcal{L}}{\partial v''}} = 0.$$

- For a beam with uniform properties ($E_y I_{33}$ constant along the beam), the governing equations may be written as

$$\boxed{E_y I_{33} v'''' - f = 0},$$

which is precisely what we expect from Euler-Bernoulli Beam Theory.

So far we've just reinvented the wheel and not really shown an example where the Variational approach really shines. We will do this next.

3. Tutorial Example

- Consider the following fixed-free beam along with a spring support at the end.
- The energy quantities are,

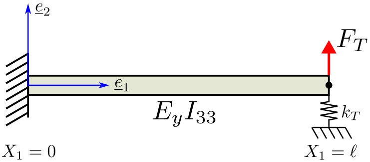
Remember the integral here.

$$\mathrm{U} = \int_0^\ell \frac{E_y I_{33}}{2} \left(v''\right)^2 dX_1 + \frac{1}{2} k_T v_T^2, \qquad \Pi = F_T v_T.$$

- Since the contributions from the load as well as the spring are from the boundaries, the bulk Equations of Motion (EoM) remains unchanged as: $E_y I_{33} v'''' = 0$.

Boundary Conditions

Clamped End	$v = 0,\ v' = 0,$	$X_1 = 0$
Loaded End	$v = v_T,\ v'' = 0,$	$X_1 = \ell$



- Solution in $X_1 \in (0, \ell)$ can be written in terms of v_T as:

$$v(X_1) = \frac{v_T}{2\ell^3} X_1^2 (3\ell - X_1).$$

3. Tutorial Example

- Substituting this back into the energy quantities and integrating it yields,

$$\mathrm{U} - \Pi = \underbrace{\frac{3E_y I_{33}}{2\ell^3} v_T^2}_{\int_0^\ell \frac{E_y I_{33}}{2}(v'')^2} + \frac{k_T}{2} v_T^2 - F_T v_T.$$

- Extremization of this quantity is trivial since everything just depends on a single unknown scalar, v_T. So we have,

$$\delta(\mathrm{U} - \Pi) = \delta v_T \frac{\partial}{\partial v_T}(\mathrm{U} - \Pi) = \delta v_T \left(\left(\frac{3E_y I_{33}}{\ell^3} + k_T \right) v_T - F_T \right).$$

- Setting this to zero for all δv_T implies

$$\boxed{v_T^* = \frac{F_T}{k_T + \frac{3E_y I_{33}}{\ell^3}}},$$

which is the equilibrium deflection.
- Plugging this back into the solution $v(X_1)$ above yields the full deformation shape.

4. Bending of Axially Loaded Beams

- We now draw upon the kinematic description from before for an axially loaded beam. The kinematics are,

$$u_1 = -X_2 v'(X_1), \quad u_2 = v(X_1).$$

- The corresponding axial strain is,

$$\begin{aligned} E_{11} &= u_{1,1} + \frac{1}{2}\left(u_{1,1}^2 + u_{2,1}^2\right) \\ &= -X_2 v'' + \frac{1}{2}\left(X_2^2 (v'')^2 + (v')^2\right). \end{aligned}$$

- Considering a loading case that leads to a uniform axial stress (σ_{11}) of $\frac{P}{A}$, the strain energy can be written as

$$\mathrm{U} = \int_0^\ell \left(\int_{\mathcal{S}} \frac{P}{A} E_{11} + \frac{E_y}{2} E_{11}^2 dA \right) dX_1$$

Retaining up to $\mathcal{O}(v^2)$ terms

$$= \int_0^\ell \frac{E_y I_{33}}{2} \left(1 + \frac{P}{E_y A}\right) (v'')^2 + \frac{P}{2} (v')^2 dX_1$$

4. Bending of Axially Loaded Beams

- Considering transverse loads $f(X_1)$, the external work done is

$$\Pi = \int_0^\ell f v dX_1.$$

- The Lagrangian density is

$$\mathcal{L} = \mathrm{U} - \Pi = \frac{E_y I_{33}}{2}\left(1 + \frac{P}{E_y A}\right)(v'')^2 + \frac{P}{2}(v')^2 - fv.$$

- The Euler-Lagrange equations (for stationarity) are

$$\boxed{E_y I_{33}\left(1 + \frac{P}{E_y A}\right) v'''' - Pv'' - f = 0}.$$

- We expect the term $\frac{P}{E_y A}$ to be small in general.

4.1. Cables and Beams

Bending of Axially Loaded Beams

- The first simple case we will consider is the case of extremely small cross-sections. Here it can be said that

$$A \sim \mathcal{O}(\epsilon^2), \qquad I_{33} \sim \mathcal{O}(\epsilon^4).$$

- For sufficiently small ϵ, contributions from I_{33} can be negligible. In general, when

$$E_y I_{33} \ll P,$$

  the following approximation may be made, wherein the equations governing transverse deflection become:

$$\boxed{Pv'' + f = 0}.$$

- This governs the deflection of cables **wherein the axial tension is the primary source of stiffness**. This is also a second order differential equation, unlike beam bending, which is a fourth order equation.

4.2. Column Buckling

Bending of Axially Loaded Beams

- Another simplification is possible for the absence of transverse loads. Here the general equations read:

$$E_y I_{33} v'''' - \frac{P}{1+\frac{P}{E_y A}} v'' = 0.$$

- Since $\frac{P}{E_y A} << 1$, we have

$$\frac{P}{1+\frac{P}{E_y A}} = P - \frac{P^2}{E_y A} + \frac{P^3}{E_y A^2} - \dots = P,$$

which leads to

$$\boxed{E_y I_{33} v'''' - P v'' = 0}.$$

- For compressive loading ($P \to -P$), this is the classical buckling equation

$$\boxed{E_y I_{33} v'''' + P v'' = 0}.$$

- Finding values of P where the trivial solution $v = 0$ is unstable leads to a Sturm-Liouville problem which will be treated in the **next module**.

References I

[1] C. Lanczos. The Variational Principles of Mechanics, Mathematical Expositions no. 4. Toronto: University of Toronto Press, 1949. ISBN: 978-1-4875-8177-0 978-1-4875-8305-7 (cit. on p. 2).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).