AS3020: Aerospace Structures

Module 5: Torsion of Beams

Instructor: Nidish Narayanaa Balaji

Dept. of Aerospace Engg., IIT-Madras, Chennai

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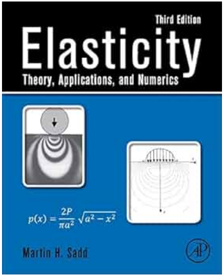
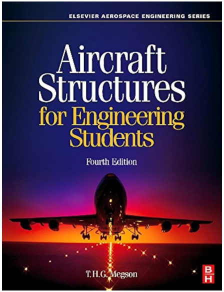


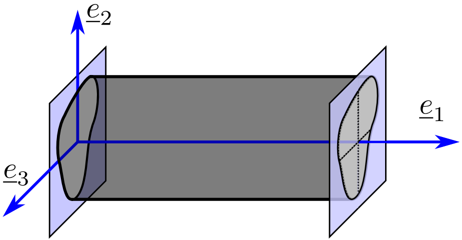
Chapter 9 in Sadd [1]

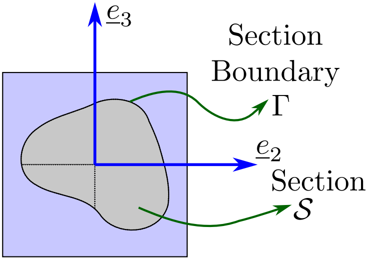


Chapters 3, 17-19 in Megson [2]

1. Solid Section Torsion

Basic Setup





- We assume:
  1. No direct stresses applied:
  $$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$$
  2. Sections "rotate rigidly":
  $$\gamma_{23} = 0 \implies \sigma_{23} = 0.$$
  3. Body is at equilibrium under constant torque applied at right end.
- We will denote the section by $\mathcal{S}$ and the section-boundary by Γ.

1.1. Stress Formulation

Solid Section Torsion

- Since we assume $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, the equilibrium equations read,

$$\sigma_{12,2} + \sigma_{13,3} = 0, \quad \sigma_{12,1} = 0, \quad \sigma_{13,1} = 0.$$

- We introduce the **Prandtl Stress Function** $\phi(X_2, X_3)$ (no dependence on X_1) such that

$$\sigma_{12} = \phi_{,3}, \quad \sigma_{13} = -\phi_{,2}.$$

  This satisfies equilibrium by definition.
- In terms of strains the above assumptions imply that we only have E_{12} and E_{13} active. **Recall** that Strain compatibility is $\epsilon_{mjk}\epsilon_{nil}E_{ij,mn} = 0$ (see Module 3).
- The non-trivial compatibility equations read,

$$\left.\begin{aligned} E_{12,23} - E_{13,22} &= 0 \\ E_{12,33} - E_{13,23} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} \phi_{,332} + \phi_{,222} &= 0 \\ \phi_{,333} + \phi_{,322} &= 0 \end{aligned}\right\} \implies \boxed{\nabla^2\phi = \text{constant}}.$$

- This is known as the **Poisson's problem**. What about Boundary Conditions?

1.1. Stress Formulation

Solid Section Torsion

- Since we assume $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, the equilibrium equations read,
$$\sigma_{12,2} + \sigma_{13,3} = 0, \quad \sigma_{12,1} = 0, \quad \sigma_{13,1} = 0.$$
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$$\sigma_{12} = \phi_{,3}, \quad \sigma_{13} = -\phi_{,2}.$$
This satisfies equilibrium by definition.
- In terms of strains the above assumptions imply t and E_{13} active. **Recall** that Strain compatibility (see Module 3).

Kinematic considerations will give us this "constant".

- The non-trivial compatibility equations read,
$$\left.\begin{aligned} E_{12,23} - E_{13,22} &= 0 \\ E_{12,33} - E_{13,23} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} \phi_{,332} + \phi_{,222} &= 0 \\ \phi_{,333} + \phi_{,322} &= 0 \end{aligned}\right\} \implies \boxed{\nabla^2\phi = \text{constant}}.$$
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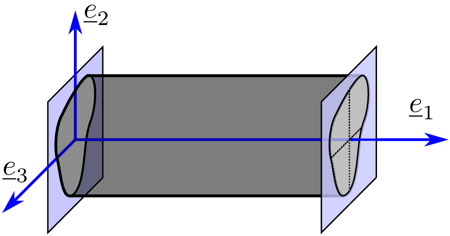
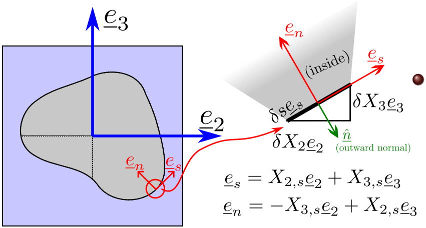
1.1. Stress Formulation

Solid Section Torsion





- We derive the coordinate transformation on the boundary as follows:

$$dX_2\underline{e}_2 + dX_3\underline{e}_3 = ds\underline{e}_s + dn\underline{e}_n$$

$$\implies \begin{bmatrix} dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_2, \underline{e}_n \rangle \\ \langle \underline{e}_3, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} ds \\ dn \end{bmatrix}$$

$$\text{and,} \begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_s \rangle \\ \langle \underline{e}_2, \underline{e}_n \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

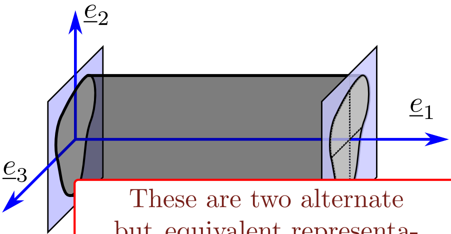
$$= \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

- Considering only Cartesian transformations (inverse has to be transpose), we will also have

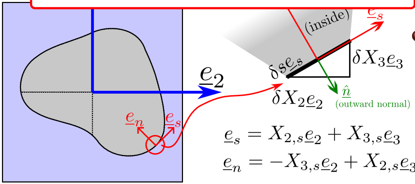
$$\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}.$$

1.1. Stress Formulation

Solid Section Torsion



These are two alternate but equivalent representations for $\underline{e}_s$ and $\underline{e}_n$ that we will invoke as convenient.



$$\underline{e}_s = X_{2,s}\underline{e}_2 + X_{3,s}\underline{e}_3$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$$

- We derive the coordinate transformation on the boundary as follows:

  $$dX_2\underline{e}_2 + dX_3\underline{e}_3 = ds\underline{e}_s + dn\underline{e}_n$$

  $$\implies \begin{bmatrix} dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_2, \underline{e}_n \rangle \\ \langle \underline{e}_3, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} ds \\ dn \end{bmatrix}$$

  $$\text{and, } \begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_s \rangle \\ \langle \underline{e}_2, \underline{e}_n \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

  $$= \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

- Considering only Cartesian transformations (inverse has to be transpose), we will also have

  $$\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}.$$

1.1. Derivation of Coordinate Transformation Relationships

Stress Formulation

- For cartesian transformations, the determinant has to be unity. So the inverse can be written as

$$\underbrace{\begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix}^{-1}}_{\mathbb{T}^{-1}} = \underbrace{\begin{bmatrix} X_{3,n} & -X_{3,s} \\ -X_{2,n} & X_{2,s} \end{bmatrix}}_{Adj(\mathbb{T})}.$$

- Also, for cartesian transformations, the inverse has to be the transpose of the matrix. So we have

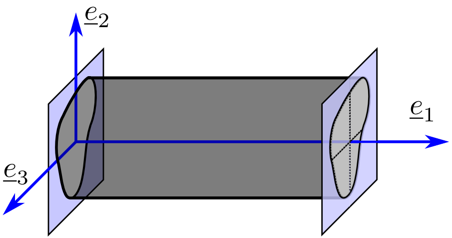
$$\underbrace{\begin{bmatrix} X_{2,s} & X_{2,n} \\ X_{3,s} & X_{3,n} \end{bmatrix}}_{\mathbb{T}^{T}} = \underbrace{\begin{bmatrix} X_{3,n} & -X_{3,s} \\ -X_{2,n} & X_{2,s} \end{bmatrix}}_{\mathbb{T}^{-1}}.$$

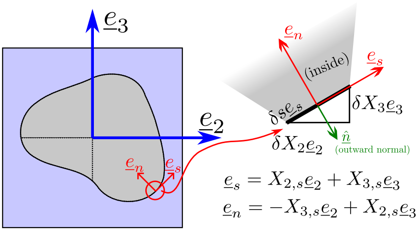
- So the following equalities make sense:

$$\boxed{\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}}, \text{ and } \boxed{\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \underbrace{\begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix}}_{\mathbb{T}^{-T}} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}}.$$

1.1. Stress Formulation

Solid Section Torsion





We invoke $\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$ here.

- Enforcing stress-free section boundary condtion leads to:

$$\begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} 0 \\ X_{3,s} \\ -X_{2,s} \end{bmatrix}}^{\hat{\underline{n}}=-\underline{e}_n} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \sigma_{12}X_{3,s} - \sigma_{13}X_{2,s} = 0$$

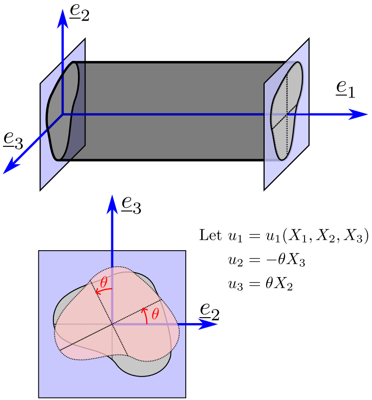
$$(\phi_{,3}X_{3,s} + \phi_{2}X_{2,s}) = \phi_{,s} = 0$$

- That is, on the section-boundary, the stress function is constant, set to 0 w.l.o.g.:

$$\phi = \cancelto{0}{constant} \quad \text{on } \Gamma.$$

1.2. Displacement Formulation

Solid Section Torsion



- The strains are,

$$\begin{aligned} E_{11} &= u_{1,1} = 0 \\ E_{22} &= -\theta_{,2} X_3 = 0 \\ E_{33} &= \theta_{,3} X_2 = 0 \\ 2E_{23} &= \theta - \theta = 0 \\ 2E_{12} &= u_{1,2} - \theta_{,1} X_3 = \frac{\sigma_{12}}{G} = \frac{\phi_{,3}}{G} \\ 2E_{13} &= u_{1,3} + \theta_{,1} X_2 = \frac{\sigma_{13}}{G} = -\frac{\phi_{,2}}{G} \end{aligned}$$

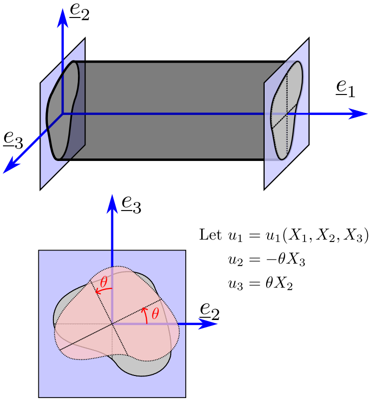
- Differentiating the strain expressions for σ_{12} and σ_{13} above allows us to write:

$$\boxed{\phi_{,kk} = -2G\theta_{,1}},$$

which gives us the "constant" required for the Poisson problem from before (along with the B.C. $\phi = 0 \text{ on } \Gamma$).

1.3. Section Moment

Solid Section Torsion



- The non-trivial shear strains are:

$$\sigma_{12} = \phi_{,3} \quad = G(u_{1,2} - X_3\theta_{,1})$$
$$\sigma_{13} = -\phi_{,2} = G(u_{1,3} + X_2\theta_{,1})$$

- The moment about $\underline{e}_1$ is

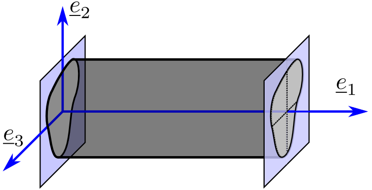
$$\boxed{M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12})dA}.$$

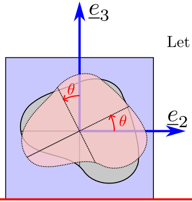
- Since σ_{12} and σ_{13} are expressed in terms of **kinematic quantities** as well as the **stress function** ϕ, we will write down relationships with both before proceeding.
- It is also obvious that $\phi_{,kk} = -2G\theta_{,1}$ implies

$$\boxed{u_{1,kk} = 0}.$$

1.3. Section Moment

Solid Section Torsion





Let $u_1 = u_1(X_1, X_2, X_3)$
$u_2 = -\theta X_3$
$u_3 = \theta X_2$

This is the governing equation in terms of the section-axial displacement field.

- The non-trivial shear strains are:

$$\sigma_{12} = \phi_{,3} \quad = G(u_{1,2} - X_3\theta_{,1})$$
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- It is also obvious that $\phi_{,kk} = -2G\theta_{,1}$ implies

$$\boxed{u_{1,kk} = 0}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA$$
$$= -\int_{\mathcal{S}} (\phi_{,2}X_2 + \phi_{,3}X_3) dA$$
$$\boxed{M_1 = 2\int_{\mathcal{S}} \phi dA}.$$

In terms of kinematic description

$$M_1 = G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2}) dA + G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2) dA}_{I_{11}}\,\theta_{,1}$$
$$= GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{1jk}X_ju_{1,k} dA$$
$$= GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{ijk}(X_ju_1)_{,k} dA - G\int_{\mathcal{S}} \cancel{\epsilon_{ijk}\delta_{jk}}u_1 dA$$
$$M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk}X_jn_ku_1 d|s|$$
$$\boxed{M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|s|}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA$$

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$$= GI_{11}\theta_{,1} + G\int \epsilon_{1jk}X_ju_{1,k}dA$$

$$\ldots u_1)_{,k}dA$$

$$\ldots \int_{\mathcal{S}}$$

This term is clearly zero for a perfectly circular section. What about other types?

$$M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk}X_jn_ku_1 d|s|$$

$$\boxed{M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|s|}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12})dA$$
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In terms of kinematic description

$$M_1 = G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2})dA + G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2)dA}_{I_{11}}\,\theta_{,1}$$
$$= GI_{11}\theta_{,1} + G\int \epsilon_{1jk}X_ju_{1,k}dA$$

This term is clearly zero for a perfectly circular section. What about other types?

Not zero in the general case.

$$\boxed{M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|s|}.$$

1.3. Section Moment: St. Venant's Warping Function

Solid Section Torsion

- For a "pure twist" condition, due to **translational symmetry**, u_1 can not depend on X_1. It also makes sense that u_1 has to be proportional to the twist θ somehow.
- Since θ depends on X_1, but $\theta_{,1}$ is a constant, St. Venant introduced a warping function $\psi(X_2, X_3)$ such that

$$\boxed{u_1 = \theta_{,1}\psi(X_2, X_3)}.$$

- Under this definition, the effective moment M_1 can be given as,

$$\boxed{M_1 = G\underbrace{\left(I_{11} + \int_\Gamma (\underline{X} \times \underline{n})_1 \psi d|s|\right)}_{J}\theta_{,1} = GJ\theta_{,1}}.$$

- Alternatively, J can also be written as,

$$\boxed{J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA}$$

1.3. Section Moment: St. Venant's Warping Function

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- Al

The product GJ is also known as **Torsional Rigidity**

$$\boxed{J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA}$$

1.4. Membrane Analogy

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\text{on}\, \Gamma,$$

along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

1.4. Membrane Analogy

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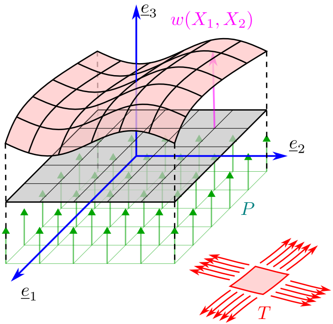
along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Uniform Planar Load Density P

- **The displacement field**
  $u_1 = 0, \quad u_2 = 0, \quad u_3 = w(X_1, X_2)$
- **The strain Field**
  $E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$
- **The Stress Field**
  $\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$



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- **The displacement field**
  $u_1 = 0, \quad u_2 = 0, \quad u_3 = w(X_1, X_2)$
- **The strain Field**
  $E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1} w_{,2}$
- **The Stress Field**
  $$\sigma_{11} = \frac{1}{t} T, \quad \sigma_{22} = \frac{1}{t} T.$$
- **Strain Energy Density (Integrated over thickness)**
  $$\mathcal{U} = \frac{1}{2}\left(w_{,1}^2 + w_{,2}^2\right) T + Pw$$
- Equations of Motion [a]:
  $\frac{\partial}{\partial X_k} \frac{\partial \mathcal{U}}{\partial w_{,k}} - \frac{\partial \mathcal{U}}{\partial w} = 0$:
  $$\boxed{T\left(w_{,11} + w_{,22}\right) - P = 0}$$

[a] Euler-Ostrogradsky

1.4. Membrane Analogy

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\text{on}\, \Gamma,$$

along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Un

The governing equations, therefore, are identical to that of a **membrane undergoing deformation under the action of a uniform area-load P.**

- **The displacem**

$u_1 = 0, \quad u_2 = 0$

- **The strain Field**

$$E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$$

- **The Stress Field**

$$\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$$

Energy Density

rated over thickness)

$$= \frac{1}{2}\left(w_{,1}^2 + w_{,2}^2\right)T + Pw$$

- Equations of Motion [a]:

$\frac{\partial}{\partial X_k}\frac{\partial \mathcal{U}}{\partial w_{,k}} - \frac{\partial \mathcal{U}}{\partial w} = 0$:

$$\boxed{T\left(w_{,11} + w_{,22}\right) - P = 0}$$

[a] Euler-Ostrogradsky

1.4. Membrane Analogy: Governing Equations of u_1 (Warping)

Solid Section Torsion

- The governing equations in terms of u_1 is the **Laplace equation**:

$$u_{1,kk} = 0,$$

and its boundary conditions **(Neumann B.C.s)** are written as (again based on zero traction at free end:

$$G\left\langle (u_{1,2} - X_3\theta_{,1})\underline{e}_2 + (u_{1,3} + X_2\theta_{,1})\underline{e}_3, \underline{e}_n \right\rangle = 0$$

$$\implies \langle u_{1,2}\underline{e}_2 + u_{1,3}\underline{e}_3, X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3\rangle$$

$$-\theta_{,1}\langle X_3\underline{e}_2 - X_2\underline{e}_3, -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3\rangle = 0$$

$$\implies \boxed{u_{1,n} = -\frac{\theta_{,1}}{2}\frac{d}{ds}\left(X_2^2 + X_3^2\right)} = -\theta_{,1}\left(X_3 \underbrace{X_{2,n}}_{-n_2} - X_2 \underbrace{X_{3,n}}_{-n_3}\right).$$

1.4. Membrane Analogy: Governing Equations of u_1 (Warping)

Solid Section Torsion

Note: We have used two different representations of $\underline{e}_n$ here:

$$\underline{e}_n = X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3, \text{ and}$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3.$$

Also, we are representing the outward normal as

$$\hat{\underline{n}} = n_2\underline{e}_2 + n_3\underline{e}_3 = -\underline{e}_n.$$

- The governing equ ... **ace equation**:

  and its boundary ... re written as (again based on zero trac ...

$$G\langle(u_{1,2} - \ldots 0$$
$$\implies \langle u_{1,2}\underline{e}_2 + \ldots$$
$$- \theta_{,1}\langle X_3\underline{e}_2 - X_2\underline{e}_3, -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3\rangle = 0$$
$$\implies \boxed{u_{1,n} = -\frac{\theta_{,1}}{2}\frac{d}{ds}\left(X_2^2 + X_3^2\right)} = -\theta_{,1}\left(X_3\underbrace{X_{2,n}}_{-n_2} - X_2\underbrace{X_{3,n}}_{-n_3}\right).$$

1.4. Membrane Analogy

Solid Section Torsion

Equations in the Stress Function

$$\nabla^2\phi = -2G\theta_{,1},$$
$$\phi = 0 \text{ on } \Gamma,$$
$$M_1 = 2\int_S \phi dA.$$

Equations in Warping

$$\nabla^2 u_1 = 0,$$
$$\frac{\partial u_1}{\partial n} = \theta_{,1}\left(X_3 n_2 - X_2 n_3\right) \text{ on } \Gamma.$$
$$M_1 = GJ\theta_{,1}$$

Relating the two

- Once we find ϕ, we can integrate the following to get u_1:

$$\frac{1}{G}\phi_{,3} = u_{1,2} - X_3\theta_{,1}$$
$$-\frac{1}{G}\phi_{,2} = u_{1,3} + X_2\theta_{,1}$$

1.4. Membrane Analogy

Solid Section Torsion

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$$-\frac{1}{G}\phi_{,2} = u_{1,3} + X_2\theta_{,1}$$

If interested, you can see the FreeFem scripts in the website for numerical implementations of these. You need to know just a little bit about weak forms to understand the code, it is very straightforward.

(not for exam)

1.5. Tutorial: Elliptical Section

Solid Section Torsion

- Let us consider an elliptical section and choose the stress function as

$$\phi = C\left(\frac{X_2^2}{a^2} + \frac{X_3^2}{b^2} - 1\right).$$

- The Laplacian of ϕ evaluates as,

$$\nabla^2\phi = 2C\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\theta_{,1} \implies C = -G\theta_{,1}\frac{a^2b^2}{a^2+b^2}.$$

- Let us first compute the total resultant twisting moment M_1 that this represents:

$$M_1 = 2\int_{\mathcal{S}}\phi = 2C\left(\frac{1}{a^2}\overbrace{\int_{\mathcal{S}}X_2^2 dA}^{\frac{\pi a^3 b}{4}} + \frac{1}{b^2}\overbrace{\int_{\mathcal{S}}X_3^2 dA}^{\frac{\pi a b^3}{4}} - \overbrace{\int_{\mathcal{S}}dA}^{\pi ab}\right) = -C\pi ab$$

$$\boxed{M_1 = G\frac{\pi a^3 b^3}{a^2+b^2}\theta_{,1}}.$$

1.5. Tutorial: Elliptical Section

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- Let us first compute the total resultant twisting moment M_1 that this represents:

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$$\boxed{M_1 = G\frac{\pi a^3 b^3}{a^2+b^2}\theta_{,1}}.$$

The torsional rigidity reads,

$$GJ = G\frac{\pi a^3 b^3}{a^2+b^2}$$

1.5. Tutorial: Elliptical Section

Solid Section Torsion

- For the axial deflection we have two equations (by equating shear stress expressions),

$$u_{1,2} = \theta_{,1}\psi_{,2} = -\frac{2a^2}{a^2+b^2}\theta_{,1}X_3 + \theta_{,1}X_3 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_3$$
$$u_{1,3} = \theta_{,1}\psi_{,3} = \frac{2b^2}{a^2+b^2}\theta_{,1}X_2 - \theta_{,1}X_2 \quad = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2$$

- Integrating them separately we have,

$$u_1 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2 + f_1(X_3)$$
$$= -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2 + f_2(X_2)$$

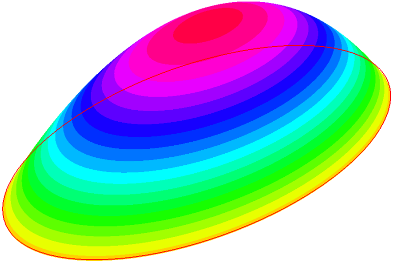
- f_1 and f_2 **have to be constant**. Setting it to zero we have,

$$\boxed{u_1 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2X_3 = -\frac{a^2-b^2}{G\pi a^3b^3}M_1X_2X_3}.$$

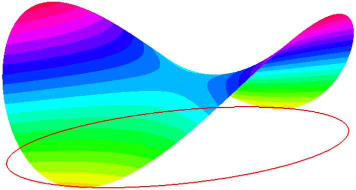
1.5. Tutorial: Elliptical Section

Solid Section Torsion

Stress Function



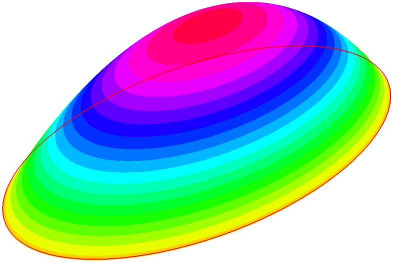
Section Warping



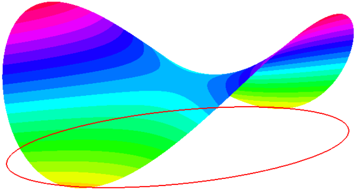
1.5. Tutorial: Elliptical Section

Solid Section Torsion

Stress Function



Section Warping



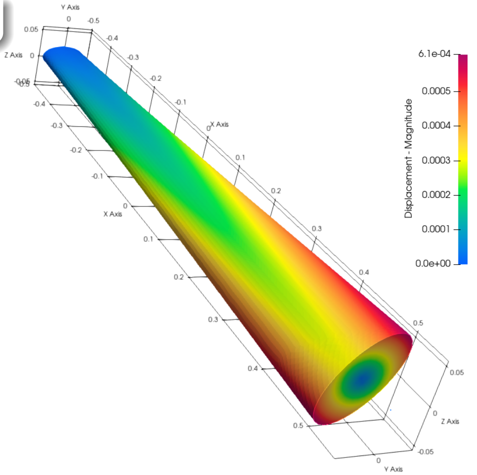
General Sections

- Torsion is amenable to analysis when the solid section boundary can be expressed in closed form **AND** its Laplacian evaluates to a constant. (See Chapter 9 in [1])
- Every assumed form of ϕ will give us a warping field. For an application wherein the section warping is also constrained, **this solution is not exact**. (St. Venant's principle can be invoked, however).
- Several analytical techniques exist (check [1] and references therein).
- Fully numerical approaches are also possible, see the FreeFem scripts in the website.

1.5. Tutorial: Elliptical Section: Results in 3D

Solid Section Torsion

Here is a 3D FE Result.
(Salome_Meca HDF Files in website)



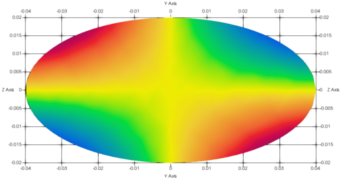
CODE aster
salome_meca 2023

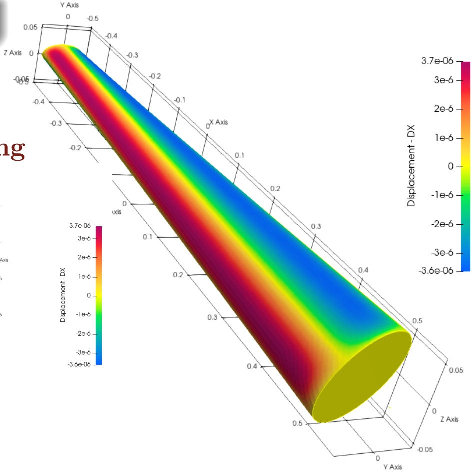
1.5. Tutorial: Elliptical Section: Results in 3D

Solid Section Torsion

Here is a 3D FE Result.
(Salome_Meca HDF Files in website)

Mid-Section Warping





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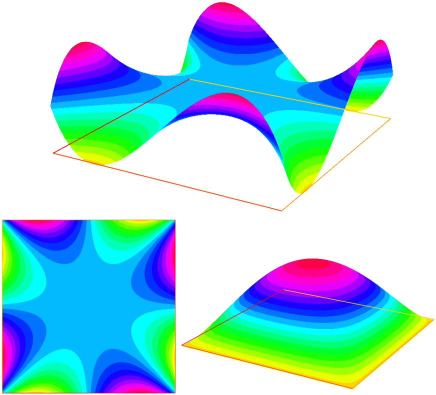
salome_meca 2023

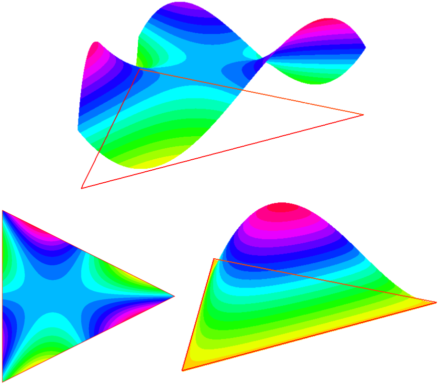
1.5. General Sections

Solid Section Torsion

Square Section



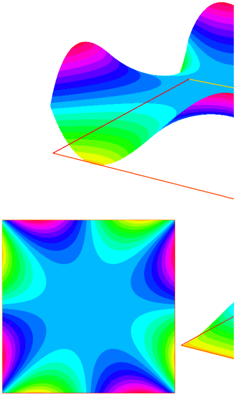
Triangular Section



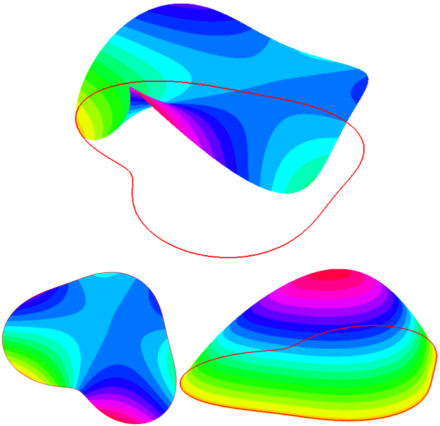
1.5. General Sections

Solid Section Torsion

Square Section



An Arbitrary Hand-drawn Section



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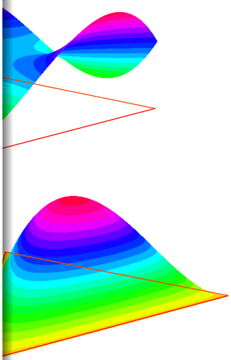
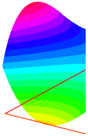
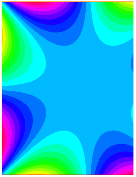


1.5. General Sections

Solid Section Torsion

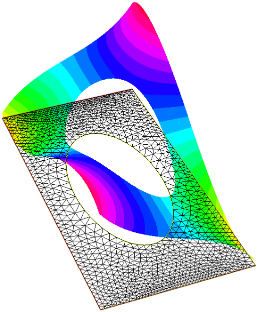
Square Sec





Sections with Holes

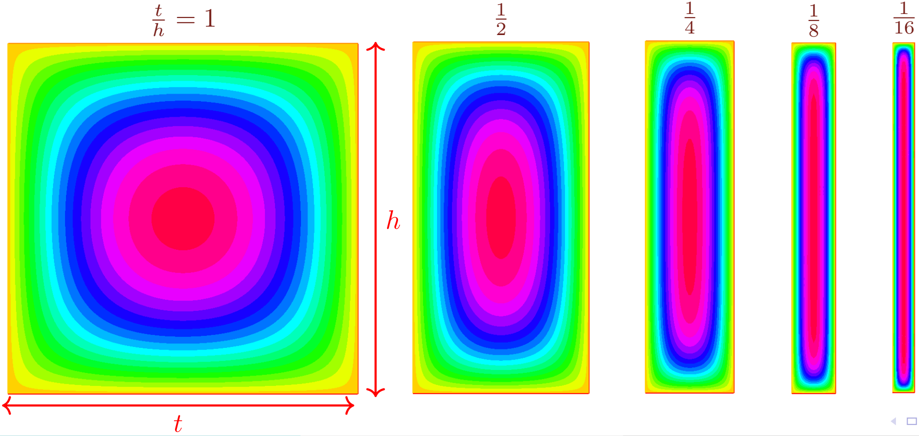
The validity of the governing equations extend beyond singly connected sections. Nothing stops us from applying it for multiply connected sections also for the warping formulation. (Some additional considerations necessary for the stress function, see sec. 9.3.3 in [1]).



1.6. Rectangular Sections

Solid Section Torsion

- Rectangular sections are slightly more involved, in general. But an important simplifcation is achieved for thin sections.
- Let us look at some numerical results for motivation (FreeFem code `b_rectangle.edp`).



1.6. Rectangular Sections

Solid Section Torsion

- Rectangul… But an important…
- Let us loo… Fem code b_rectang…

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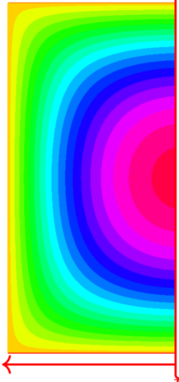
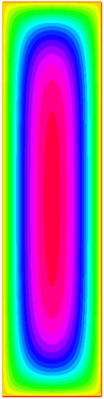
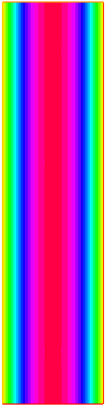


t

For the thin cases, we will approximate the shear function as

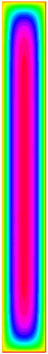


$\approx$

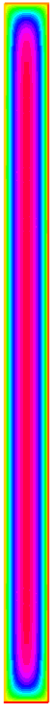


This will form the basis for the study of thin sections.

$\frac{1}{8}$



$\frac{1}{16}$



1.6. Rectangular Sections: Thin Strip Idealization

Solid Section Torsion

- Idealizing the rectangle as a "strip" (t/h is very small), we can write the stress function Poisson problem as,

$$\phi_{,22} = -2G\theta', \quad \text{with} \quad \phi = 0 \text{ at } X_2 \in \left\{-\frac{t}{2}, \frac{t}{2}\right\}, X_3 \in \left\{-\frac{h}{2}, \frac{h}{2}\right\},$$

solved by $\boxed{\phi(X_2, X_3) = -G\theta' \left(X_2^2 - \left(\frac{t}{2}\right)^2\right)}$.

- This implies the following shear stress and resultant moment:

$$\sigma_{12} = \overbrace{0}^{\phi_{,3}}, \qquad \sigma_{13} = \overbrace{2GX_2\theta'}^{-\phi_{,2}}, \qquad M_1 = 2\int_{\mathcal{S}} \phi dA = G\overbrace{\frac{ht^3}{3}}^{J}\theta'.$$

- The shear strain is $\gamma_{13} = u_{1,3} + u_{3,1} = u_{1,3} + X_2\theta_{,1}$, which implies $\boxed{u_1 = \theta' X_2 X_3}$ as the warping field (setting integration constant to zero).

1.6. Rectangular Sections: Thin Strip Idealization

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  $\frac{h}{2}\Big\}$,

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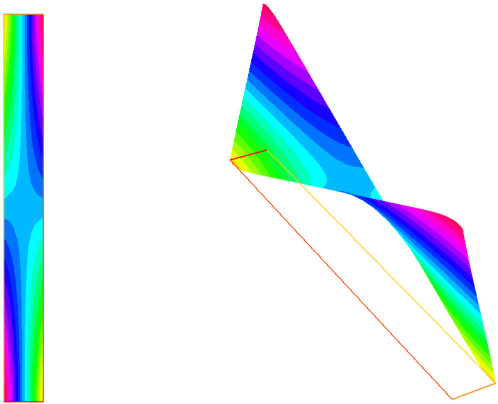
  $\boxed{u_1 = \theta' X_2 X_3}$ as the warping field (setting integration constant to zero).

Warping Profile



2. Torsion of Thin-Walled Sections

- Using the same notation as in Module 4, the equilibrium equations, for a thin-walled section undergoing pure torsion ($\sigma_{11} = \sigma_{ss} = \sigma_{nn} = \sigma_{sn} = 0$) can be written as

$$\cancelto{0}{\sigma_{11,1}} + \sigma_{1s,s} = 0, \quad \sigma_{1s,1} = 0.$$

- This implies, when in "pure torsion", $\sigma_{1s,s}$ is constant along the section arc.
- Since $q(s) = \int \sigma_{1s,s} dX_n$, this shows that **shear flow is constant across the section when it is under pure torsion.**
- The resultant moment of a shear flow distribution $q(s)$ can be given by

$$M_1 = \int_{\mathcal{S}} \underline{X} \times (q(s) ds \underline{e}_s) = q \int_{\mathcal{S}} p ds,$$

where p is the perpendicular distance to the point on the skin under consideration.

2. Torsion of Thin-Walled Sections

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> An important simplification occurs when $\mathcal{S}$ is a *closed section*. This leads to the **Bredt-Batho Formula**:
>
> $$M_1 = 2\mathcal{A}q.$$

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where p is the perpendicular distance to the point on the skin under consideration.

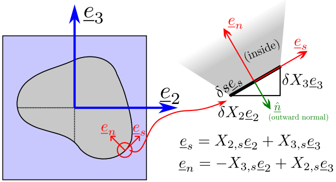
2.1. Transformation of Displacement Field to Skin-local Coordinates

Torsion of Thin-Walled Sections

We will consider the bending-torsion combined displacement field:

$$u_2 = v - X_3\theta$$
$$u_3 = w + X_3\theta,$$

and transform this to the **skin local coordinate system**.



- The section displacement field transforms as,

$$\begin{bmatrix} u_s \\ u_n \end{bmatrix} = \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}.$$

- The **tangential component** of displacement along the boundary Γ can be written as,

$$u_s = X_{2,s}(v - X_3\theta) + X_{3,s}(w + X_2\theta) = X_{2,s}v + X_{3,s}w + \theta \underbrace{(X_{3,s}X_2 - X_{2,s}X_3)}_{-X_n = p}$$

$$\implies \boxed{u_s = p\theta + vX_{2,s} + wX_{3,s}}.$$

2.1. Transformation of Displacement Field to Skin-local Coordinates

Torsion of Thin-Walled Sections

- The transformed displacement field combining bending and torsion is:

$$\left.\begin{aligned} u_1 &= -X_3 v' - X_2 w' + \theta' \psi \\ u_2 &= v - X_3 \theta \\ u_3 &= w + X_2 \theta \end{aligned}\right\} \implies \begin{aligned} &u_1 && \text{(unchanged)} \\ &u_s && = p\theta + vX_{2,s} + wX_{3,s} \\ &u_n && = X_s \theta - vX_{3,s} + wX_{2,s} \end{aligned}$$

- The shear strain along a thin section between the $\underline{e}_1$, $\underline{e}_s$ directions is

$$\gamma_{1s} = u_{1,s} + u_{s,1} = u_{1,s} + p\theta' + X_{2,s}v' + X_{3,s}w' = \frac{\tau}{G} = \frac{q(s)}{Gt}.$$

- Integrating this over the skin, we get

$$\begin{aligned} \int_0^s \frac{q(x)}{Gt} dx =& (u_1(s) - u_1(0)) + \theta' \int_0^s p dx + v' \int_0^s X_{2,x} dx + w' \int_0^s X_{3,x} dx \\ =& (u_1(s) - u_1(0)) + \theta' 2\mathcal{A}_{Os}(s) + v'(X_2(s) - X_2(0)) + w'(X_3(s) - X_3(0)). \end{aligned}$$

- Over a **completely closed section** we have, $\boxed{\oint \frac{q(s)}{Gt} ds = 2\mathcal{A}\theta'}$

2.2. Closed Sections: Bredt-Batho Theory

Torsion of Thin-Walled Sections

- For closed sections under *pure torsion*, we will set $v = w = 0$.
- So q is constant over the section and is written with the *Bredt-Batho Formula* based on the resultant twisting moment M_1 as

$$M_1 = 2\mathcal{A}q \implies q = \frac{M_1}{2\mathcal{A}}.$$

- The shear flow integral reads,

$$q \underbrace{\int_0^s \frac{1}{Gt} dx}_{\delta_{Os}(s)} = (u_1(s) - u_1(0)) + \theta' \underbrace{\int_0^s p dx}_{2\mathcal{A}_{Os}(s)} .$$

For the whole section, this becomes

$$q \oint \frac{1}{Gt} ds = \theta' 2\mathcal{A} \implies \theta' = \frac{q}{2\mathcal{A}} \oint \frac{1}{Gt} ds.$$

- So we can write the warping as

$$u_1(s) - u_1(0) = \overbrace{\frac{M_1 \delta}{2\mathcal{A}}}^{q\delta} \left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}} \right)$$

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Torsion of Thin-Walled Sections

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$$M_1 = 2\mathcal{A}q \implies q = \frac{M_1}{\ldots}.$$

- Th…

F…

The integration constant $u_1(0)$ can be found by enforcing $\sigma_{11} = 0$ on the section after assuming $\sigma_{11} \propto u_1$. So $\oint u_1(s)ds = 0$ in the section, leading to:

$$u_1(0) = \frac{\oint u_{10}(s)tds}{\oint tds},$$

where $u_{10}(s)$ is the warping distriution assuming $u_1(0) = 0$.

- So we can write the warping as

$$u_1(s) - u_1(0) = \overbrace{\frac{M_1\delta}{2\mathcal{A}}}^{q\delta}\left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}}\right)$$

2.2. Closed Sections: Bredt-Batho Theory

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- The shear flow integral reads,

$$q\underbrace{\int_0^s \frac{1}{Gt}dx}_{\delta_{Os}(s)} = (u_1(s) - u_1(0)) + \theta' \underbrace{\int_0^s p dx}_{2\mathcal{A}_{Os}(s)}.$$

For the whole section, this becomes

$$q\oint \frac{1}{Gt}ds = \theta' 2\mathcal{A} \implies \boxed{\theta' = \frac{q}{2\mathcal{A}}\oint \frac{1}{Gt}ds.}$$

- So we can write the warping as

$$u_1(s) - u_1(0) = \frac{\overbrace{M_1\delta}^{q\delta}}{2\mathcal{A}}\left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}}\right)$$

2.2. Closed Sections: Bredt-Batho Theory

Torsion of Thin-Walled Sections

- For closed sections under *pure torsion*, we will set $v = w = 0$.
- ... is written with the *Bredt-Batho* ... ing moment M_1 as

$$\implies \boxed{q = \frac{M_1}{2\mathcal{A}}}.$$

- ... $- u_1(0)) + \theta' \underbrace{\int_0^s p dx}_{2\mathcal{A}_{Os}(s)}$.

$$\implies \boxed{\theta' = \frac{q}{2\mathcal{A}} \oint \frac{1}{Gt} ds}.$$

- ... $2\mathcal{A}\left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}}\right)$

Combining these two, we get the torsional rigidity:

$$M_1 = 2\mathcal{A}q = \frac{4A^2}{\delta}\theta'.$$

For constant G, t, we get,

$$M_1 = \frac{4A^2}{|\Gamma|} G t \theta' = G J \theta'$$

$$\implies \boxed{J = \frac{4tA^2}{|\Gamma|}}.$$

$|\Gamma|$ is the section circumference.

2.2. Closed Sections: The Neuber Beam

Torsion of Thin-Walled Sections

- A natural question arises: what should I do if I want to minimize/eliminate warping?
- We want to set $u_1(s) - u_1(0) = 0$, $\forall\, s \in \Gamma$. This implies:

$$\frac{\delta_{Os}(s)}{\delta} = \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}},$$

which is satisfied iff

$$\frac{1}{\delta} \overbrace{\frac{1}{Gt}}^{\frac{d\delta_{Os}(s)}{ds}} = \frac{p}{2\mathcal{A}}.$$

- This implies that the quantity pGt (modulus as well as thickness can vary along section) has to be a constant:

$$pGt = \frac{2\mathcal{A}}{\delta}.$$

- It is known as a Neuber Beam if this is satisfied. (eg., circular sections)

2.2. Closed Sections: The Shear Center

- Based on relating the kinematics to stress (through linear elastic constitutive relationships), we have written the shear flow integral as:

$$\oint \frac{q(s; \xi_2, \xi_3)}{Gt} ds = 2\mathcal{A}\theta'.$$

- Suppose, for a closed section, we evaluated the shear flow by the approach in Module 4. **Recall** that we required the resultant moment M_1 to be zero for this:

$$\oint p \overbrace{\left(q_b(s; \xi_2, \xi_3) + q_0(\xi_2, \xi_3)\right)}^{q(s;\xi_2,\xi_3)} ds = 0.$$

- We can not take it for granted that the section does not twist when no moment is applied. So we add this additional consideration in our definition of shear center. We posit that the resultant twist angle must also be zero when the shear resultants act along the shear center:

$$\theta' = 0 \implies \oint \frac{q_b(s; \xi_2, \xi_3) + q_0(\xi_2, \xi_3)}{Gt} ds = 0$$

- Considering V_2, V_3 separately, we can get 3 equations in the 3 unknowns and can solve it.

2.2. Closed Sections: The Shear Center

Torsion of Thin-Walled Sections

One possible sequence of analysis is this (for shear):

1. We choose some convenient point as origin, say $\mathcal{O}$.
2. We first obtain the "baseline" shear flow $q_b(s)$ using some arbitrary starting point for the shear flow integral.
3. We estimate q_0 by requiring zero twist:

$$\oint \frac{q_b(s)+q_0}{Gt} ds = 0 \implies \boxed{q_0 = -\frac{\oint \frac{q_b(s)}{Gt} ds}{\oint \frac{1}{Gt} ds}}.$$

4. We write down the resultant moment as

$$\oint p(q_b(s)+q_0(s)) ds = V_2(-\xi_3) + V_3(\xi_2).$$

The shear center coordinates (ξ_2, ξ_3) are estimated by comparing the coefficients of V_2 & V_3 in the above.

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3. We estimate q_0

$$\cdots \frac{\oint \frac{q_b(s)}{Gt} ds}{\oint \frac{1}{Gt} ds}.$$

Question: We never required the zero twist condition for open sections. Does this mean open sections can undergo twisting even when $M_1 = 0$?

4. We write down the resultant moment as

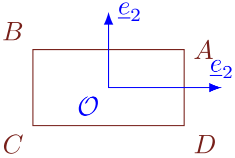
$$\oint p(q_b(s) + q_0(s)) ds = V_2(-\xi_3) + V_3(\xi_2).$$

The shear center coordinates (ξ_2, ξ_3) are estimated by comparing the coefficients of V_2 & V_3 in the above.

2.2. Closed Sections: Tutorial on Rectangular Closed Sections

Torsion of Thin-Walled Sections

- Consider this rectangular Section:



- We will write out the warping quantity $\frac{1}{2\mathcal{A}\theta'}(u(s) - u(0)) = \frac{\delta_{OS}(s)}{\delta} - \frac{\mathcal{A}_{OS}(s)}{\mathcal{A}}$ as a table in the following fashion:

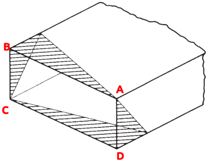
Section	$\delta_{OS}(s)$	$\mathcal{A}_{OS}(s)$	$\frac{\delta_{OS}(s)}{\delta} - \frac{\mathcal{A}_{OS}(s)}{\mathcal{A}}$	$\frac{1}{2\mathcal{A}\theta'}(u_{end} - u_{start})$
A→B	$\frac{\frac{a}{2}-X_2}{Gt}$	$\frac{b}{4}\left(\frac{a}{2} - X_2\right)$	$\frac{a-b}{4a(a+b)}\left(\frac{a}{2} - X_2\right)$	$\frac{a-b}{4(a+b)}$
B→C	$\frac{\frac{b}{2}-X_3}{Gt}$	$\frac{a}{4}\left(\frac{b}{2} - X_3\right)$	$-\frac{a-b}{4b(a+b)}\left(\frac{b}{2} - X_3\right)$	$-\frac{a-b}{4a(a+b)}$
C→D	$\frac{\frac{a}{2}+X_2}{Gt}$	$\frac{b}{4}\left(\frac{a}{2} + X_2\right)$	$\frac{a-b}{4a(a+b)}\left(\frac{a}{2} + X_2\right)$	$\frac{a-b}{4a(a+b)}$
D→A	$\frac{\frac{b}{2}+X_3}{Gt}$	$\frac{a}{4}\left(\frac{b}{2} + X_3\right)$	$-\frac{a-b}{4a(a+b)}\left(\frac{b}{2} + X_3\right)$	$-\frac{a-b}{4a(a+b)}$

2.2. Closed Sections: Tutorial on Rectangular Closed Section

- Letting u_A be some constant, we have the following:

$$u_B = u_A + 2\mathcal{A}\theta' \frac{a-b}{4(a+b)}, \quad uC = u_A, \quad u_D = u_A + 2\mathcal{A}\theta' \frac{a-b}{4(a+b)}.$$

- In each member, the warping function is distributed linearly in each member such that the warped shape looks like:



Figures from [2]

- Imposing zero net translation of section we get,

$$\oint u(s)ds = u_A 2(a+b) + \frac{a-b}{4} := 0 \implies u_A = -\frac{a-b}{8(a+b)}.$$

2.2. Closed Sections: Tutorial on Rectangular Closed Section

Torsion of Thin-Walled Sections

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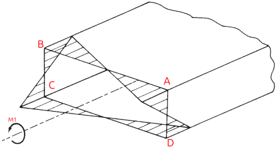

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2.3. Open Sections

Torsion of Thin-Walled Sections

- We will invoke the thin-strip idealization for this. The main results from the idealization are:

$$\phi = -G\theta' \left(X_2^2 - \frac{t^2}{4} \right); \quad M_1 = G\frac{ht^3}{3}\theta';$$

$$\sigma_{12} = 0, \quad \sigma_{13} = 2GX_2\theta', \quad u_1 = \theta' X_2 X_3.$$

- For general thin-walled sections, the torsion constant J is generalized as,

$$J = \frac{1}{3}\int_{\mathcal{S}} t^3 ds, \quad \text{s.t.} \quad M_1 = GJ\theta'.$$

Thin Section Kinematics

The kinematics of thin sections can be given as

$$u_s = -X_n\theta + vX_{2,s} + wX_{3,s} \xrightarrow{X_n=-p} p\theta + vX_{2,s} + wX_{3,s}$$

$$u_n = X_s\theta - vX_{3,s} + wX_{2,s} \xrightarrow{X_s=s} s\theta - vX_{3,s} + wX_{2,s}.$$

2.3. Open Sections: Warping

Torsion of Thin-Walled Sections

- Along the centerline $\sigma_{1n} = \sigma_{1s} = 0$ (**Note**: shear flow is zero under the idealization!). So we have (on the centerline),

$$\gamma_{1s} = 0 = u_{1,s} + u_{s,1} = u_{1,s} + p\theta',$$

where p is the perpendicular distance to the point on the skin. This can be integrated to

$$u_1(s) - u_1(0) = -\theta' \int_0^s p ds = -2\theta' \mathcal{A}_{Os}(s).$$

- $u_1(0)$ can be fixed based on enforcing the zero straight-stress ($\sigma_{11} = 0$, $\sigma_{11} \propto u_1$) assumption which leads to

$$\int_\Gamma u_1(s) ds = 0 \implies u_1(0) = \frac{1}{|\Gamma|} 2\theta' \int_\Gamma \mathcal{A}_{Os}(s) ds.$$

$|\Gamma|$ is the total *circumference*.

2.3. Open Sections

Torsion of Thin-Walled Sections

- For points off of the centerline, we consider $\sigma_{1n} = 0$, which implies,

$$\gamma_{1n} = u_{1,n} + u_{n,1} = u_{1,n} + s\theta' = 0 \implies u_{1,n} = -s\theta',$$

  where s is the position of the point along the skin (measured relative to the central line).

- This can be integrated to

$$u_1 = -\theta' ns + u_1(n = 0),$$

  where n is the position with respect to the centerline along $\underline{e}_n$.
    - Note that while $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$, we have $\underline{e}_s \times \underline{e}_n = \underline{e}_1$. Hence the negative sign in comparison to the thin-strip expression.

- $u_1(n = 0) = u_0 - 2\theta' \mathcal{A}_{Os}(s)$ from the centerline considerations above.

2.3. Open Sections

Torsion of Thin-Walled Sections

- In summary, the warping can be written in terms of section-local coordinates as,

$$\boxed{u_1 = \underbrace{u_0 - 2\mathcal{A}_{Os}(s)\theta'}_{u_1(n=0)} - \theta' n s}.$$

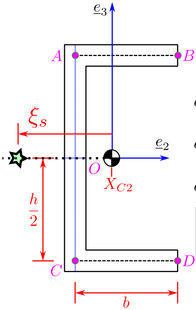
- The first term in the above, representing center-line warping, is known as **primary warping**, and the second term, representing section warping, is known as **secondary warping**.
- For sufficiently thin sections, the latter is usually neglected.

2.3. Open Sections Tutorial: C-Section

- Let us consider the C-Section from Module 4.
- We will shift the origin to the shear center and consider the integrals.
- The torsional rigidity is given by:

$$GJ = \frac{Gt^3}{3}\int_\Gamma ds = G\frac{t^3(h+2b)}{3}$$

- Warping is worked out as,



$$\boxed{I_{22} \approxeq \frac{(h^3+6bh^2)t}{12}} + \mathcal{O}(t^2)$$

$$q_{BA}(X_2) = -\frac{htV_3}{2I_{22}}(b-X_2)$$

$$q_{AC}(X_3) = -\frac{htV_3}{2I_{22}}\left(b+\frac{h}{4}\right) + \frac{tV_3}{2I_{22}}X_3^2$$

$$q_{CD}(X_2) = -\frac{htV_3}{2I_{22}}(b-X_2)$$

$$\boxed{M_1 = \oint pq ds = -\frac{b^2h^2tV_3}{4I_{22}} := -V_3\xi_s}$$

$$\boxed{\xi_s \approxeq \frac{3b^2}{h+6b} + \mathcal{O}(t)}$$

	$\mathcal{A}_{Os}(s)$	end
$B \to A$	$\frac{h}{2}(b+\xi_s-X_2)$	$\frac{bh}{2}$
$A \to C$	$-\xi_s(\frac{h}{2}-X_3)$	$-\xi_s h$
$C \to D$	$\frac{h}{2}(X_2-\xi_s)$	$\frac{bh}{2}$

- Using the table we can write:

$$u_b(s) = -\theta'\begin{cases}\frac{h}{2}(b+\xi_s-X_2) & B\to A\\ \frac{bh}{2}-\xi_s(\frac{h}{2}-X_3) & A\to C\\ \frac{bh}{2}-\frac{h}{2}(X_2-2\xi_s) & C\to D\end{cases}.$$

2.3. Open Sections Tutorial: C-Section I

Torsion of Thin-Walled Sections

- Since warping is linear in each segment, it is sufficient to look at points A, B, C, D to visualize it completely.
- Here we have:

$$u_B = 0, \quad u_A = -\theta' \frac{bh}{2}, \quad u_C = -\theta' \frac{bh}{2}\left(1 - 2\frac{\xi_s}{b}\right), \quad u_D = -\theta' \frac{bh}{2}\left(2 - 2\frac{\xi_s}{b}\right).$$

- The integral of warping over the complete section comes out to be

$$\begin{aligned}\int_\Gamma u_b ds &= -\theta' \left(\frac{b^2h}{4} + \frac{bh^2}{2}(1 - \frac{\xi_s}{b}) + \frac{b^2h}{4}(3 - 4\frac{\xi_s}{b})\right)\\ &= -\theta' \frac{bh(h+2b)}{2}\left(1 - \frac{\xi_s}{b}\right)\end{aligned}$$

- Requiring $\int_\Gamma u ds = 0$ implies, since $u = u_b + u_0$,

$$u_0 = -\frac{1}{|\Gamma|}\int_\Gamma u_b ds = \theta' \frac{bh}{2}\left(1 - \frac{\xi_s}{b}\right).$$

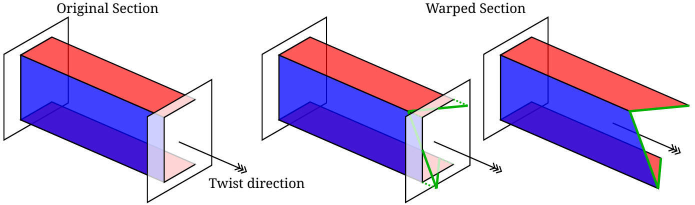
2.3. Open Sections Tutorial: C-Section II

- Notice that u_o is exactly the negative of the warping at the mid-point between points A and C (marked $\mathcal{O}$ in figure). The warping at this point is given by:

$$u_{\mathcal{O}} = \frac{u_A + u_C}{2} = -\theta' \frac{bh}{2}\left(1 - \frac{\xi_s}{b}\right).$$

- This implies that the section warps in such a manner as to ensure that point $\mathcal{O}$ does not move at all ($u_o + u_{\mathcal{O}} = 0$).
- Finally the warping function at the corner points come out to be,

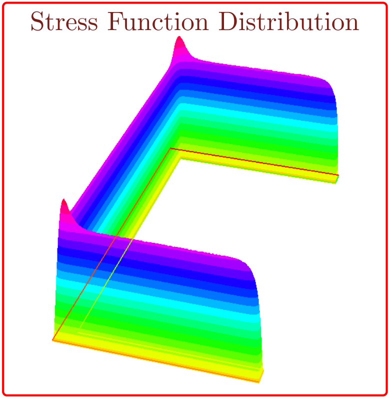
$$u_B = \theta' \frac{bh}{2}\left(1 - \frac{\xi_s}{b}\right), \quad u_A = -\theta' \frac{\xi_s h}{2}, \quad u_C = \theta' \frac{\xi_s h}{2}, \quad u_D = -\theta' \frac{bh}{2}\left(1 - \frac{\xi_s}{b}\right)$$

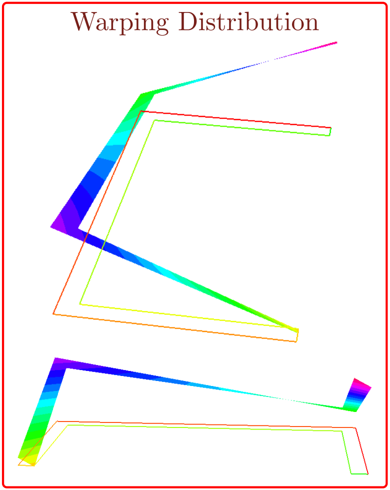


2.3. Open Sections Tutorial: C-Section

Torsion of Thin-Walled Sections

- Let us also illustrate the above with exact (numerical) results.





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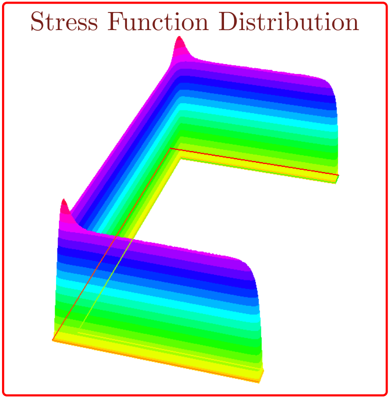

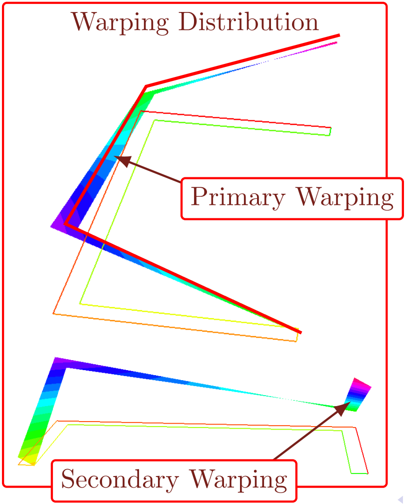

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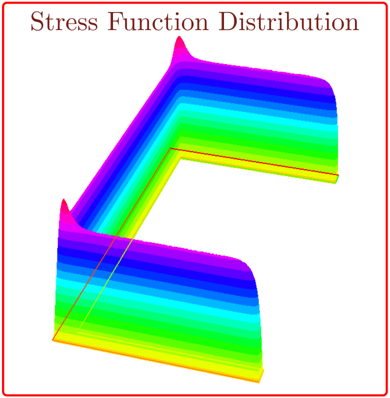

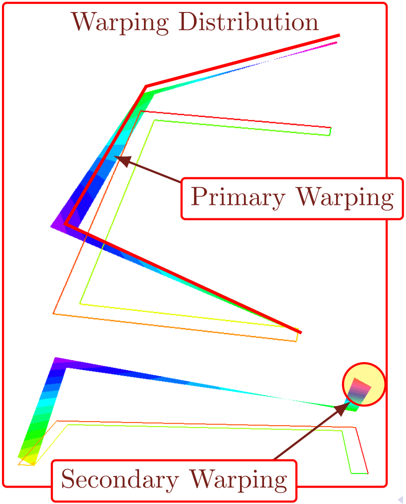

2.4. Combined Cells

Torsion of Thin-Walled Sections

- It is instructive to now take stock of what we have obtained so far. The moment-twist relationship is generically written by

$$M_1 = GJ\theta',$$

with J being the torsion constant.

Solid Sections

$$J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA$$

Closed Sections

$$J = \frac{4t\mathcal{A}^2}{|\Gamma|}$$

Open Sections

$$J = \frac{t^3|\Gamma|}{3}$$

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Let us consider the implications on a **Circular Section** of radius R.

Solid Section $J_s = I_{11} = \frac{\pi R^4}{2}$.

Closed Section $J_c = \frac{4t\times(\pi R^2)^2}{2\pi R} = 2\pi R^3 t$

Open Section $J_o = \frac{t^3}{3} 2\pi R = \frac{2\pi}{3} R t^3$

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Open Section $J_o = \frac{t^3}{3}2\pi R = \frac{2\pi}{3}Rt^3$

For $J_c = J_s$, we need $t = \frac{1}{4}R = 0.25R.$

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For $J_o = J_s$, we need $t = \sqrt[3]{\frac{3}{4}} R \approx 0.91R$.

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Open Section $J_o = \frac{t^3}{3} 2\pi R = \frac{2\pi}{3} R t^3$

For a given thickness,

$$\frac{J_o}{J_c} = \frac{1}{3}\left(\frac{t}{R}\right)^2 = \mathcal{O}(t^2).$$

2.4. Combined Cells

Torsion of Thin-Walled Sections

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$$M_1 = GJ\theta',$$

with J being the torsion constant.

So open sections can safely be ignored for torsion calculations in the combined context!

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Solid Sections

$J = I_{11} + \int_S X_2\psi_{,3} - X_3 \ldots$

Open Sections

$$J = \frac{t^3|\Gamma|}{3}$$

For shear, we can follow exactly the same procedure as in module 4 for combined sections.

Let us consider the … radius R.

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For a given thickness,

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3. Summary of Final Expressions

Solid Sections

$$J = I_{11} + \int_S X_2\psi_{,3} - X_3\psi_{,2} dA$$

$$u_1 = \theta'\psi(X_2, X_3)$$

Thin Strip Idealization

$$J = \frac{ht^3}{3}$$

$$u_1 = X_2 X_3 \theta'$$

Closed Sections

$$GJ = \frac{4\mathcal{A}^2}{\delta}$$

$$u_1(s) = u_0 + 2\mathcal{A}\theta' \left(\frac{\delta_{Os}(s)}{\delta} - \frac{\mathcal{A}_{Os}(s)}{\mathcal{A}} \right)$$

Open Sections

$$GJ = \frac{1}{3} \int_S G t^3 ds$$

$$u_1(s) = u_0 - 2\theta' \mathcal{A}_{Os}(s) - \theta' ns$$

$$\delta_{Os}(s) = \int_0^s \frac{1}{Gt} dx; \quad \mathcal{A}_{Os}(s) = \frac{1}{2} \int_0^s p dx$$

References I

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