AS3020: Aerospace Structures

Module 5: Torsion of Beams

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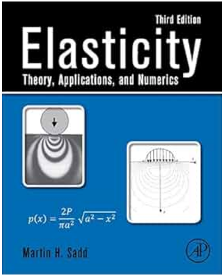
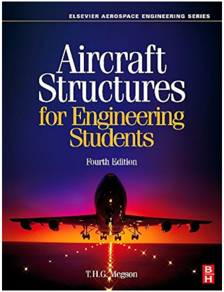


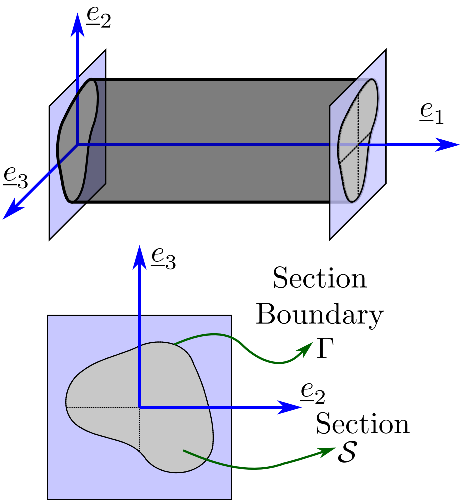
Chapter 7 in Sun [1]



Chapters 3, 18, 19 in Megson [2]

1. Solid Section Torsion

Basic Setup



- We assume:
  1. No direct stresses applied:
     $$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$$
  2. Sections "rotate rigidly":
     $$\gamma_{23} = 0 \implies \sigma_{23} = 0.$$
  3. Body is at equilibrium under constant torque applied at right end.
- We will denote the section by $\mathcal{S}$ and the section-boundary by Γ.

1.1. Stress Formulation

Solid Section Torsion

- Since we assume $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, the equilibrium equations read,
$$\sigma_{12,2} + \sigma_{13,3} = 0, \quad \sigma_{12,1} = 0, \quad \sigma_{13,1} = 0.$$
- We introduce the **Prandtl Stress Function** $\phi(X_2, X_3)$ (no dependence on X_1) such that
$$\sigma_{12} = \phi_{,3}, \quad \sigma_{13} = -\phi_{,2}.$$
This satisfies equilibrium by definition.
- In terms of strains the above assumptions imply that we only have E_{12} and E_{13} active. **Recall** that Strain compatibility is $\epsilon_{mjk}\epsilon_{nil}E_{ij,mn} = 0$ (see Module 3).
- The non-trivial compatibility equations read,
$$\left.\begin{aligned} E_{12,23} - E_{13,22} &= 0 \\ E_{12,33} - E_{13,23} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} \phi_{,332} + \phi_{,222} &= 0 \\ \phi_{,333} + \phi_{,322} &= 0 \end{aligned}\right\} \implies \boxed{\nabla^2\phi = \text{constant}}.$$
- This is known as the **Poisson's problem**. What about Boundary Conditions?

1.1. Stress Formulation

Solid Section Torsion

- Since we assume $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, the equilibrium equations read,
$$\sigma_{12,2} + \sigma_{13,3} = 0, \quad \sigma_{12,1} = 0, \quad \sigma_{13,1} = 0.$$
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This satisfies equilibrium by definition.
- In terms of strains the above assumptions imply t and E_{13} active. **Recall** that Strain compatibility (see Module 3).
- The non-trivial compatibility equations read,

Kinematic considerations will give us this "constant".

$$\left.\begin{aligned} E_{12,23} - E_{13,22} &= 0 \\ E_{12,33} - E_{13,23} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} \phi_{,332} + \phi_{,222} &= 0 \\ \phi_{,333} + \phi_{,322} &= 0 \end{aligned}\right\} \implies \boxed{\nabla^2\phi = \text{constant}}.$$

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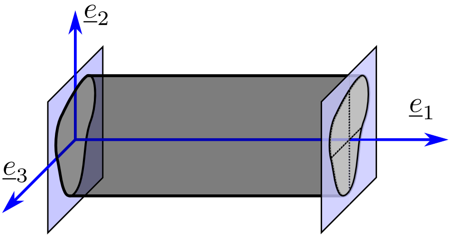
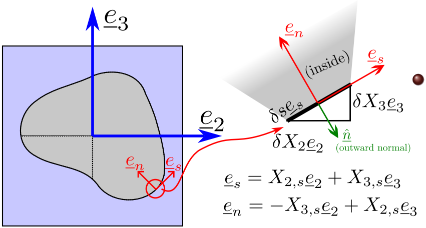
1.1. Stress Formulation

Solid Section Torsion





- We derive the coordinate transformation on the boundary as follows:

$$dX_2\underline{e}_2 + dX_3\underline{e}_3 = ds\underline{e}_s + dn\underline{e}_n$$

$$\implies \begin{bmatrix} dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_2, \underline{e}_n \rangle \\ \langle \underline{e}_3, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} ds \\ dn \end{bmatrix}$$

$$\text{and, } \begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_s \rangle \\ \langle \underline{e}_2, \underline{e}_n \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

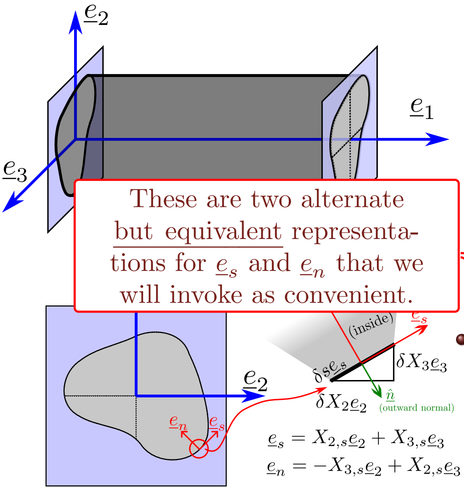
$$= \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

- Considering only Cartesian transformations (inverse has to be transpose), we will also have

$$\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}.$$

1.1. Stress Formulation

Solid Section Torsion



These are two alternate but equivalent representations for $\underline{e}_s$ and $\underline{e}_n$ that we will invoke as convenient.

$$\underline{e}_s = X_{2,s}\underline{e}_2 + X_{3,s}\underline{e}_3$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$$

- We derive the coordinate transformation on the boundary as follows:

$$dX_2\underline{e}_2 + dX_3\underline{e}_3 = ds\underline{e}_s + dn\underline{e}_n$$

$$\implies \begin{bmatrix} dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_2, \underline{e}_n \rangle \\ \langle \underline{e}_3, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} ds \\ dn \end{bmatrix}$$

$$\text{and, } \begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_s \rangle \\ \langle \underline{e}_2, \underline{e}_n \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

$$= \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

- Considering only Cartesian transformations (inverse has to be transpose), we will also have

$$\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}.$$

1.1. Derivation of Coordinate Transformation Relationships

Stress Formulation

- For cartesian transformations, the determinant has to be unity. So the inverse can be written as

$$\underbrace{\begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix}^{-1}}_{\mathbb{T}^{-1}} = \underbrace{\begin{bmatrix} X_{3,n} & -X_{3,s} \\ -X_{2,n} & X_{2,s} \end{bmatrix}}_{Adj(\mathbb{T})}.$$

- Also, for cartesian transformations, the inverse has to be the transpose of the matrix. So we have

$$\underbrace{\begin{bmatrix} X_{2,s} & X_{2,n} \\ X_{3,s} & X_{3,n} \end{bmatrix}}_{\mathbb{T}^{T}} = \underbrace{\begin{bmatrix} X_{3,n} & -X_{3,s} \\ -X_{2,n} & X_{2,s} \end{bmatrix}}_{\mathbb{T}^{-1}}.$$

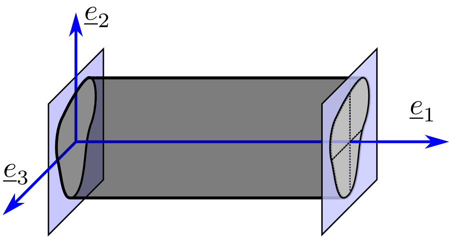
- So the following equalities make sense:

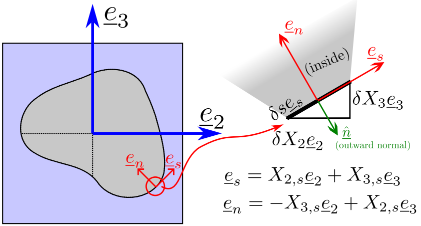
$$\boxed{\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}}, \text{ and } \boxed{\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{2,s} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}}.$$

$\mathbb{T}^{-T}$

1.1. Stress Formulation

Solid Section Torsion





$$\underline{e}_s = X_{2,s}\underline{e}_2 + X_{3,s}\underline{e}_3$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$$

> We invoke $\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$ here.

- Enforcing stress-free section boundary condtion leads to:

$$\begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} 0 \\ X_{3,s} \\ -X_{2,s} \end{bmatrix}}^{\underline{\hat{n}}=-\underline{e}_n} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \sigma_{12}X_{3,s} - \sigma_{13}X_{2,s} = 0$$

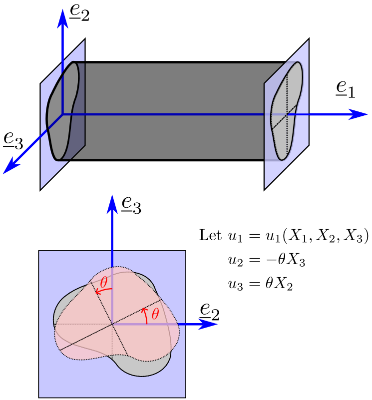
$$(\phi_{,3}X_{3,s} + \phi_{2}X_{2,s}) = \phi_{,s} = 0$$

- That is, on the section-boundary, the stress function is constant, set to 0 w.l.o.g.:

$$\phi = \cancel{constant}^{\,0} \quad \text{on } \Gamma.$$

1.2. Displacement Formulation

Solid Section Torsion



- The strains are,

$$
\begin{aligned}
E_{11} &= u_{1,1} = 0\\
E_{22} &= -\theta_{,2} X_3 = 0\\
E_{33} &= \theta_{,3} X_2 = 0\\
2E_{23} &= \theta - \theta = 0\\
2E_{12} &= u_{1,2} - \theta_{,1} X_3 = \frac{\sigma_{12}}{G} = \frac{\phi_{,3}}{G}\\
2E_{13} &= u_{1,3} + \theta_{,1} X_2 = \frac{\sigma_{13}}{G} = -\frac{\phi_{,2}}{G}
\end{aligned}
$$

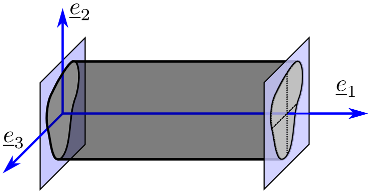
- Differentiating the strain expressions for σ_{12} and σ_{13} above allows us to write:

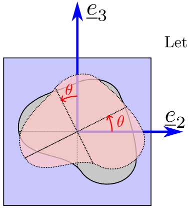
$$\boxed{\phi_{,kk} = -2G\theta_{,1}},$$

which gives us the "constant" required for the Poisson problem from before (along with the B.C. $\phi = 0 \text{ on } \Gamma$).

1.3. Section Moment

Solid Section Torsion





Let $u_1 = u_1(X_1, X_2, X_3)$
$u_2 = -\theta X_3$
$u_3 = \theta X_2$

- The non-trivial shear strains are:

$$\sigma_{12} = \phi_{,3} \quad = G(u_{1,2} - X_3\theta_{,1})$$
$$\sigma_{13} = -\phi_{,2} = G(u_{1,3} + X_2\theta_{,1})$$

- The moment about $\underline{e}_1$ is

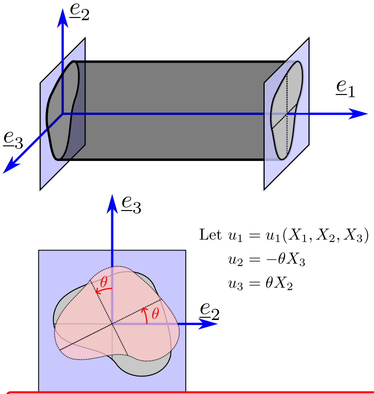
$$\boxed{M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA}.$$

- Since σ_{12} and σ_{13} are expressed in terms of **kinematic quantities** as well as the **stress function** ϕ, we will write down relationships with both before proceeding.
- It is also obvious that $\phi_{,kk} = -2G\theta_{,1}$ implies

$$\boxed{u_{1,kk} = 0}.$$

1.3. Section Moment

Solid Section Torsion



- The non-trivial shear strains are:

$$\sigma_{12} = \phi_{,3} \quad = G(u_{1,2} - X_3\theta_{,1})$$
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- It is also obvious that $\phi_{,kk} = -2G\theta_{,1}$ implies

$$\boxed{u_{1,kk} = 0}.$$

This is the governing equation in terms of the section-axial displacement field.

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12})dA$$
$$= -\int_{\mathcal{S}} (\phi_{,2}X_2 + \phi_{,3}X_3)dA$$
$$\boxed{M_1 = 2\int_{\mathcal{S}} \phi dA}.$$

In terms of kinematic description

$$\begin{aligned}
M_1 =& G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2})dA \\
&+ G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2)dA}_{I_{11}}\,\theta_{,1} \\
=& GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{1jk}X_ju_{1,k}dA \\
=& GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{ijk}(X_ju_1)_{,k}dA \\
&- G\int_{\mathcal{S}} \cancel{\epsilon_{ijk}\delta_{jk}}u_1dA \\
M_1 =& GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk}X_jn_ku_1d|s|
\end{aligned}$$

$$\boxed{M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} (\underline{X} \times \underline{n})_1u_1d|s|}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA$$

$$= -\int_{\mathcal{S}} (\phi_{,2}X_2 + \phi_{,3}X_3) dA$$

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In terms of kinematic description

$$M_1 = G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2}) dA$$

$$+ G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2) dA}_{I_{11}}\,\theta_{,1}$$

$$= GI_{11}\theta_{,1} + G\int \epsilon_{1jk}X_ju_{1,k} dA$$

$$\ldots u_1)_{,k} dA$$

This term is clearly zero for a perfectly circular section. What about other types?

$$M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk}X_jn_ku_1 d|s|$$

$$\boxed{M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|s|}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA$$

$$= -\int_{\mathcal{S}} (\phi_{,2}X_2 + \phi_{,3}X_3) dA$$

$$\boxed{M_1 = 2\int_{\mathcal{S}} \phi dA}.$$

In terms of kinematic description

$$M_1 = G\int_{\mathcal{S}} (X_2 u_{1,3} - X_3 u_{1,2}) dA + G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2) dA}_{I_{11}}\,\theta_{,1}$$

$$= G I_{11}\theta_{,1} + G\int \epsilon_{1jk} X_j u_{1,k} dA$$

This term is clearly zero for a perfectly circular section. What about other types?

Not zero in the general case.

$$\boxed{M_1 = G I_{11}\theta_{,1} + \boxed{G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|s|}}.$$

1.3. Section Moment: St. Venant's Warping Function

Solid Section Torsion

- For a "pure twist" condition, due to **translational symmetry**, u_1 can not depend on X_1. It also makes sense that u_1 has to be proportional to the twist θ somehow.
- Since θ depends on X_1, but $\theta_{,1}$ is a constant, St. Venant introduced a warping function $\psi(X_2, X_3)$ such that

$$\boxed{u_1 = \theta_{,1}\psi(X_2, X_3)}.$$

- Under this definition, the effective moment M_1 can be given as,

$$\boxed{M_1 = G \underbrace{\left(I_{11} + \int_\Gamma (\underline{X} \times \underline{n})_1 \psi d|s| \right)}_{J} \theta_{,1} = GJ\theta_{,1}}.$$

- Alternatively, J can also be written as,

$$\boxed{J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA}$$

1.3. Section Moment: St. Venant's Warping Function

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- Al

The product GJ is also known as **Torsional Rigidity**

$$\boxed{J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA}$$

1.4. Membrane Analogy

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\text{on}\, \Gamma,$$

along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

1.4. Membrane Analogy

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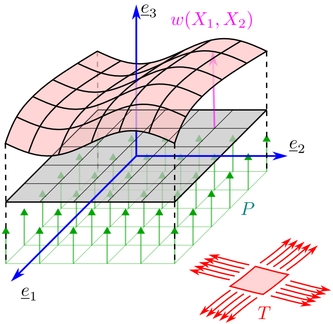
along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Uniform Planar Load Density P

- **The displacement field**
  $u_1 = 0, \quad u_2 = 0, \quad u_3 = w(X_1, X_2)$
- **The strain Field**
  $E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$
- **The Stress Field**
  $\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$



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Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Uniform Planar Load Density P

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- **The strain Field**
  $E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$
- **The Stress Field**
  $$\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$$
- **Strain Energy Density (Integrated over thickness)**
  $$\mathcal{U} = \frac{1}{2}\left(w_{,1}^2 + w_{,2}^2\right) T + Pw$$
- Equations of Motion [a]:
  $\frac{\partial}{\partial X_k}\frac{\partial \mathcal{U}}{\partial w_{,k}} - \frac{\partial \mathcal{U}}{\partial w} = 0$:
  $$\boxed{T\left(w_{,11} + w_{,22}\right) - P = 0}$$

[a] Euler-Ostrogradsky

1.4. Membrane Analogy

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\mathrm{on}\, \Gamma,$$

along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Un

The governing equations, therefore, are identical to that of a **membrane undergoing deformation under the action of a uniform area-load P.**

- **The displacem**

$u_1 = 0, \quad u_2 = 0$

- **The strain Field**

$$E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$$

- **The Stress Field**

$$\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$$

Energy Density
rated over thickness)

$$= \frac{1}{2}\left(w_{,1}^2 + w_{,2}^2\right)T + Pw$$

- Equations of Motion [a]:
$\frac{\partial}{\partial X_k}\frac{\partial \mathcal{U}}{\partial w_{,k}} - \frac{\partial \mathcal{U}}{\partial w} = 0$:

$$\boxed{T\left(w_{,11} + w_{,22}\right) - P = 0}$$

[a] Euler-Ostrogradsky

1.4. Membrane Analogy: Governing Equations of u_1 (Warping)

Solid Section Torsion

- The governing equations in terms of u_1 is the **Laplace equation**:

$$u_{1,kk} = 0,$$

and its boundary conditions **(Neumann B.C.s)** are written as (again based on zero traction at free end:

$$G \left\langle (u_{1,2} - X_3\theta_{,1})\underline{e}_2 + (u_{1,3} + X_2\theta_{,1})\underline{e}_3, \underline{e}_n \right\rangle = 0$$

$$\implies \langle u_{1,2}\underline{e}_2 + u_{1,3}\underline{e}_3, X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3 \rangle$$

$$- \theta_{,1}\langle X_3\underline{e}_2 - X_2\underline{e}_3, -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3 \rangle = 0$$

$$\implies \boxed{u_{1,n} = -\theta_{,1}\frac{d}{ds}\left(X_2^2 + X_3^2\right)} = -\frac{\theta_{,1}}{2}\left(X_3X_{2,n} - X_2X_{3,n}\right).$$

1.4. Membrane Analogy: Governing Equations of u_1 (Warping)

Solid Section Torsion

- The governing equations in terms of u_1 is the **Laplace equation**:

Note: We have used two different representations of $\underline{e}_n$ here:

$$\underline{e}_n = X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3, \text{ and}$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$$

and its boundary … re written as (again based on zero trac…

$$G\left\langle (u_{1,2} - X_3\theta_{,1})\underline{e}_2 + (u_{1,3} + X_2\theta_{,1})\underline{e}_3, \underline{e}_n \right\rangle = 0$$
$$\implies \left\langle u_{1,2}\underline{e}_2 + u_{1,3}\underline{e}_3, X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3 \right\rangle$$
$$-\theta_{,1}\left\langle X_3\underline{e}_2 - X_2\underline{e}_3, -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3 \right\rangle = 0$$
$$\implies \boxed{u_{1,n} = -\theta_{,1}\frac{d}{ds}\left(X_2^2 + X_3^2\right)} = -\frac{\theta_{,1}}{2}\left(X_3X_{2,n} - X_2X_{3,n}\right).$$

1.4. Membrane Analogy

Solid Section Torsion

Equations in the Stress Function

$$\nabla^2\phi = -2G\theta_{,1},$$
$$\phi = 0 \text{ on } \Gamma,$$
$$M_1 = 2\int_S \phi dA.$$

Equations in Warping

$$\nabla^2 u_1 = 0,$$
$$\frac{\partial u_1}{\partial n} = -\theta_{,1}\frac{d}{ds}\left(X_2^2 + X_3^2\right) \text{ on } \Gamma.$$
$$M_1 = GJ\theta_{,1}$$

Relating the two

- Once we find ϕ, we can integrate the following to get u_1:

$$\frac{1}{G}\phi_{,3} = u_{1,2} - X_3\theta_{,1}$$
$$-\frac{1}{G}\phi_{,2} = u_{1,3} + X_2\theta_{,1}$$

1.5. Tutorial: Elliptical Section

- Let us consider an elliptical section and choose the stress function as

$$\phi = C\left(\frac{X_2^2}{a^2} + \frac{X_3^2}{b^2} - 1\right).$$

- The Laplacian of ϕ evaluates as,

$$\nabla^2\phi = 2C\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\theta_{,1} \implies C = -G\theta_{,1}\frac{a^2b^2}{a^2+b^2}.$$

- Let us first compute the total resultant twisting moment M_1 that this represents:

$$M_1 = 2\int_{\mathcal{S}}\phi = 2C\left(\frac{1}{a^2}\overbrace{\int_{\mathcal{S}} X_2^2 dA}^{\frac{\pi a^3 b}{4}} + \frac{1}{b^2}\overbrace{\int_{\mathcal{S}} X_3^2 dA}^{\frac{\pi a b^3}{4}} - \overbrace{\int_{\mathcal{S}} dA}^{\pi ab}\right) = -C\pi ab$$

$$\boxed{M_1 = G\frac{\pi a^3 b^3}{a^2+b^2}\theta_{,1}}.$$

1.5. Tutorial: Elliptical Section

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$$\boxed{M_1 = G\frac{\pi a^3 b^3}{a^2+b^2}\theta_{,1}}.$$

The torsional rigidity reads,

$$GJ = G\frac{\pi a^3 b^3}{a^2+b^2}$$

1.5. Tutorial: Elliptical Section

- For the axial deflection we have two equations (by equating shear stress expressions),

$$u_{1,2} = \theta_{,1}\psi_{,2} = -\frac{2a^2}{a^2+b^2}\theta_{,1}X_3 + \theta_{,1}X_3 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_3$$
$$u_{1,3} = \theta_{,1}\psi_{,3} = \frac{2b^2}{a^2+b^2}\theta_{,1}X_2 - \theta_{,1}X_2 \quad = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2$$

- Integrating them separately we have,

$$u_1 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2 + f_1(X_3)$$
$$= -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2 + f_2(X_2)$$

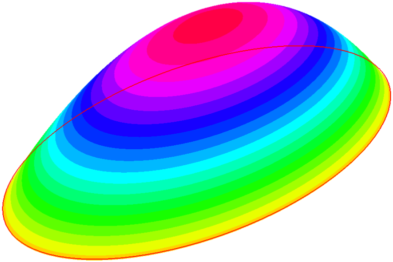
- f_1 and f_2 **have to be constant**. Setting it to zero we have,

$$\boxed{u_1 = -\frac{a^2-b^2}{a^2+b^2}\theta_{,1}X_2X_3 = -\frac{a^2-b^2}{G\pi a^3b^3}M_1X_2X_3}.$$

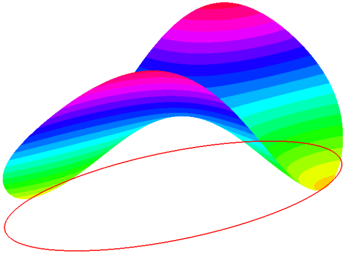
1.5. Tutorial

Solid Section Torsion

Stress Function



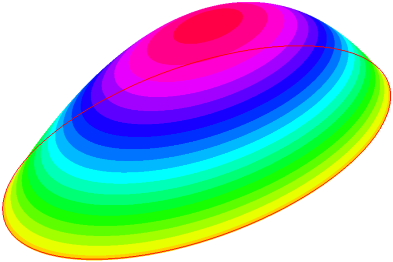
Section Warping



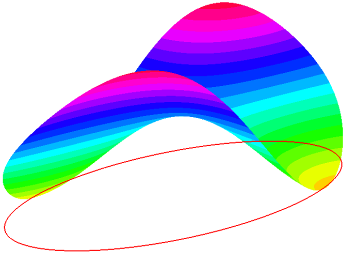
1.5. Tutorial

Solid Section Torsion

Stress Function



Section Warping



General Sections

- Torsion is amenable to analysis when the solid section boundary can be expressed in closed form **AND** its Laplacian evaluates to a constant. (See Chapter 9 in [3])
- Every assumed form of ϕ will give us a warping field. For an application wherein the section warping is also constrained, **this solution is not exact**. (St. Venant's principle can be invoked, however).
- Several analytical techniques exist (check [3] and references therein).
- Fully numerical approaches are also possible, see the FreeFem scripts in the website.

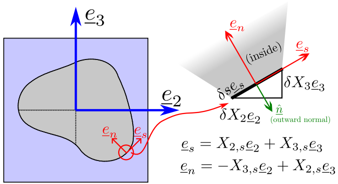
2. Thin Section Torsion

Transformation of Displacement Field to Skin-local Coordinates

We will consider the bending-torsion combined displacement field:

$$u_2 = v - X_3\theta$$
$$u_3 = w + X_3\theta,$$

and transform this to the **skin local coordinate system**.



- The section displacement field transforms as,

$$\begin{bmatrix} u_s \\ u_n \end{bmatrix} = \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{3,n} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}.$$

- The **tangential component** of displacement along the boundary Γ can be written as,

$$u_s = X_{2,s}(v - X_3\theta) + X_{3,s}(w + X_2\theta) = X_{2,s}v + X_{3,s}w + \theta \underbrace{(X_{3,s}X_2 - X_{2,s}X_3)}_{-X_n}$$

$$\implies \boxed{u_s = p\theta + vX_{2,s} + wX_{3,s}}.$$

2. Thin Section Torsion

Transformation of Displacement Field to Skin-local Coordinates

- The transformed displacement field combining bending and torsion is:

$$\left.\begin{aligned} u_1 &= -X_3 v' - X_2 w' + \theta' \psi \\ u_2 &= v - X_3 \theta \\ u_3 &= w + X_2 \theta \end{aligned}\right\} \implies \begin{aligned} &u_1 && \text{(unchanged)} \\ &u_s && = p\theta + v X_{2,s} + w X_{3,s} \\ &u_n && = X_s \theta - v X_{3,s} + w X_{2,s} \end{aligned}$$

- The shear strain along a thin section between the $\underline{e}_1$, $\underline{e}_s$ directions is

$$\gamma_{1s} = u_{1,s} + u_{s,1} = u_{1,s} + p\theta' + X_{2,s} v' + X_{3,s} w' = \frac{\tau}{G} = \frac{q(s)}{Gt}.$$

- Integrating this, we get

$$\begin{aligned} \int_0^s \frac{q(x)}{Gt} dx =& (u_1(s) - u_1(0)) + \theta' \int_0^s p dx + v' \int_0^s X_{2,x} dx + \int_0^s X_{3,x} dx \\ =& (u_1(s) - u_1(0)) + \theta' 2\mathcal{A}_{sweep}(s) + v'(X_2(s) - X_2(0)) \\ & + w'(X_3(s) - X_3(0)). \end{aligned}$$

- Over a **complete closed section** we have, $\boxed{\oint \frac{q(s)}{Gt} ds = 2\mathcal{A}\theta'}$

2.1. Open Sections

Thin Section Torsion

2.2. Closed Sections

Thin Section Torsion

2.3. Combined Cells

Thin Section Torsion

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[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).

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