AS3020: Aerospace Structures

Module 5: Torsion of Beams

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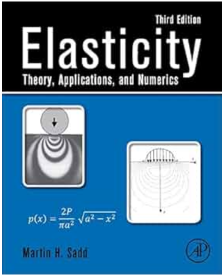
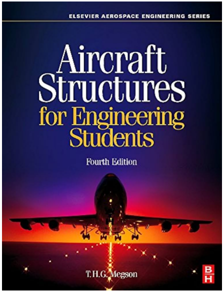
Table of Contents



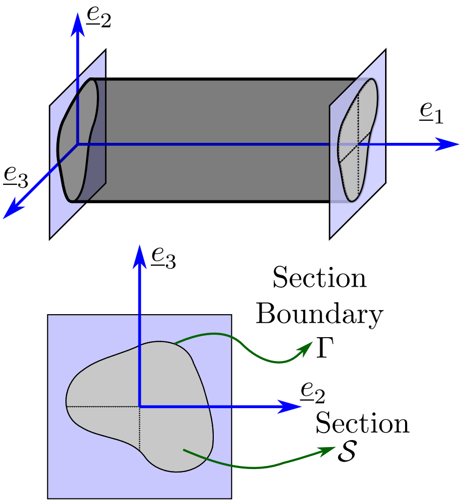
Chapter 7 in Sun [1]



Chapters 3, 18, 19 in Megson [2]

1. Solid Section Torsion

Basic Setup



- We assume:
  1. No direct stresses applied:
  $$\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$$
  2. Sections "rotate rigidly":
  $$\gamma_{23} = 0 \implies \sigma_{23} = 0.$$
  3. Body is at equilibrium under constant torque applied at right end.
- We will denote the section by $\mathcal{S}$ and the section-boundary by Γ.

1.1. Stress Formulation

Solid Section Torsion

- Since we assume $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, the equilibrium equations read,
$$\sigma_{12,2} + \sigma_{13,3} = 0, \quad \sigma_{12,1} = 0, \quad \sigma_{13,1} = 0.$$
- We introduce the **Prandtl Stress Function** $\phi(X_2, X_3)$ (no dependence on X_1) such that
$$\sigma_{12} = \phi_{,3}, \quad \sigma_{13} = -\phi_{,2}.$$
This satisfies equilibrium by definition.
- In terms of strains the above assumptions imply that we only have E_{12} and E_{13} active. **Recall** that Strain compatibility is $\epsilon_{mjk}\epsilon_{nil}E_{ij,mn} = 0$ (see Module 3).
- The non-trivial compatibility equations read,
$$\left.\begin{aligned} E_{12,23} - E_{13,22} &= 0 \\ E_{12,33} - E_{13,23} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} \phi_{,332} + \phi_{,222} &= 0 \\ \phi_{,333} + \phi_{,322} &= 0 \end{aligned}\right\} \implies \boxed{\nabla^2\phi = \text{constant}}.$$
- This is known as the **Poisson's problem**. What about Boundary Conditions?

1.1. Stress Formulation

Solid Section Torsion

- Since we assume $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, the equilibrium equations read,
$$\sigma_{12,2} + \sigma_{13,3} = 0, \quad \sigma_{12,1} = 0, \quad \sigma_{13,1} = 0.$$
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$$\sigma_{12} = \phi_{,3}, \quad \sigma_{13} = -\phi_{,2}.$$
This satisfies equilibrium by definition.
- In terms of strains the above assumptions imply t and E_{13} active. **Recall** that Strain compatibility (see Module 3).
- The non-trivial compatibility equations read,

Kinematic considerations will give us this "constant".

$$\left.\begin{aligned} E_{12,23} - E_{13,22} &= 0 \\ E_{12,33} - E_{13,23} &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} \phi_{,332} + \phi_{,222} &= 0 \\ \phi_{,333} + \phi_{,322} &= 0 \end{aligned}\right\} \implies \boxed{\nabla^2\phi = \text{constant}}.$$

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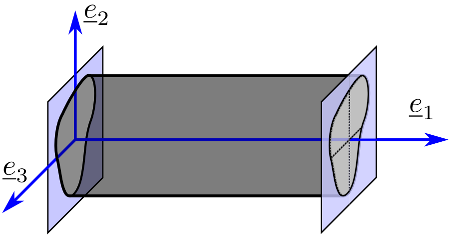
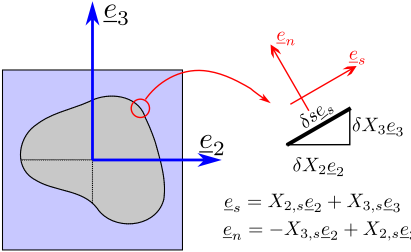
1.1. Stress Formulation

Solid Section Torsion





$$\underline{e}_s = X_{2,s}\underline{e}_2 + X_{3,s}\underline{e}_3$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$$

- We derive the coordinate transformation on the boundary as follows:

$$dX_2\underline{e}_2 + dX_3\underline{e}_3 = ds\underline{e}_s + dn\underline{e}_n$$

$$\implies \begin{bmatrix} dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_2, \underline{e}_n \rangle \\ \langle \underline{e}_3, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} ds \\ dn \end{bmatrix}$$

$$\text{and, } \begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_s \rangle \\ \langle \underline{e}_2, \underline{e}_n \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

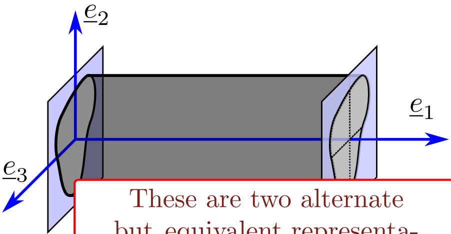
$$= \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

- Considering only Cartesian transformations (inverse has to be transpose), we will also have

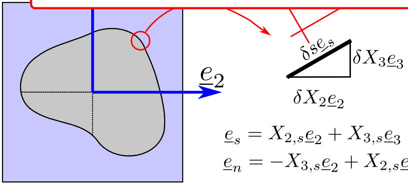
$$\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}.$$

1.1. Stress Formulation

Solid Section Torsion



These are two alternate but equivalent representations for $\underline{e}_s$ and $\underline{e}_n$ that we will invoke as convenient.



$$\underline{e}_s = X_{2,s}\underline{e}_2 + X_{3,s}\underline{e}_3$$
$$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$$

- We derive the coordinate transformation on the boundary as follows:

  $$dX_2\underline{e}_2 + dX_3\underline{e}_3 = ds\underline{e}_s + dn\underline{e}_n$$

  $$\implies \begin{bmatrix} dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_2, \underline{e}_n \rangle \\ \langle \underline{e}_3, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} ds \\ dn \end{bmatrix}$$

  $$\text{and, } \begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} \langle \underline{e}_2, \underline{e}_s \rangle & \langle \underline{e}_3, \underline{e}_s \rangle \\ \langle \underline{e}_2, \underline{e}_n \rangle & \langle \underline{e}_3, \underline{e}_n \rangle \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

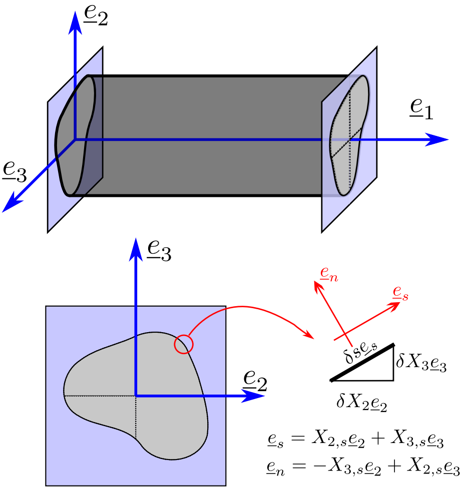
  $$= \begin{bmatrix} X_{2,s} & X_{3,s} \\ X_{2,n} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}$$

- Considering only Cartesian transformations (inverse has to be transpose), we will also have

  $$\begin{bmatrix} \underline{e}_s \\ \underline{e}_n \end{bmatrix} = \begin{bmatrix} X_{3,n} & -X_{2,n} \\ -X_{3,s} & X_{3,n} \end{bmatrix} \begin{bmatrix} \underline{e}_2 \\ \underline{e}_3 \end{bmatrix}.$$

1.1. Stress Formulation

Solid Section Torsion



We invoke
$\underline{e}_n = -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3$ here.

- Enforcing stress-free section boundary condtion leads to:

$$\begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -X_{3,s} \\ X_{2,s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies -\sigma_{12} X_{3,s} + \sigma_{13} X_{2,s} = 0$$

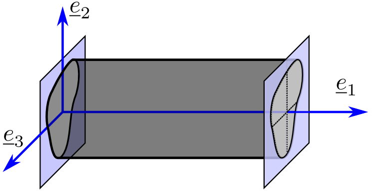
$$-(\phi_{,3} X_{3,s} + \phi_2 X_{2,s}) = -\phi_{,s} = 0$$

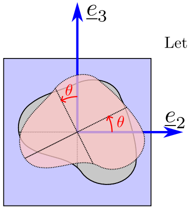
- That is, on the section-boundary, the stress function is constant, set to 0 w.l.o.g.:

$$\phi = \cancelto{0}{constant} \quad \text{on } \Gamma.$$

1.2. Displacement Formulation

Solid Section Torsion





Let $u_1 = u_1(X_1, X_2, X_3)$
$u_2 = -\theta X_3$
$u_3 = \theta X_2$

- The strains are,

$$\begin{aligned} E_{11} &= u_{1,1} = 0 \\ E_{22} &= -\theta_{,2} X_3 = 0 \\ E_{33} &= \theta_{,3} X_2 = 0 \\ 2E_{23} &= \theta - \theta = 0 \\ 2E_{12} &= u_{1,2} - \theta_{,1} X_3 = \frac{\sigma_{12}}{G} = \frac{\phi_{,3}}{G} \\ 2E_{13} &= u_{1,3} + \theta_{,1} X_2 = \frac{\sigma_{13}}{G} = -\frac{\phi_{,2}}{G} \end{aligned}$$

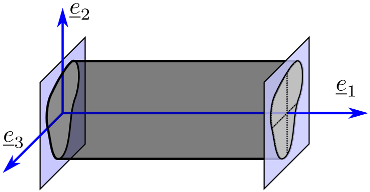
- Differentiating the strain expressions for σ_{12} and σ_{13} above allows us to write:

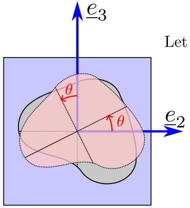
$$\boxed{\phi_{,kk} = -2G\theta_{,1}},$$

which gives us the "constant" required for the Poisson problem from before (along with the B.C. $\phi = 0 \text{ on } \Gamma$).

1.3. Section Moment

Solid Section Torsion





- The non-trivial shear strains are:

$$\sigma_{12} = \phi_{,3} \quad = G(u_{1,2} - X_3\theta_{,1})$$
$$\sigma_{13} = -\phi_{,2} = G(u_{1,3} + X_2\theta_{,1})$$

- The moment about $\underline{e}_1$ is

$$\boxed{M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12})dA}.$$

- Since σ_{12} and σ_{13} are expressed in terms of **kinematic quantities** as well as the **stress function** ϕ, we will write down relationships with both before proceeding.
- Firstly, it is obvious that $\phi_{,kk} = 0$ implies

$$u_{1,kk} = 0.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12})dA$$
$$= -\int_{\mathcal{S}} (\phi_{,2}X_2 + \phi_{,3}X_3)dA$$
$$\boxed{M_1 = -2\int_{\mathcal{S}} \phi dA}.$$

In terms of kinematic description

$$\begin{aligned}
M_1 =& G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2})dA \\
&+ G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2)dA}_{I_{11}}\,\theta_{,1} \\
=& GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{1jk}X_j u_{1,k} dA \\
=& GI_{11}\theta_{,1} + G\int_{\mathcal{S}} \epsilon_{ijk}(X_j u_1)_{,k} dA \\
&- G\int_{\mathcal{S}} \cancel{\epsilon_{ijk}\delta_{jk}} u_1 dA \\
M_1 =& GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk}X_j n_k u_1 d|\ell|
\end{aligned}$$

$$\boxed{M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|\ell|}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12})dA$$

$$= -\int_{\mathcal{S}} (\phi_{,2}X_2 + \phi_{,3}X_3)dA$$

$$\boxed{M_1 = -2\int_{\mathcal{S}} \phi dA}.$$

In terms of kinematic description

$$M_1 = G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2})dA$$

$$+ G\underbrace{\int_{\mathcal{S}} (X_2^2 + X_3^2)dA}_{I_{11}}\,\theta_{,1}$$

$$= GI_{11}\theta_{,1} + G\int \epsilon_{1jk}X_ju_{1,k}dA$$

$$u_1)_{,k}dA$$

This term is clearly zero for a perfectly circular section. What about other types?

$$M_1 = GI_{11}\theta_{,1} + G\int_{\Gamma} \epsilon_{1jk}X_jn_ku_1d|\ell|$$

$$\boxed{M_1 = GI_{11}\theta_{,1} + \boxed{G\int_{\Gamma} (\underline{X}\times\underline{n})_1u_1d|\ell|}}.$$

1.3. Section Moment

Solid Section Torsion

In terms of stress function

$$M_1 = \int_{\mathcal{S}} (X_2\sigma_{13} - X_3\sigma_{12}) dA$$

$$= -\int_{\mathcal{S}} (\phi_{,2}X_2 + \phi_{,3}X_3) dA$$

$$\boxed{M_1 = -2\int_{\mathcal{S}} \phi dA}.$$

In terms of kinematic description

$$M_1 = G\int_{\mathcal{S}} (X_2u_{1,3} - X_3u_{1,2}) dA + \underbrace{G\int_{\mathcal{S}} (X_2^2 + X_3^2) dA}_{I_{11}}\, \theta_{,1}$$

$$= GI_{11}\theta_{,1} + G\int \epsilon_{1jk}X_j u_{1,k} dA$$

This term is clearly zero for a perfectly circular section. What about other types?

Not zero in the general case.

$$\boxed{M_1 = GI_{11}\theta_{,1} + \boxed{G\int_{\Gamma} (\underline{X} \times \underline{n})_1 u_1 d|\ell|}}.$$

1.3. Section Moment: St. Venant's Warping Function

- For a "pure twist" condition, due to **translational symmetry**, u_1 can not depend on X_1. It also makes sense that u_1 has to be proportional to the twist θ somehow.
- Since θ depends on X_1, but $\theta_{,1}$ is a constant, St. Venant's introduced a warping function $\psi(X_2, X_3)$ such that

$$\boxed{u_1 = \theta_{,1}\psi(X_2, X_3)}.$$

- Under this definition, the effective moment M_1 can be given as,

$$\boxed{M_1 = G\underbrace{\left(I_{11} + \int_\Gamma (\underline{X} \times \underline{n})_1 \psi d|\ell|\right)}_{J}\theta_{,1} = GJ\theta_{,1}}.$$

- Alternatively, J can also be written as,

$$\boxed{J = I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2} dA}$$

1.4. Membrane Analogy: Contributions from Warping

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\text{on}\, \Gamma,$$

along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.

1.4. Membrane Analogy: Contributions from Warping

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is

$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\text{on}\, \Gamma,$$

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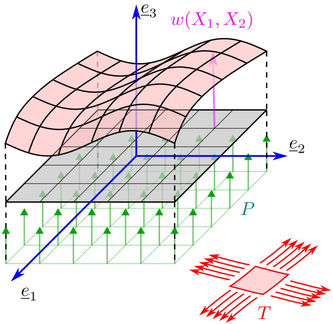
Transverse Deflections of a Membrane under Isotropic Linear Tension Density T and Uniform Planar Load Density P

- **The displacement field**
  $u_1 = 0, \quad u_2 = 0, \quad u_3 = w(X_1, X_2)$
- **The strain Field**
  $E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$
- **The Stress Field**
  $\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$



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  $E_{11} = \frac{w_{,1}^2}{2}, \quad E_{22} = \frac{w_{,2}^2}{2}, \quad 2E_{12} = w_{,1}w_{,2}$
- **The Stress Field**
  $$\sigma_{11} = \frac{1}{t}T, \quad \sigma_{22} = \frac{1}{t}T.$$

- **Strain Energy Density (Integrated over thickness)**

$$\mathcal{U} = \frac{1}{2}\left(w_{,1}^2 + w_{,2}^2\right) T + Pw$$

- Equations of Motion [a]:
  $\frac{\partial}{\partial X_k}\frac{\partial \mathcal{U}}{\partial w_{,k}} - \frac{\partial \mathcal{U}}{\partial w} = 0$:

$$\boxed{T\left(w_{,11} + w_{,22}\right) - P = 0}$$

[a] Euler-Ostrogradsky

1.4. Membrane Analogy: Contributions from Warping

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is
$$\phi_{,kk} + 2G\theta_{,1} = 0, \qquad \phi = 0 \,\text{on}\, \Gamma,$$
along with $M_1 = 2\int_{\mathcal{S}} \phi dA$.
- The governing equations, therefore, are identical to that of a **membrane undergoing deformation under the action of a uniform area-load** P.
- Also note that the governing equations in terms of u_1 is the Laplacian problem:
$$u_{1,kk} = 0,$$
and its boundary conditions are written as
$$\begin{aligned} &\langle (u_{1,2} - X_3\theta_{,1})\underline{e}_2 + (u_{1,3} + X_2\theta_{,1})\underline{e}_3, \underline{e}_n \rangle = 0 \\ \implies &\langle u_{1,2}\underline{e}_2 + u_{1,3}\underline{e}_3, X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3 \rangle \\ &- \theta_{,1}\langle X_3\underline{e}_2 - X_2\underline{e}_3, -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3 \rangle = 0 \\ \implies &\boxed{u_{1,n} = -\theta_{,1}\frac{d}{ds}\left(X_2^2 + X_3^2\right)}. \end{aligned}$$

1.4. Membrane Analogy: Contributions from Warping

Solid Section Torsion

- The governing equations in terms of Prandtl Stress function is
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$$\langle (u_{1,2} - X_3\theta_{,1})\underline{e}_2 + (u_{1,3} + X_2\theta_{,1})\underline{e}_3, \underline{e}_n \rangle = 0$$
$${}_{1,2}\underline{e}_2 + u_{1,3}\underline{e}_3, X_{2,n}\underline{e}_2 + X_{3,n}\underline{e}_3 \rangle$$
$$\theta_{,1}\langle X_3\underline{e}_2 - X_2\underline{e}_3, -X_{3,s}\underline{e}_2 + X_{2,s}\underline{e}_3 \rangle = 0$$
$$\implies \boxed{u_{1,n} = -\theta_{,1}\frac{d}{ds}\left(X_2^2 + X_3^2\right)}.$$

Note: We have used two different representations of $\underline{e}_s$ here

2.1. Open Sections

Thin Section Torsion

2.2. Closed Sections

Thin Section Torsion

2.3. Combined Cells

Thin Section Torsion

References I

[1] C. T. Sun. Mechanics of Aircraft Structures, 2nd edition. Hoboken, N.J: Wiley, June 2006. ISBN: 978-0-471-69966-8 (cit. on p. 2).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).