



AS3020: Aerospace Structures

Module 4: Bending of Beam-Like Structures

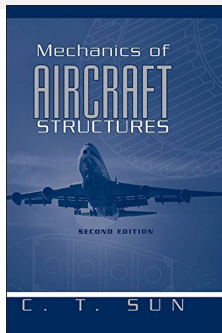
Instructor: Nidish Narayanaa Balaji

Dept. of Aerospace Engg., IIT-Madras, Chennai

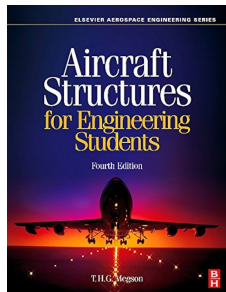
September 19, 2024

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Chapters 4-5 in Sun [1]

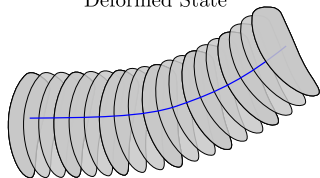


Chapters 16-20 in Megson [2]

1. Unsymmetrical Bending



Deformed State



- Displacement Field

$$u_1 = -X_2\theta_3 + X_3\theta_2, \quad u_2 = v, \quad u_3 = w.$$

- Zero shear $\implies \theta_3 = v'$, $\theta_2 = -w'$
- Direct stress

$$\begin{aligned} \sigma_{11} &= E_y [X_3 \quad -X_2] \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix} \\ &= \frac{[X_3 \quad -X_2]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}M_2 + I_{23}M_3 \\ I_{23}M_2 + I_{22}M_3 \end{bmatrix} \end{aligned}$$

- Equilibrium Considerations:

$$M_{2,1} = V_3, \quad V_{2,1} + F_2 = 0$$

$$M_{3,1} = -V_2, \quad V_{3,1} + F_3 = 0.$$

2. Shear Stress and Flow in Sections

- If shear strain is assumed zero, can we still have shear stress?
- We posit: $\gamma_{12} = 0$, $\gamma_{13} = 0$, $\gamma_{23} = 0$. As point quantities, the shear stresses may still be small ($\tau_{12} = G\sigma_{12}$).
- But the **integral quantities** are taken to be finite:

$$\int \sigma_{12} dA = V_2, \quad \int \sigma_{13} dA = V_3,$$

$$\int \sigma_{12} dX_3 = q_2, \quad \int \sigma_{13} dX_2 = q_3.$$

- Invoking plane stress assumption at the section, the governing equation is,

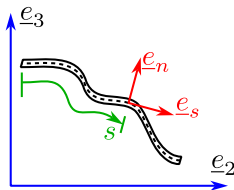
$$\sigma_{11,1} + \sigma_{1s,s} = 0.$$

- Integrating the above from $s = 0$ to s , we get the **Shear flow formula**:

$$q(s) - q_0 = - \frac{\left[\int_0^s tX_3 ds \quad - \int_0^s tX_2 ds \right]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

2. Shear Stress and Flow in Sections

Thin Section: Plane Stress Assumption



- We define the above section-local coordinate system and transform the elasticity equations to $\sigma_{11,1} + \sigma_{1n,n} + \sigma_{1s,s} = 0$. Applying **plane stress** assumption (for thin sections) drops the σ_{1n} term, leading to:

$$\sigma_{11,1} + \sigma_{1s,s} = 0 \implies t\sigma_{11,1} + q_{,s} = 0,$$

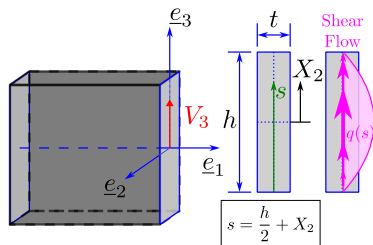
where we have integrated along the \underline{e}_n direction once.

- Following through with the integral along \underline{e}_s , this leads to the **shear flow formula**

$$q(s) - q_0 = - \frac{\left[\int_0^s tX_3 ds \quad - \int_0^s tX_2 ds \right]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

2. Shear Stress and Flow in Sections

- Consider the rectangular section with height h and thickness t :



$$\begin{aligned}
 q(s) &= -\frac{V_3}{I_{22}} \int_0^s t X_3 ds = -\frac{tV_3}{I_{22}} \int_{-\frac{h}{2}}^{X_3} X_3 dX_2 \\
 &= -\frac{tV_3}{2I_{22}} \left(X_2^2 - \frac{h^2}{4} \right)
 \end{aligned}$$

- Remember that V_3 is NOT any externally **applied force**. It is merely **the resultant of all the shear stresses in the section**.
- So V_3 and $q(s)$ point in the same direction in this example. It is incorrect to think that $q(s)$ is balancing out V_3 .

2. Shear Stress and Flow in Sections

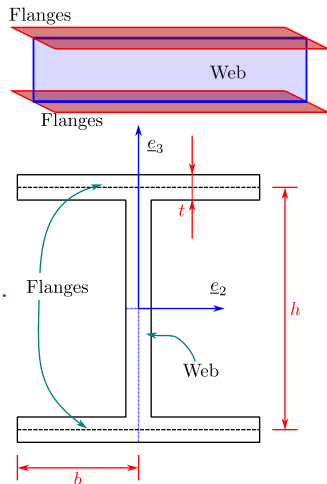
The “I” section

- Consider the shear distribution through an I-section as shown here
- The shear distribution looks like it is “flowing”, with more “flow” occurring in the thin vertical web and less in the flanges.
- The second moment of area I_{22} sums up as,

$$I_{22} = \underbrace{\frac{h^3 t}{12}}_{web} + 2 \times \underbrace{\left(\frac{2bt^3}{12} + 2bt \times \frac{h^2}{4} \right)}_{flange} \approx \left(\frac{h^3}{12} + bh^2 \right) t.$$

- I_{33} sums up as,

$$I_{33} = \underbrace{\frac{ht^3}{12}}_{web} + 2 \times \underbrace{\left(\frac{2b^3 t}{3} \right)}_{flange} \approx \underbrace{\frac{4b^3 t}{3}}_{\approx 0 \text{ for small } b}.$$

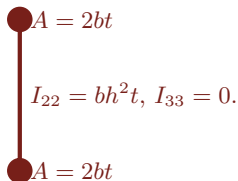


2. Shear Stress and Flow in Sections

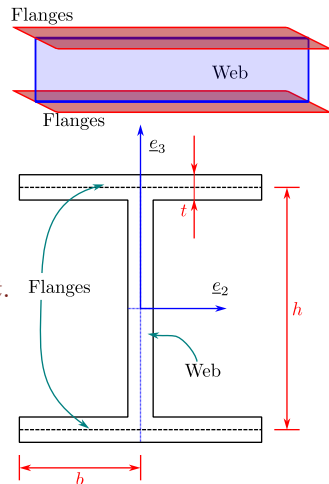
The “I” section

Idealization

- Both I_{22} and I_{33} are dominated by flange contributions, implying that bending is supported primarily by the flanges.
- This motivates the following idealization for the I-section:



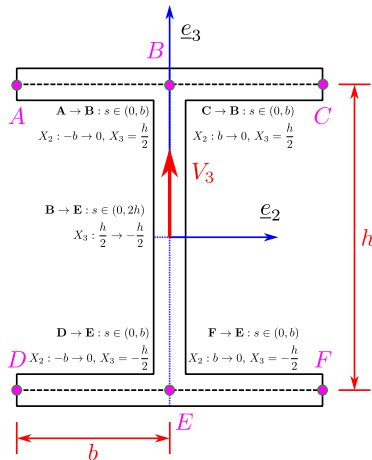
- The lumped area elements denoted ● are sometimes referred to as “**Booms**” in the section.
- Thickness in the web (denoted t) is taken to be zero for bending-stress calculations.



2. Shear Stress and Flow in Sections

The "I" section

- Let us consider the case with $V_2 = 0$, $V_3 \neq 0$.
- The segments $A \rightarrow B$, $C \rightarrow B$, $D \rightarrow E$, and $F \rightarrow E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).

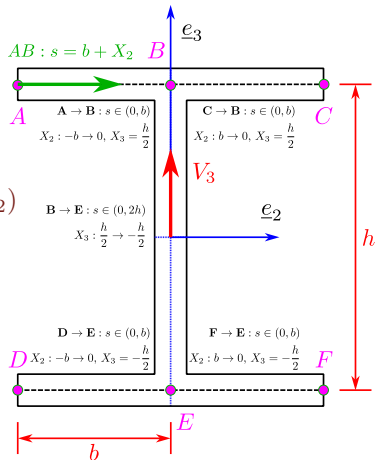


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$$\mathbf{A} \rightarrow \mathbf{B} : q_{AB}(s) \equiv q_{AB}(X_2) = -\frac{tV_3}{I_{22}} \int_0^s X_3 ds = -\frac{htV_3}{2I_{22}}(b + X_2)$$



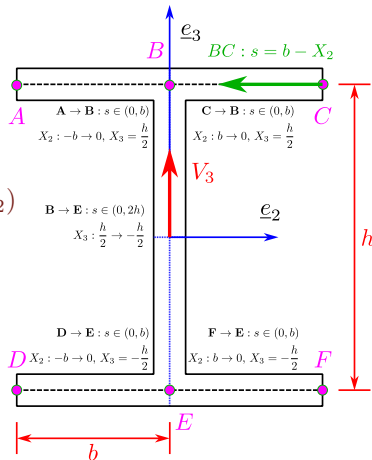
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2. Shear Stress and Flow in Sections

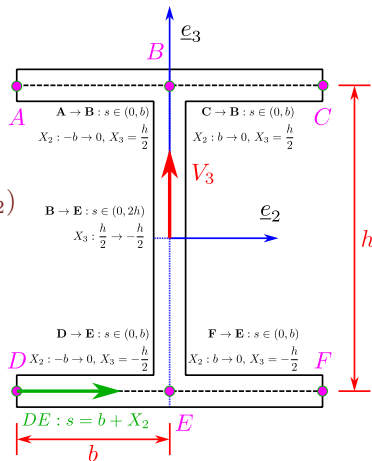
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$$\mathbf{D} \rightarrow \mathbf{E} : q_{DE}(X_2) = \frac{htV_3}{2I_{22}}(b + X_2)$$



2. Shear Stress and Flow in Sections

The "I" section

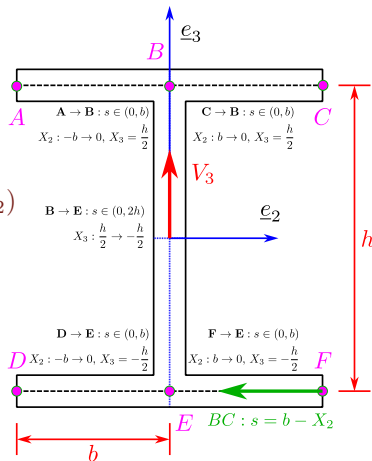
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$$\mathbf{D} \rightarrow \mathbf{E} : q_{DE}(X_2) = \frac{htV_3}{2I_{22}}(b + X_2)$$

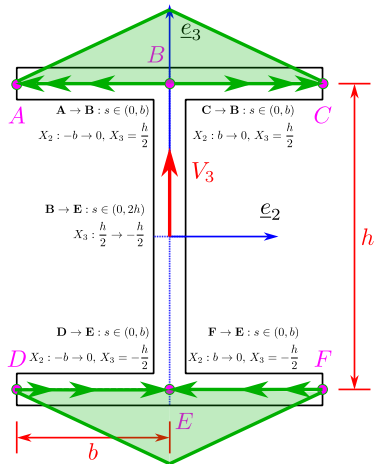
$$\mathbf{F} \rightarrow \mathbf{E} : q_{FE}(X_2) = \frac{htV_3}{2I_{22}}(b - X_2)$$



2. Shear Stress and Flow in Sections

The "I" section

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- The segments $A \rightarrow B$, $C \rightarrow B$, $D \rightarrow E$, and $F \rightarrow E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).
- In summary we have linear relationships at the flanges.

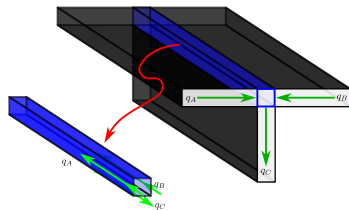


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- In summary we have linear relationships at the flanges.
- Before looking at the web ($B \rightarrow E$), we have to observe the balance at the "T" junction.

Balance at the T-junction



$$\sum F = q_A + q_B - q_C = 0 \Rightarrow q_C = q_A + q_B$$

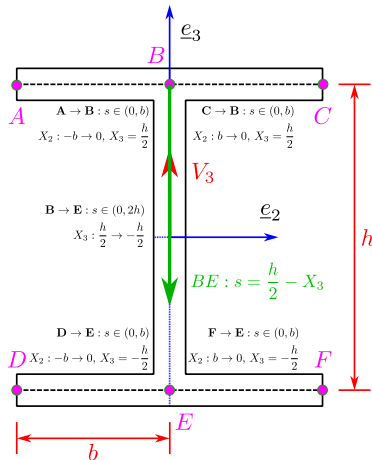
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- The segments $A \rightarrow B$, $C \rightarrow B$, $D \rightarrow E$, and $F \rightarrow E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).
- On $B \rightarrow E$, we have

$$q_{BE}(0) = q_{AB}(0) + q_{CB}(0) = -\frac{bhtV_3}{I_{22}}.$$
- The integration evaluates as,

$$\begin{aligned} q_{BE}(X_3) &= -\frac{bhtV_3}{I_{22}} + \frac{tV_3}{2I_{22}}(X_3^2 - \frac{h^2}{4}) \\ &= -\frac{htV_3}{I_{22}}(b + \frac{h}{8}) + \frac{tV_3}{2I_{22}}X_3^2. \end{aligned}$$



2. Shear Stress and Flow in Sections

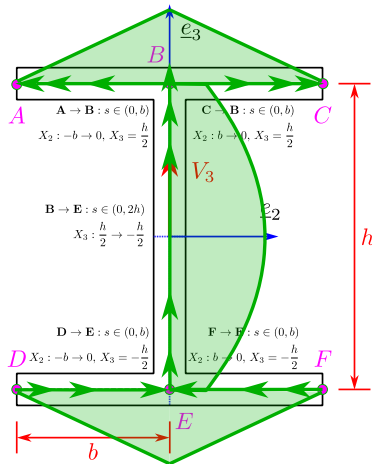
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- We now have the complete shear flow in the section.



2. Shear Stress and Flow in Sections

The “I” section: Order of Magnitude Analysis

- Let us consider the “total” shear forces experienced by each member.
- Flange AB**

$$V_{AB} = -\frac{htV_3}{2I_{22}} \int_{-b}^0 (b + X_2) dX_2 = -\frac{b^2htV_3}{4I_{22}}$$

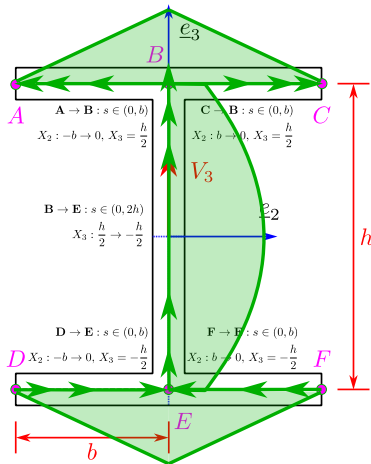
- Web BE**

$$V_{BE} = \frac{h^2(h + 12b)tV_3}{12I_{22}} = V_3$$

- For $b = \frac{h}{2}$, we have,

$$V_{AB} = -\frac{h^3tV_3}{16I_{22}} \approx -\frac{V_3}{8}$$

$$V_{BE} = V_3$$

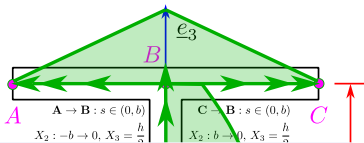


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Idealization

- Web BE**

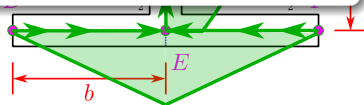
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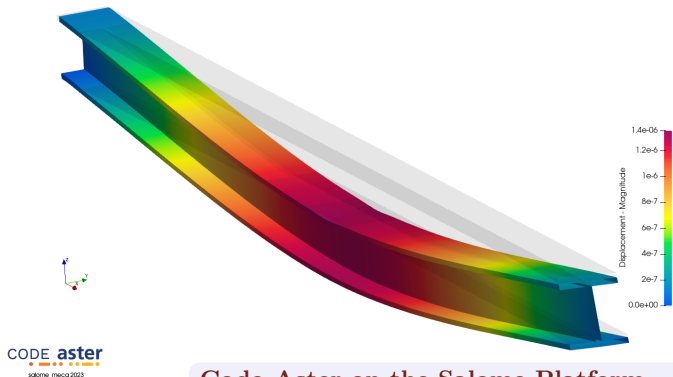
$$V_{BE} = V_3$$

Since $V_{AB} \ll V_{BE}$, we understand that the **web is primarily responsible for restoring shear loads**, with negligible contributions from the flanges.



2. Shear Stress and Flow in Sections

An "I" section beam subjected to 3-point bending: Finite Element Results

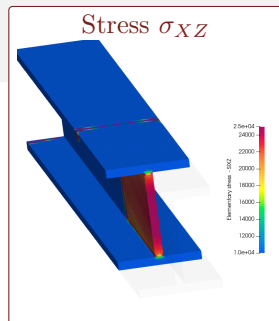
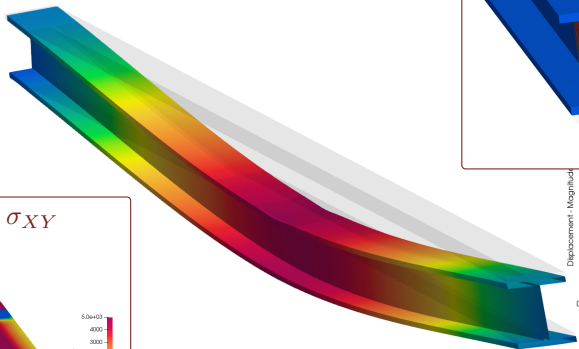
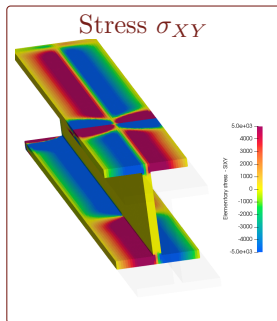


Code_Aster on the Salome Platform

Free and Open Source (FOSS) FE solver that comes with a fully functional frontend (Salome)! Please Do Explore!

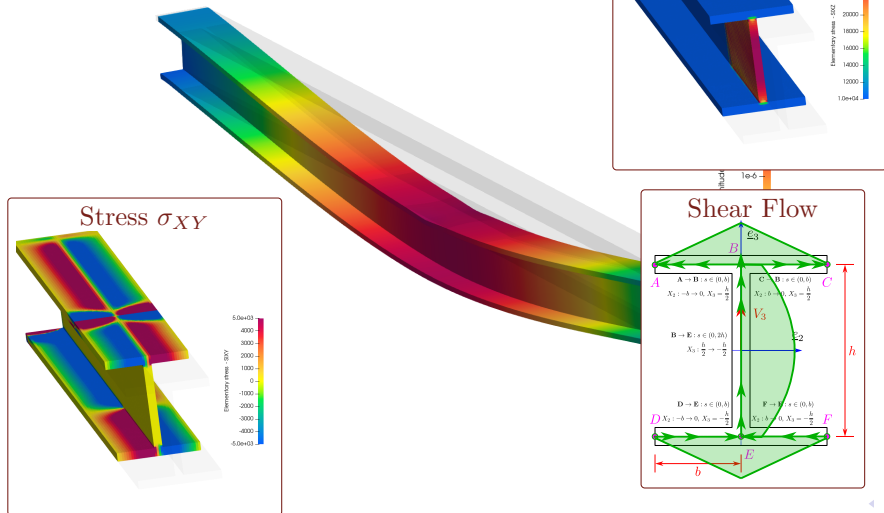
2. Shear Stress and Flow in Sections

An "I" section beam subjected to 3-point bending: Finite Element Results



2. Shear Stress and Flow in Sections

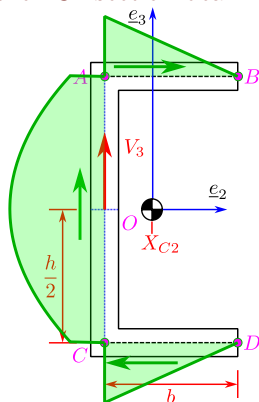
An "I" section beam subjected to 3-point bending: Finite Element Results



2.1. Shear Center

Shear Stress and Flow in Sections

- Shear Center is the point of shear load such that $\sum m_1 = 0$. Although in a lot of symmetric sections this is coincident with the centroid, **this is NOT always the case.**
- Consider the “C” section beam:



$$I_{22} \approx \frac{(h^3 + 6bh^2)t}{12} + \mathcal{O}(t^2)$$

$$q_{BA}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

$$q_{AC}(X_3) = -\frac{htV_3}{2I_{22}}\left(b + \frac{h}{4}\right) + \frac{tV_3}{2I_{22}}X_3^2$$

$$q_{CD}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

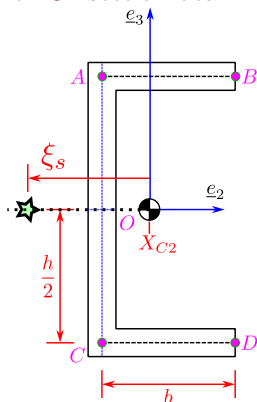
$$M_1 = \oint pqds = -\frac{b^2h^2tV_3}{4I_{22}} := V_3\xi_s$$

$$\xi_s \approx -\frac{3b^2}{h + 6b} + \mathcal{O}(t)$$

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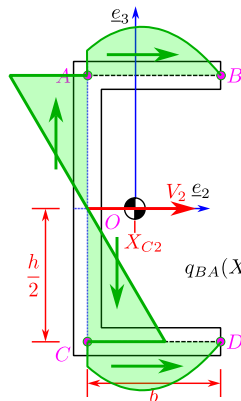
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- Consider the “C” section beam:



$$XC_2 = \frac{(bt)\frac{b}{2} \times 2}{2bt + ht} = \frac{b}{2 + h/b}$$

$$q(s) - q(0) = -\frac{tV_2}{I_{33}} \int_0^s X_2 ds$$

Shear Flow

$$q_{BA}(X_2) = \frac{tV_2}{2I_{33}} \frac{(X_2^2(h+2b)^2 - b^2(h+b)^2)}{(h+2b)^2}$$

$$q_{AC}(X_3) = -\frac{tb^2V_2}{(2b+h)I_{33}} X_3$$

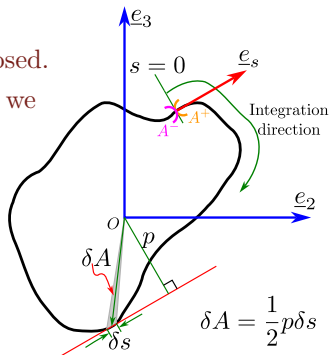
$$q_{CD}(X_2) = -q_{BA}(X_2)$$

2.2. Closed Sections

Shear Stress and Flow in Sections

- The shear-flow integral formula makes no reference to whether the section is open or closed.
- Considering the generic closed section shown, we start the integral at some **arbitrary** point A , denoting the point right before it as A^- . The integral is then written as,

$$q(s) - q_{A^-} = - \underbrace{\int_0^s t \sigma_{11,1} ds}_{q_b(s)}$$

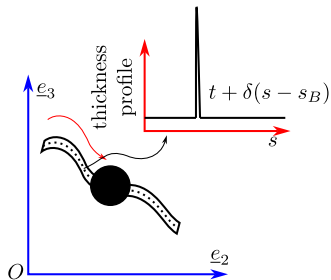


- When no twisting is expected at the section, the moment along \underline{e}_1 about O has to be zero. This is computed as,

$$\int_{A^-}^{A^+} dM_1 = \oint pq(s) ds = q_{A^-} \overbrace{\oint pq(s) ds}^{2A} + \oint pq_b(s) ds \implies \boxed{q_{A^-} = -\frac{1}{2A} \oint pq_b(s) ds}$$

3. Stringer-Web Idealization

- A stringer, when looked at from a section, looks like a feature with a sudden increase in thickness.
- Consider the r^{th} “Boom” to be located at (X_{r2}, X_{r3}) , with area A_r .



- So the shear flow integral can be generalized to,

$$q(s) - q(0) = - \frac{\left[t \int_0^s X_3 ds + \sum_r A_r X_{r3} \quad -t \int_0^s X_2 ds - \sum_r A_r X_{r2} \right]}{I_{22} I_{33} - I_{23}^2} \begin{bmatrix} I_{33} V_3 - I_{23} V_2 \\ I_{23} V_3 - I_{22} V_2 \end{bmatrix}$$

- When thickness t is negligible in comparison to the boom sections, this further simplifies to,

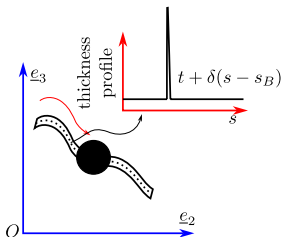
$$q(s) - q(0) = - \frac{\left[\sum_r A_r X_{r3} \quad - \sum_r A_r X_{r2} \right]}{I_{22} I_{33} - I_{23}^2} \begin{bmatrix} I_{33} V_3 - I_{23} V_2 \\ I_{23} V_3 - I_{22} V_2 \end{bmatrix}$$

3. Stringer-Web Idealization

- Considering the integral right across the boom, we have,

$$q^+ - q^- = - \frac{[A_r X_{r3} \quad -A_r X_{r2}]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}.$$

- For the sections without a boom, there is **no change in the shear flow**.
- Therefore, the shear flow is constant in the webs that join two booms.



General Design Principle

- As a general principle, the stringers/booms are added to support bending.
- The web thicknesses are chosen to ensure the shear stresses don't exceed failure threshold (we should like to have $t = 0$ to minimize weight!).

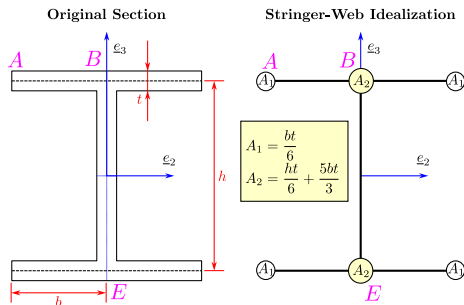
3. Stringer-Web Idealization

Idealization of the I-Section

- We idealize the I-section by lumping area elements A_1 and A_2 . A_1 and A_2 are estimated by matching the second area moments:

$$\underbrace{I_{22,ideal}}_{4 \times \left(\frac{A_1 h^2}{4}\right) + 2 \times \left(\frac{A_2 h^2}{4}\right)} = \underbrace{I_{22,orig}}_{\left(\frac{h^3}{12} + bh^2\right) t + \mathcal{O}(t^2)}$$

$$\underbrace{I_{33,ideal}}_{4 \times (A_1 b^2)} = \underbrace{I_{33,ideal}}_{\frac{4b^3 t}{3} + \mathcal{O}(t^2)}$$



- The total sectional area of the original section is $ht + 4bt$.
- The total area of the new section (assuming the web thickness is drastically reduced) is $4A_1 + 2A_2 = \frac{ht}{3} + 4bt$, which is a **slight reduction**.
- Looking at this from a manufacturing standpoint, this shows that a **web-stringer construction can achieve similar bending stiffness with lesser material expenditure**.

3. Stringer-Web Idealization

Idealization of the I-Section: Shear Flow Comparisons

- For the idealized section AB , the shear flow is given by,

$$q_{AB,ideal} = -\frac{V_3}{I_{22,ideal}} A_1 \frac{h}{2} = -\frac{V_3 b}{h^2 + 12bh}.$$

The average flow for the original section is,

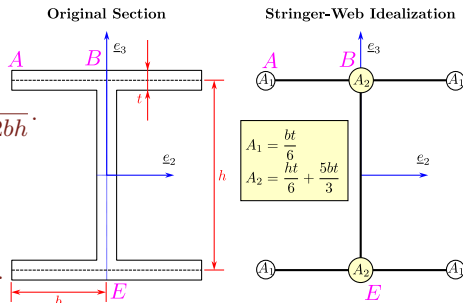
$$q_{AB,avg} = \frac{1}{b} \left(-\frac{b^2 ht V_3}{4I_{22}} \right) = \frac{3V_3 b}{h^2 + 12bh}.$$

- On BE , the idealized flow is

$$q_{BE,ideal} = 2q_{AB,ideal} - \frac{V_3}{I_{22,ideal}} A_2 \frac{h}{2} = -\frac{V_3}{h},$$

which is the same for the original section also.

- In the stringer-web section, therefore, the flanges carry lesser average shear than the original section, and the web carries the same shear.



4. Shear Lag

Example from Sun [1] (Section 3.1)

- We already saw the St. Venant's principle. Can we say something more specific about “how fast” the end effects start getting smoothed out in the stress field? Consider:
- The strain relationship leads to:

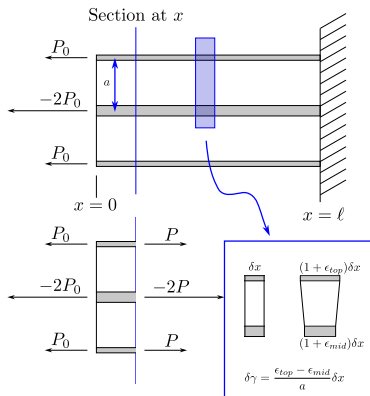
$$\begin{aligned}\frac{\partial \gamma}{\partial x} &= \frac{1}{a} (\epsilon_{top} - \epsilon_{mid}) \\ &= \frac{P}{aE_y} \left(\frac{1}{A_{top}} + \frac{2}{A_{mid}} \right).\end{aligned}$$

- From the equilibrium equations we have,

$$\tau = \frac{1}{t} \frac{\partial P}{\partial x}.$$

- Invoking $\tau = G\gamma$ we get,

$$G \frac{\partial \gamma}{\partial x} = \frac{1}{t} \frac{\partial^2 P}{\partial x^2} = \frac{G}{aE_y} \left(\frac{1}{A_{top}} + \frac{2}{A_{mid}} \right) P.$$



4. Shear Lag

Example from Sun [1] (Section 3.1)

- We already saw the St. Venant's principle. Can we say something more specific about "how fast" the end effects start getting smoothed out in the stress field? Consider:

Axial Stress Decay

- The stress is of the form

$$P_{,xx} - \lambda^2 P = 0,$$

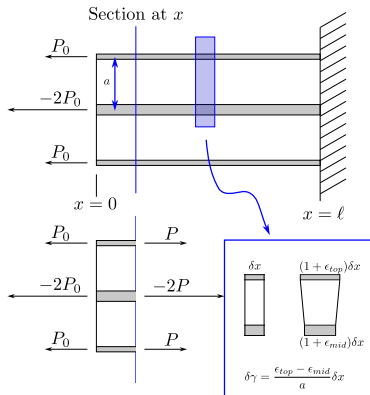
that is solved by

$$P(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}.$$

- From This shows an exponential decay of the axial stress $\frac{P}{A}$ along the axis.

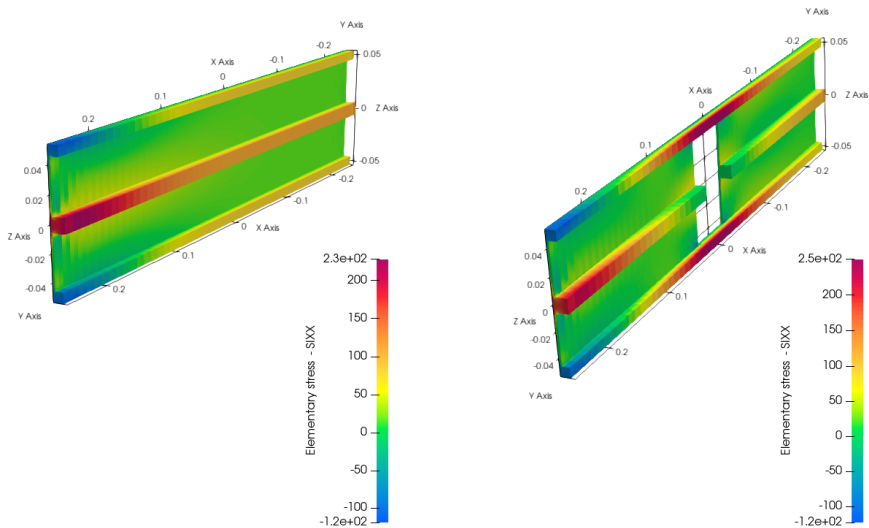
- Invoking $\tau = G\gamma$ we get,

$$G \frac{\partial \gamma}{\partial x} = \frac{1}{t} \frac{\partial^2 P}{\partial x^2} = \frac{G}{a E_y} \left(\frac{1}{A_{top}} + \frac{2}{A_{mid}} \right) P.$$



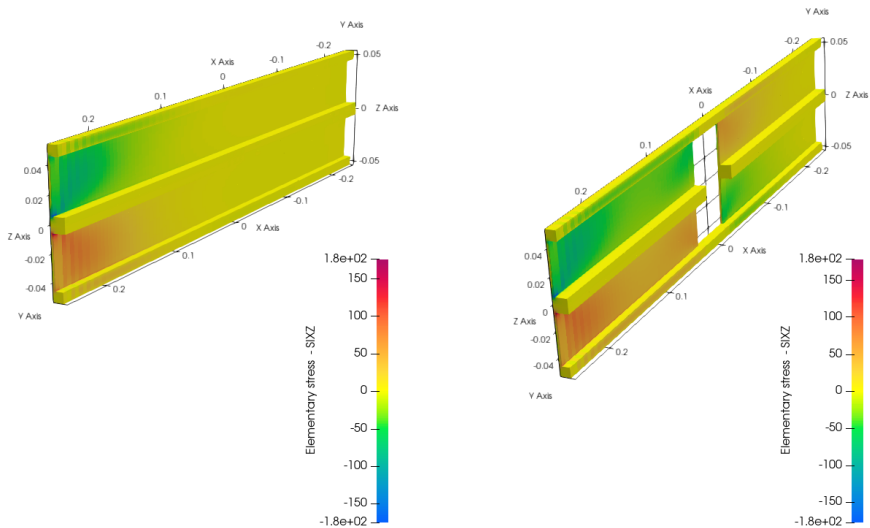
4. Shear Lag

Example from Sun [1] (Section 3.1): FE Results



4. Shear Lag

Example from Sun [1] (Section 3.1): FE Results



References I

- [1] C. T. Sun. *Mechanics of Aircraft Structures*, 2nd edition. Hoboken, N.J: Wiley, June 2006. ISBN: 978-0-471-69966-8 (cit. on pp. 2, 31–34).
- [2] T. H. G. Megson. *Aircraft Structures for Engineering Students*, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).