AS3020: Aerospace Structures

Module 4: Bending of Beam-Like Structures

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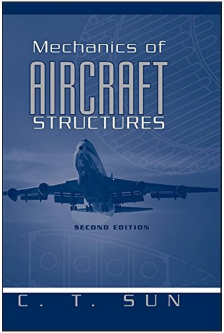
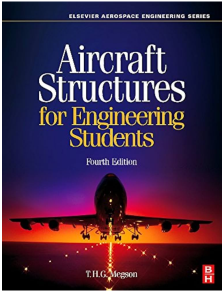
Table of Contents

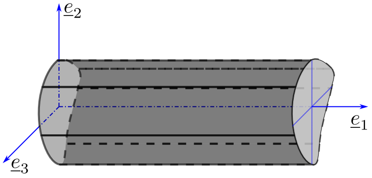


Chapters 4-5 in Sun [1]

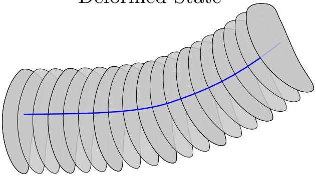


Chapters 16-20 in Megson [2]

1. Unsymmetrical Bending



Deformed State



- Displacement Field

$$u_1 = -X_2\theta_3 + X_3\theta_2, \quad u_2 = v, \quad u_3 = w.$$

- Zero shear $\implies \theta_3 = v\prime, \quad \theta_2 = -w\prime$
- Direct stress

$$\sigma_{11} = E_y \begin{bmatrix} X_3 & -X_2 \end{bmatrix} \begin{bmatrix} \theta_2\prime \\ \theta_3\prime \end{bmatrix}$$
$$= \frac{\begin{bmatrix} X_3 & -X_2 \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}M_2 + I_{23}M_3 \\ I_{23}M_2 + I_{22}M_3 \end{bmatrix}$$

- Equilibrium Considerations:

$$M_{2,1} = V_3, \qquad V_{2,1} + F_2 = 0$$
$$M_{3,1} = -V_2, \quad V_{3,1} + F_3 = 0.$$

2. Shear Stress and Flow in Sections

- If shear strain is assumed zero, can we still have shear stress?
- We posit: $\gamma_{12} = 0$, $\gamma_{13} = 0$, $\gamma_{23} = 0$. As point quantities, the shear stresses may still be small ($\tau_{12} = G\sigma_{12}$).
- But the **integral quantities** are taken to be finite:

$$\int \sigma_{12} dA = V_2, \qquad \int \sigma_{13} dA = V_3,$$
$$\int \sigma_{12} dX_3 = q_2, \qquad \int \sigma_{13} dX_2 = q_3.$$

- Invoking plane stress assumption at the section, the governing equation is,

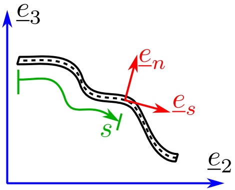
$$\sigma_{11,1} + \sigma_{1s,s} = 0.$$

- Integrating the above from $s = 0$ to s, we get the **Shear flow formula**:

$$q(s) - q_0 = -\frac{\left[\int_0^s tX_3 ds \quad -\int_0^s tX_2 ds\right]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

2. Shear Stress and Flow in Sections

Thin Section: Plane Stress Assumption



- We define the above section-local coordinate system and transform the elasticity equations to $\sigma_{11,1} + \sigma_{1n,n} + \sigma_{1s,s} = 0$. Applying **plane stress** assumption (for thin sections) drops the σ_{1n} term, leading to:

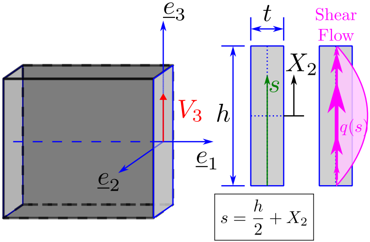
$$\sigma_{11,1} + \sigma_{1s,s} = 0 \implies t\sigma_{11,1} + q_{,s} = 0,$$

  where we have integrated along the $\underline{e}_n$ direction once.
- Following through with the integral along $\underline{e}_s$, this leads to the **shear flow formula**

$$q(s) - q_0 = -\frac{\begin{bmatrix} \int_0^s tX_3 ds & -\int_0^s tX_2 ds \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

2. Shear Stress and Flow in Sections

- Consider the rectangular section with height h and thickness t:



$$q(s) = -\frac{V_3}{I_{22}} \int_0^s t X_3 ds = -\frac{tV_3}{I_{22}} \int_{-\frac{h}{2}}^{X_3} X_3 dX_2$$

$$= -\frac{tV_3}{2I_{22}} (X_2^2 - \frac{h^2}{4})$$

- Remember that V_3 is NOT any externally **applied force**. It is merely **the resultant of all the shear stresses in the section**.
- So V_3 and $q(s)$ point in the same direction in this example. It is incorrect to think that $q(s)$ is balancing out V_3.

2. Shear Stress and Flow in Sections

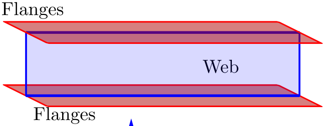
The "I" section

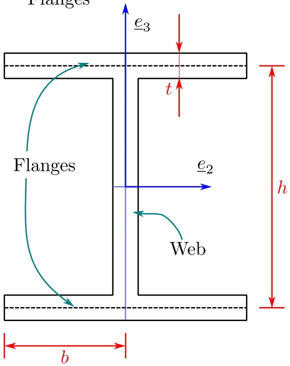
- Consider the shear distribution through an I-section as shown here
- The shear distribution looks like it is "flowing", with more "flow" occurring in the thin vertical web and less in the flanges.
- The second moment of area I_{22} sums up as,

$$I_{22} = \overbrace{\frac{h^3 t}{12}}^{web} + 2 \times \overbrace{\left(\frac{2bt^3}{12} + 2bt \times \frac{h^2}{4}\right)}^{flange} \approx (\frac{h^3}{12} + bh^2)t.$$

- I_{33} sums up as,

$$I_{33} = \underbrace{\frac{ht^3}{12}}_{web} + 2 \times \overbrace{\left(\frac{2b^3 t}{3}\right)}^{flange} \approx \underbrace{\frac{4b^3 t}{3}}_{\approx 0 \text{ for small b}} .$$



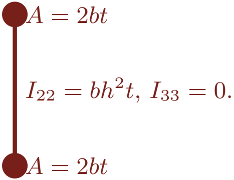


2. Shear Stress and Flow in Sections

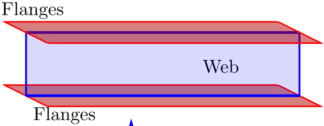
The "I" section

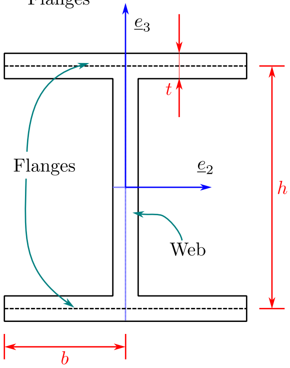
Idealization

- Both I_{22} and I_{33} are dominated by flange contributions, implying that **bending is supported primarily by the flanges.**
- This motivates the following idealization for the I-section:



- The lumped area elements denoted ● are sometimes referred to as **"Booms"** in the section.
- Thickness in the web (denoted ▬) is taken to be zero for bending-stress calculations.

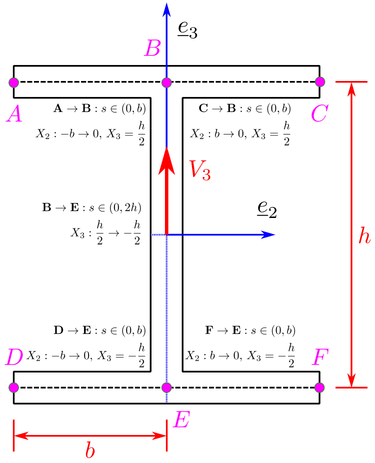




2. Shear Stress and Flow in Sections

The "I" section

- Let us consider the case with $V_2 = 0$, $V_3 \neq 0$.
- The segments $A \to B$, $C \to B$, $D \to E$, and $F \to E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).



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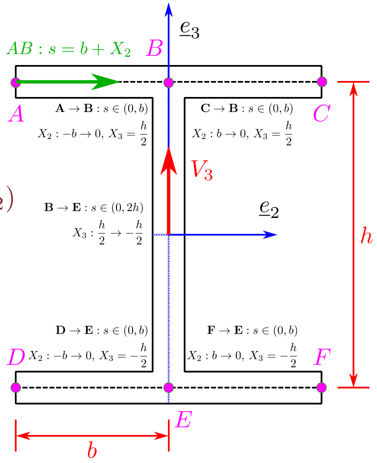
$$\mathbf{A} \to \mathbf{B} \; : \; q_{AB}(s) \equiv q_{AB}(X_2) =$$

$$-\frac{tV_3}{I_{22}} \int_0^s \cancel{X_3}^{\frac{h}{2}} ds = -\frac{htV_3}{2I_{22}}(b + X_2)$$



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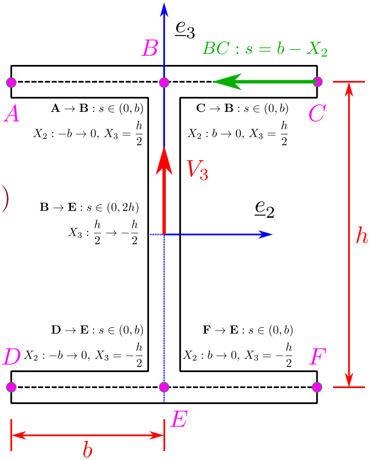
$$\mathbf{C} \to \mathbf{B} \; : \; q_{CB}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$



2. Shear Stress and Flow in Sections

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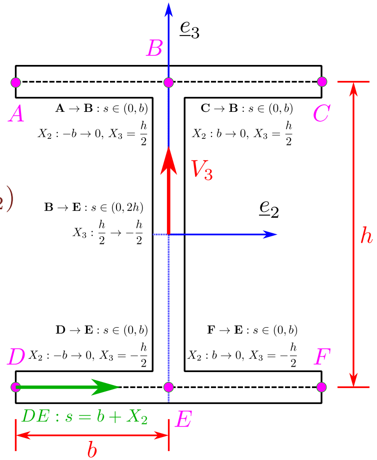
$$\mathbf{C \to B} \;:\; q_{CB}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

$$\mathbf{D \to E} \;:\; q_{DE}(X_2) = \frac{htV_3}{2I_{22}}(b + X_2)$$



2. Shear Stress and Flow in Sections

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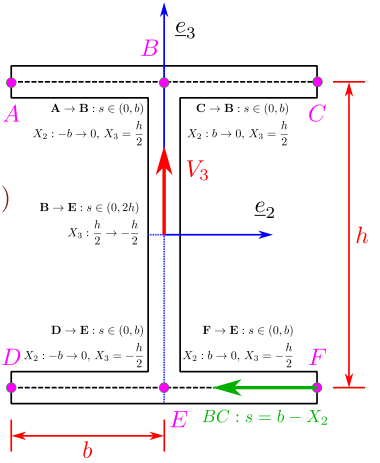
$$\mathbf{A \to B} \; : \; q_{AB}(s) \equiv q_{AB}(X_2) = -\frac{tV_3}{I_{22}} \int_0^s \cancelto{\frac{h}{2}}{X_3} ds = -\frac{htV_3}{2I_{22}}(b + X_2)$$

$$\mathbf{C \to B} \; : \; q_{CB}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

$$\mathbf{D \to E} \; : \; q_{DE}(X_2) = \frac{htV_3}{2I_{22}}(b + X_2)$$

$$\mathbf{F \to E} \; : \; q_{FE}(X_2) = \frac{htV_3}{2I_{22}}(b - X_2)$$



2. Shear Stress and Flow in Sections

The "I" section

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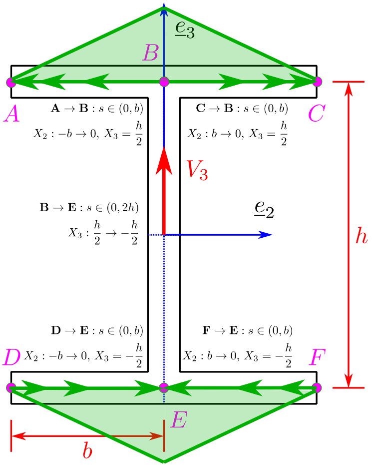
- The segments $A \to B$, $C \to B$, $D \to E$, and $F \to E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).
- In summary we have linear relationships at the flanges.



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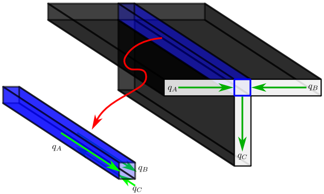
- In summary we have linear relationships at the flanges.
- Before looking at the web ($B \to E$), we have to observe the balance at the "T" junction.

Balance at the T-junction



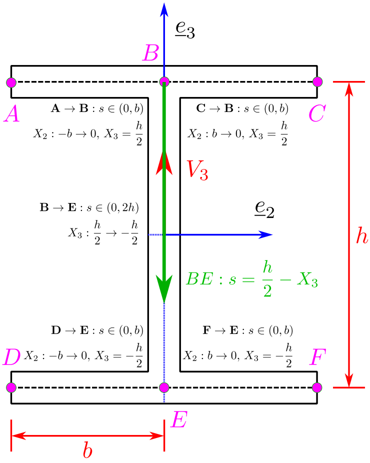
$\sum F = q_A + qB - qC = 0 => q_C = q_A + q_B$

2. Shear Stress and Flow in Sections

The "I" section

- Let us consider the case with $V_2 = 0$, $V_3 \neq 0$.
- The segments $A \to B$, $C \to B$, $D \to E$, and $F \to E$ are exposed in their free ends, simplifying the shear flow integral ($q = 0$ at free ends).
- On $\mathbf{B} \to \mathbf{E}$, we have $q_{BE}(0) = q_{AB}(b) + q_{CB}(b) = -\frac{bhtV_3}{I_{22}}$.
- The integration evaluates as,

$$q_{BE}(X_3) = -\frac{bhtV_3}{I_{22}} + \frac{tV_3}{2I_{22}}(X_3^2 - \frac{h^2}{4})$$
$$= -\frac{htV_3}{I_{22}}(b + \frac{h}{8}) + \frac{tV_3}{2I_{22}}X_3^2.$$



2. Shear Stress and Flow in Sections

The “I” section

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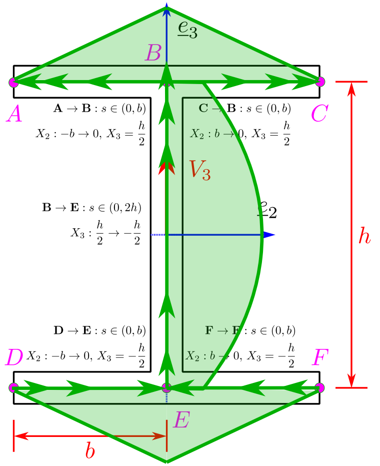
$$= -\frac{htV_3}{I_{22}}(b + \frac{h}{8}) + \frac{tV_3}{2I_{22}}X_3^2.$$

- We now have the complete shear flow in the section.



2. Shear Stress and Flow in Sections

The "I" section: Order of Magnitude Analysis

- Let us consider the "total" shear forces experienced by each member.
- **Flange AB**

$$V_{AB} = -\frac{htV_3}{2I_{22}} \int_{-b}^{0} (b + X_2) dX_2 = -\frac{b^2 htV_3}{4I_{22}}$$

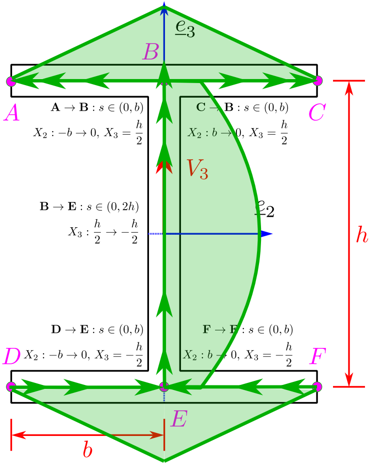
- **Web BE**

$$V_{BE} = \frac{h^2(h + 12b)tV_3}{12I_{22}} = V_3$$

- For $b = \frac{h}{2}$, we have,

$$V_{AB} = -\frac{h^3 tV_3}{16I_{22}} \approx -\frac{V_3}{8}$$

$$V_{BE} = V_3$$



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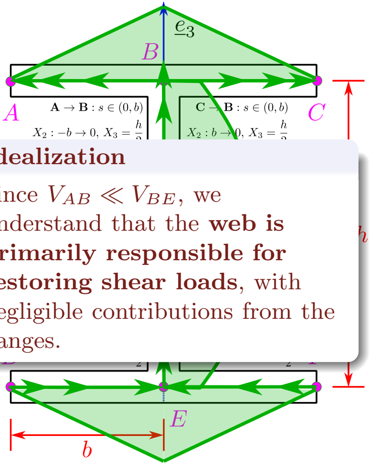
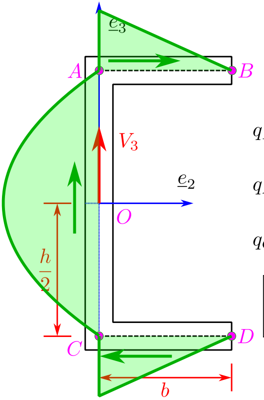


Idealization

Since $V_{AB} \ll V_{BE}$, we understand that the **web is primarily responsible for restoring shear loads**, with negligible contributions from the flanges.

2. Shear Stress and Flow in Sections

- Shear Center is the point of shear load such that $\sum m_1 = 0$. Although in a lot of symmetric sections this is coincident with the centroid, **this is NOT always the case**.
- Consider the "C" section beam:



$$\boxed{I_{22} \approxeq \frac{(h^3 + 6bh^2)t}{12}} + \mathcal{O}(t^2)$$

$$q_{BA}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

$$q_{AC}(X_3) = -\frac{htV_3}{2I_{22}}\left(b + \frac{h}{4}\right) + \frac{tV_3}{2I_{22}}X_3^2$$

$$q_{CD}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

$$\boxed{M_1 = \oint pq ds = -\frac{b^2h^2tV_3}{4I_{22}} := V_3\xi_s}$$

$$\boxed{\xi_s \approxeq -\frac{3b^2}{h + 6b} + \mathcal{O}(t)}$$

2. Shear Stress and Flow in Sections

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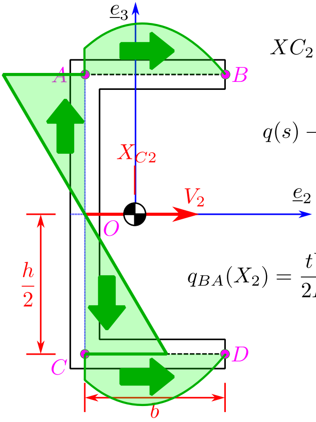
- Consider the "C" section beam:



$$XC_2 = \frac{(bt)\frac{b}{2} \times 2}{2bt + ht} = \frac{b}{2 + h/b}$$

$$q(s) - q(0) = -\frac{tV_2}{I_{33}} \int_0^s X_2 ds$$

Shear Flow

$$q_{BA}(X_2) = \frac{tV_2}{2I_{33}} \frac{(X_2^2(h+2b)^2 - b^2(h+b)^2)}{(h+2b)^2}$$

$$q_{AC}(X_3) = -\frac{tb^2V_2}{(2b+h)I_{33}} X_3$$

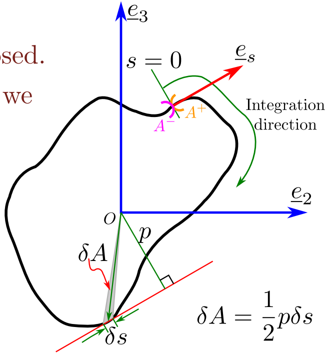
$$q_{CD}(X_2) = -q_{BA}(X_2)$$

2.2. Closed Sections

Shear Stress and Flow in Sections

- The shear-flow integral formula makes no reference to whether the section is open or closed.
- Considering the generic closed section shown, we start the integral at some **arbitrary** point A, denoting the point right before it as A^-. The integral is then written as,

$$q(s) - q_{A^-} = \underbrace{-\int_0^s t\sigma_{11,1} ds}_{q_b(s)}.$$

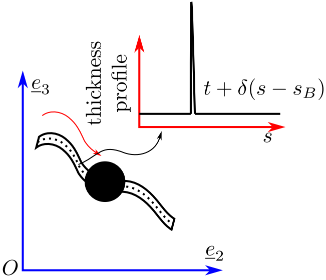


- When no twisting is expected at the section, the moment along $\underline{e}_1$ about O has to be zero. This is computed as,

$$\int_{A^-}^{A^+} dM_1 = \oint pq(s)ds = q_{A^-} \overbrace{\oint pq(s)ds}^{2A} + \oint pq_b(s)ds \implies \boxed{q_{A^-} = -\frac{1}{2A}\oint pq_b(s)ds}.$$

3. Stringer-Web Idealization

- A stringer, when looked at from a section, looks like a feature with a sudden increase in thickness.
- Consider the r^{th} "Boom" to be located at (X_{r_2}, X_{r_3}), with area A_r.



- So the shear flow integral can be generalized to,

$$q(s) - q(0) = -\frac{\begin{bmatrix} t\int_0^s X_3 ds + \sum_r A_r X_{r_3} & -t\int_0^s X_2 ds - \sum_r A_r X_{r_2} \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

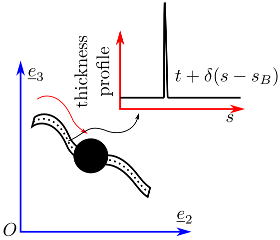
- When thickness t is negligible in comparison to the boom sections, this further simplifies to,

$$q(s) - q(0) = -\frac{\begin{bmatrix} \sum_r A_r X_{r_3} & -\sum_r A_r X_{r_2} \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

3. Stringer-Web Idealization

- Considering the integral right across the boom, we have,

$$q^+ - q^- = -\frac{\begin{bmatrix} A_r X_{r_3} & -A_r X_{r_2} \end{bmatrix}}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}.$$



- For the sections without a boom, there is **no change in the shear flow**.
- Therefore, the shear flow is constant in the webs that join two booms.

General Design Principle

- As a general principle, the stringers/booms are added to support bending.
- The web thicknesses are chosen to ensure the shear stresses don't exceed failure threshold (we should like to have $t = 0$ to minimize weight!).

4. Shear Lag

Example from Sun [1] (Section 3.1)

- We already saw the St. Venant's principle. Can we say something more specific about "how fast" the end effects start getting smoothed out in the stress field? Consider:

- The strain relationship leads to:

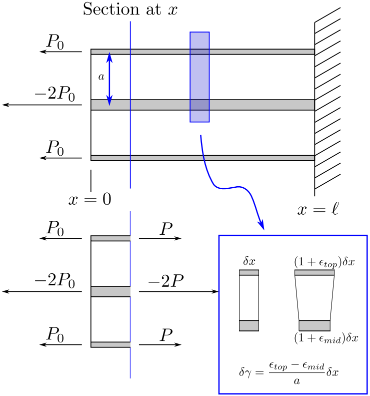
$$\frac{\partial \gamma}{\partial x} = \frac{1}{a}\left(\epsilon_{top} - \epsilon_{mid}\right) = \frac{P}{aE_y}\left(\frac{1}{A_{top}} + \frac{2}{A_{mid}}\right).$$

- From the equilibrium equations we have,

$$\tau = \frac{1}{t}\frac{\partial P}{\partial x}.$$

- Invoking $\tau = G\gamma$ we get,

$$G\frac{\partial \gamma}{\partial x} = \frac{1}{t}\frac{\partial^2 P}{\partial x^2} = \frac{G}{aE_y}\left(\frac{1}{A_{top}} + \frac{2}{A_{mid}}\right)P.$$



4. Shear Lag

Example from Sun [1] (Section 3.1)

- We already saw the St. Venant's principle. Can we say something more specific about "how fast" the end effects start getting smoothed out in the stress field? Consider:
- The s
- From ... ve,
- Invoking $\tau = G\gamma$ we get,

$$G\frac{\partial \gamma}{\partial x} = \boxed{\frac{1}{t}\frac{\partial^2 P}{\partial x^2} = \frac{G}{aE_y}\left(\frac{1}{A_{top}} + \frac{2}{A_{mid}}\right)P.}$$

Axial Stress Decay

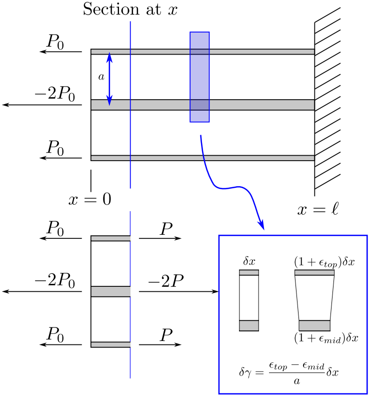
This is of the form

$$P_{,xx} - \lambda^2 P = 0,$$

that is solved by

$$\boxed{P(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}.}$$

This shows an exponential decay of the axial stress $\frac{P}{A}$ along the axis.



References I

[1] C. T. Sun. Mechanics of Aircraft Structures, 2nd edition. Hoboken, N.J: Wiley, June 2006. ISBN: 978-0-471-69966-8 (cit. on pp. 2, 25, 26).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).