



# AS3020: Aerospace Structures

## Module 4: Bending of Beam-Like Structures

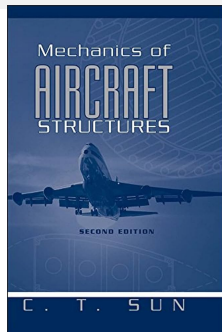
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**Dept. of Aerospace Engg., IIT-Madras, Chennai**

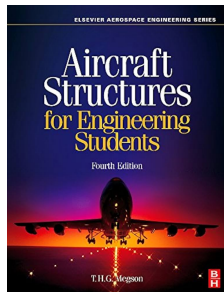
September 29, 2024

# Table of Contents

- 1 Unsymmetrical Bending
  - Equations of Motion in Terms of Displacement
- 2 Shear Stress and Flow in Sections
  - Shear Center
  - Closed Sections
    - Closed Section Shear Center
- 3 Stringer-Web Idealization
- 4 Shear Lag



*Chapters 4-5 in Sun [1]*

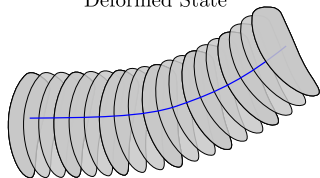


*Chapters 16-20  
in Megson [2]*

# 1. Unsymmetrical Bending



Deformed State



- Displacement Field

$$u_1 = -X_2\theta_3 + X_3\theta_2, \quad u_2 = v, \quad u_3 = w.$$

- Zero shear  $\implies \theta_3 = v'$ ,  $\theta_2 = -w'$
- Direct stress

$$\begin{aligned} \sigma_{11} &= E_y [X_3 \quad -X_2] \begin{bmatrix} \theta_2' \\ \theta_3' \end{bmatrix} \\ &= \frac{[X_3 \quad -X_2]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}M_2 + I_{23}M_3 \\ I_{23}M_2 + I_{22}M_3 \end{bmatrix} \end{aligned}$$

- Equilibrium Considerations:

$$M_{2,1} = V_3, \quad V_{2,1} + F_2 = 0$$

$$M_{3,1} = -V_2, \quad V_{3,1} + F_3 = 0.$$

# 1.1. Equations of Motion in Terms of Displacement

## Unsymmetrical Bending

- The moments are related to  $\sigma_{11}$  through (note the signs)

$$M_3 = - \int X_2 \sigma_{11} dA, \quad M_2 = \int X_3 \sigma_{11} dA.$$

- Since  $M_{2,1} = V_3$  and  $M_{3,1} = -V_2$  (from equilibrium considerations), we can write the above as

$$\begin{bmatrix} V_2 \\ V_3 \end{bmatrix} = \int \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \sigma_{11,1} dA.$$

- Invoking force equilibrium ( $V_{2,1} = -F_2$ ,  $V_{3,1} = -F_3$ ) we have,

$$\begin{aligned} - \begin{bmatrix} F_2 \\ F_3 \end{bmatrix} &= \int \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \sigma_{11,11} dA = E_y \int \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \begin{bmatrix} -X_2 & X_3 \end{bmatrix} \begin{bmatrix} \theta_3''' \\ \theta_2''' \end{bmatrix} dA \\ &= -E_y \int \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \begin{bmatrix} X_2 & X_3 \end{bmatrix} \begin{bmatrix} v'''' \\ w'''' \end{bmatrix} dA \end{aligned}$$

# 1.1. Equations of Motion in Terms of Displacement

## Unsymmetrical Bending

- Since deflections  $v, w$  do not depend on the section, the above simplifies to,

$$E_y \underbrace{\begin{bmatrix} \int X_2^2 dA & \int X_2 X_3 dA \\ \int X_2 X_3 dA & \int X_3^2 dA \end{bmatrix}}_{\mathbb{I}} \underbrace{\begin{bmatrix} v'''' \\ w'''' \end{bmatrix}}_{\underline{u}''''} - \underbrace{\begin{bmatrix} F_2 \\ F_3 \end{bmatrix}}_{\underline{F}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which, in array-matrix notation is written as,

$$E_y \mathbb{I} \underline{u}'''' - \underline{F} = \underline{0}.$$

- We have introduced the array  $\underline{u} = [v \ w]^T$  here.
- The above is a fourth order differential equation in terms of the deflections that governs the unsymmetrical bending.

## 2. Shear Stress and Flow in Sections

- If shear strain is assumed zero, can we still have shear stress?
- We posit:  $\gamma_{12} = 0$ ,  $\gamma_{13} = 0$ ,  $\gamma_{23} = 0$ . As point quantities, the shear stresses may still be small ( $\tau_{12} = G\sigma_{12}$ ).
- But the **integral quantities** are taken to be finite:

$$\int \sigma_{12} dA = V_2, \quad \int \sigma_{13} dA = V_3,$$

$$\int \sigma_{12} dX_3 = q_2, \quad \int \sigma_{13} dX_2 = q_3.$$

- Invoking plane stress assumption at the section, the governing equation is,

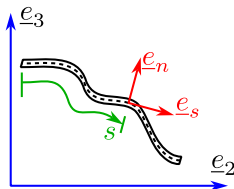
$$\sigma_{11,1} + \sigma_{1s,s} = 0.$$

- Integrating the above from  $s = 0$  to  $s$ , we get the **Shear flow formula**:

$$q(s) - q_0 = - \frac{\left[ \int_0^s tX_3 ds \quad - \int_0^s tX_2 ds \right]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

## 2. Shear Stress and Flow in Sections

Thin Section: Plane Stress Assumption



- We define the above section-local coordinate system and transform the elasticity equations to  $\sigma_{11,1} + \sigma_{1n,n} + \sigma_{1s,s} = 0$ . Applying **plane stress** assumption (for thin sections) drops the  $\sigma_{1n}$  term, leading to:

$$\sigma_{11,1} + \sigma_{1s,s} = 0 \implies t\sigma_{11,1} + q_{,s} = 0,$$

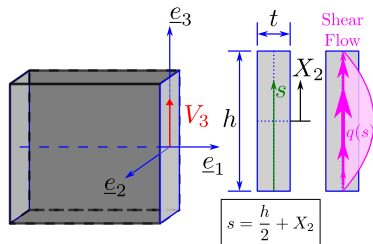
where we have integrated along the  $\underline{e}_n$  direction once.

- Following through with the integral along  $\underline{e}_s$ , this leads to the **shear flow formula**

$$q(s) - q_0 = - \frac{\left[ \int_0^s tX_3 ds \quad - \int_0^s tX_2 ds \right]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

## 2. Shear Stress and Flow in Sections

- Consider the rectangular section with height  $h$  and thickness  $t$ :



$$\begin{aligned}
 q(s) &= -\frac{V_3}{I_{22}} \int_0^s t X_3 ds = -\frac{tV_3}{I_{22}} \int_{-\frac{h}{2}}^{X_3} X_3 dX_2 \\
 &= -\frac{tV_3}{2I_{22}} \left( X_2^2 - \frac{h^2}{4} \right)
 \end{aligned}$$

- Remember that  $V_3$  is NOT any externally **applied force**. It is merely **the resultant of all the shear stresses in the section**.
- So  $V_3$  and  $q(s)$  point in the same direction in this example. It is incorrect to think that  $q(s)$  is balancing out  $V_3$ .



## 2. Shear Stress and Flow in Sections

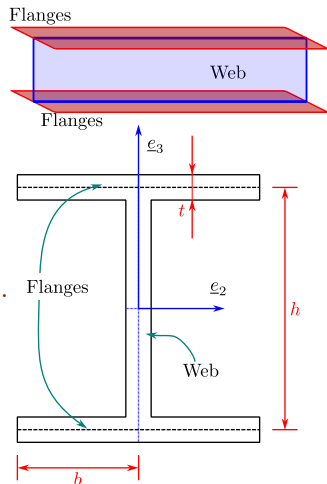
The "I" section

- Consider the shear distribution through an I-section as shown here
- The shear distribution looks like it is "flowing", with more "flow" occurring in the thin vertical web and less in the flanges.
- The second moment of area  $I_{22}$  sums up as,

$$I_{22} = \underbrace{\frac{h^3 t}{12}}_{web} + 2 \times \underbrace{\left( \frac{2bt^3}{12} + 2bt \times \frac{h^2}{4} \right)}_{flange} \approx \left( \frac{h^3}{12} + bh^2 \right) t.$$

- $I_{33}$  sums up as,

$$I_{33} = \underbrace{\frac{ht^3}{12}}_{web} + 2 \times \underbrace{\left( \frac{2b^3 t}{3} \right)}_{flange} \approx \underbrace{\frac{4b^3 t}{3}}_{\approx 0 \text{ for small } b}.$$

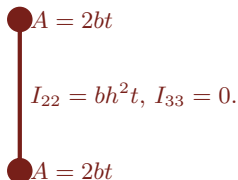


## 2. Shear Stress and Flow in Sections

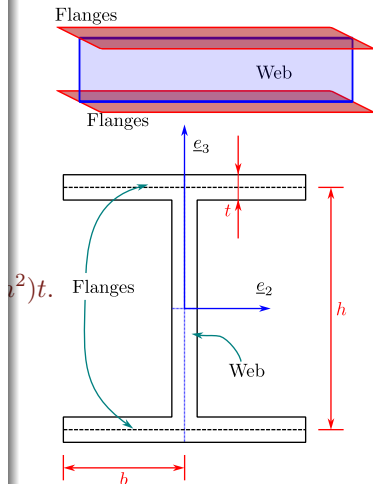
The “I” section

### Idealization

- Both  $I_{22}$  and  $I_{33}$  are dominated by flange contributions, implying that bending is supported primarily by the flanges.
- This motivates the following idealization for the I-section:



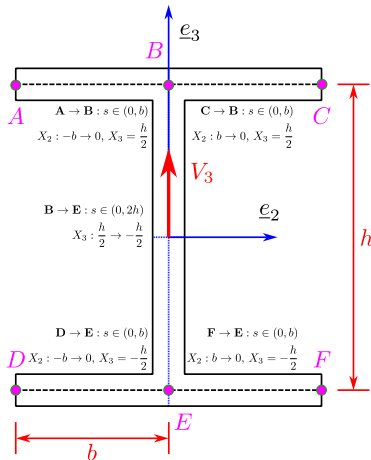
- The lumped area elements denoted ● are sometimes referred to as “**Booms**” in the section.
- Thickness in the web (denoted —) is taken to be zero for bending-stress calculations.



## 2. Shear Stress and Flow in Sections

### The "I" section

- Let us consider the case with  $V_2 = 0$ ,  $V_3 \neq 0$ .
- The segments  $A \rightarrow B$ ,  $C \rightarrow B$ ,  $D \rightarrow E$ , and  $F \rightarrow E$  are exposed in their free ends, simplifying the shear flow integral ( $q = 0$  at free ends).

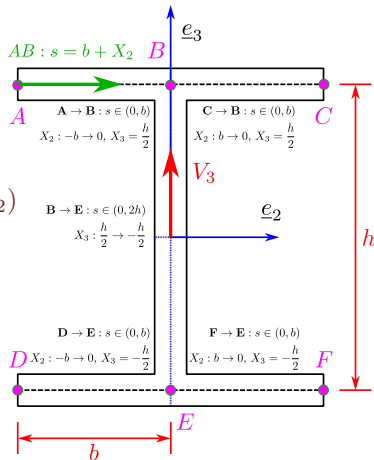


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$$\mathbf{A} \rightarrow \mathbf{B} : q_{AB}(s) \equiv q_{AB}(X_2) = -\frac{tV_3}{I_{22}} \int_0^s X_3 ds = -\frac{htV_3}{2I_{22}}(b + X_2)$$



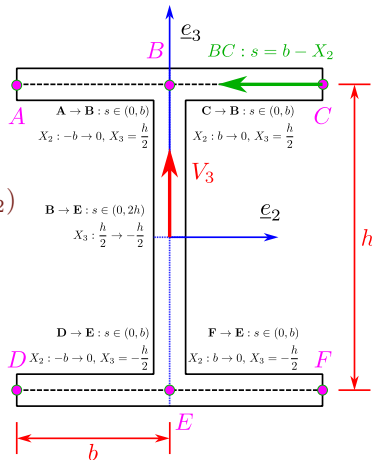
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## 2. Shear Stress and Flow in Sections

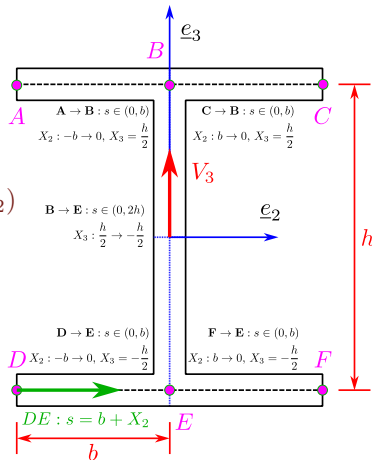
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$$\mathbf{D} \rightarrow \mathbf{E} : q_{DE}(X_2) = \frac{htV_3}{2I_{22}}(b + X_2)$$



## 2. Shear Stress and Flow in Sections

### The "I" section

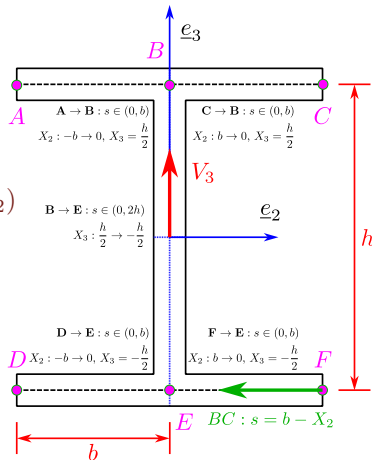
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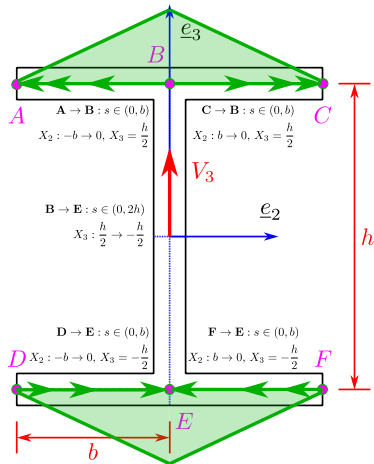
$$\mathbf{F} \rightarrow \mathbf{E} : q_{FE}(X_2) = \frac{htV_3}{2I_{22}}(b - X_2)$$



## 2. Shear Stress and Flow in Sections

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- In summary we have linear relationships at the flanges.



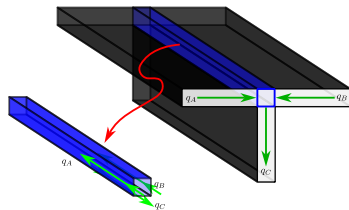


## 2. Shear Stress and Flow in Sections

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- In summary we have linear relationships at the flanges.
- Before looking at the web ( $B \rightarrow E$ ), we have to observe the balance at the "T" junction.

### Balance at the T-junction



$$\sum F = q_A + q_B - q_C = 0 \Rightarrow q_C = q_A + q_B$$

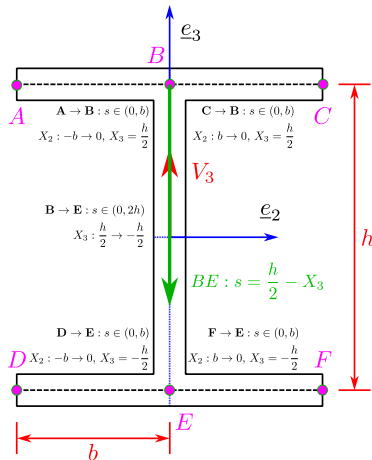
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- On  $B \rightarrow E$ , we have  

$$q_{BE}(0) = q_{AB}(0) + q_{CB}(0) = -\frac{bhtV_3}{I_{22}}.$$
- The integration evaluates as,

$$\begin{aligned} q_{BE}(X_3) &= -\frac{bhtV_3}{I_{22}} + \frac{tV_3}{2I_{22}}(X_3^2 - \frac{h^2}{4}) \\ &= -\frac{htV_3}{I_{22}}(b + \frac{h}{8}) + \frac{tV_3}{2I_{22}}X_3^2. \end{aligned}$$



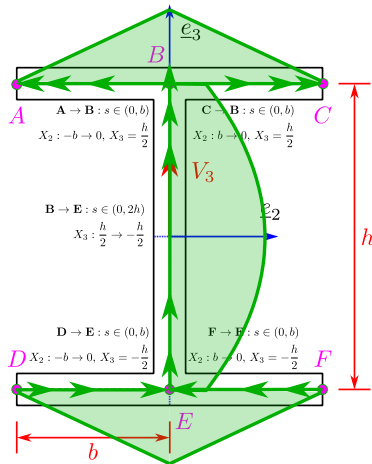
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- We now have the complete shear flow in the section.



## 2. Shear Stress and Flow in Sections

The “I” section: Order of Magnitude Analysis

- Let us consider the “total” shear forces experienced by each member.
- Flange AB**

$$V_{AB} = -\frac{htV_3}{2I_{22}} \int_{-b}^0 (b + X_2) dX_2 = -\frac{b^2htV_3}{4I_{22}}$$

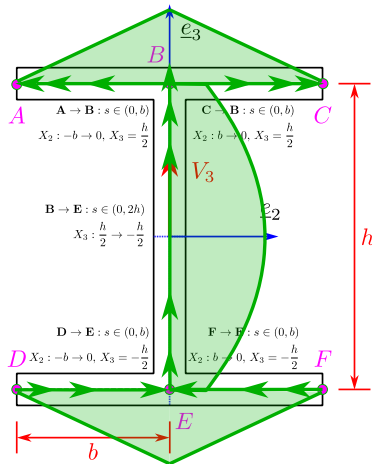
- Web BE**

$$V_{BE} = \frac{h^2(h + 12b)tV_3}{12I_{22}} = V_3$$

- For  $b = \frac{h}{2}$ , we have,

$$V_{AB} = -\frac{h^3tV_3}{16I_{22}} \approx -\frac{V_3}{8}$$

$$V_{BE} = V_3$$

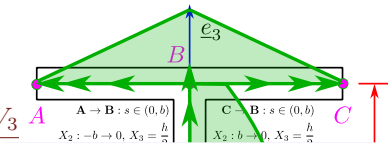


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**Idealization**

Since  $V_{AB} \ll V_{BE}$ , we understand that the **web is primarily responsible for restoring shear loads**, with negligible contributions from the flanges.

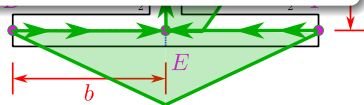
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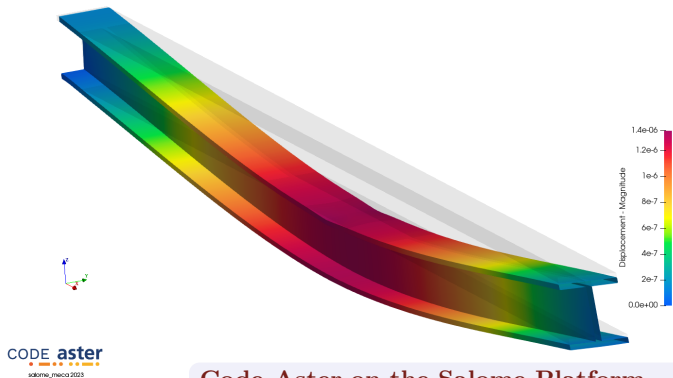
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## 2. Shear Stress and Flow in Sections

An "I" section beam subjected to 3-point bending: Finite Element Results

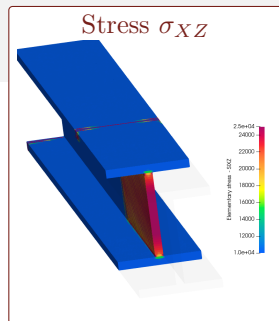
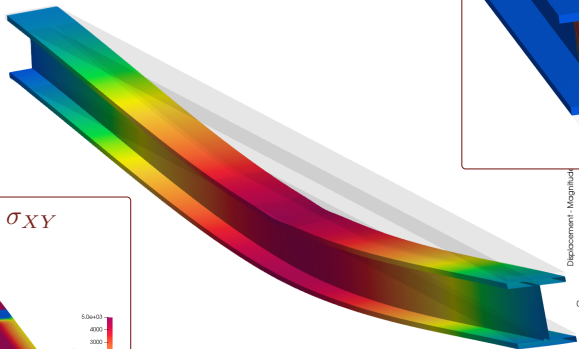
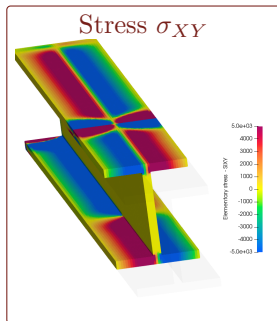


### Code\_Aster on the Salome Platform

Free and Open Source (FOSS) FE solver that comes with a fully functional frontend (Salome)! [Please Do Explore!](#)

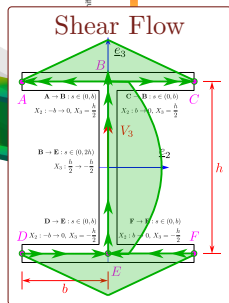
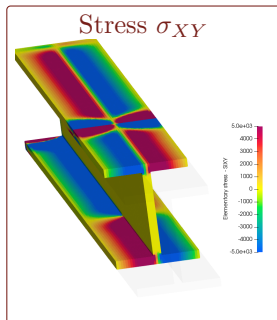
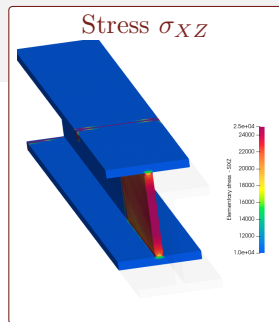
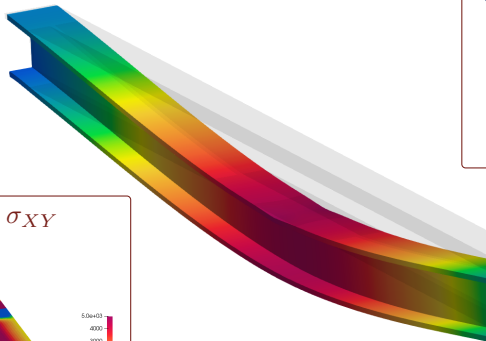
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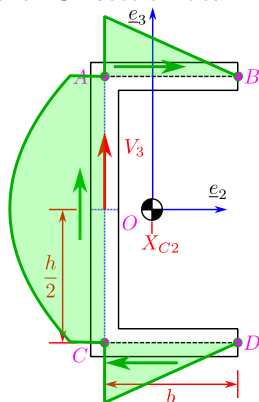




## 2.1. Shear Center

### Shear Stress and Flow in Sections

- Shear Center is the point of shear load such that  $\sum m_1 = 0$ . Although in a lot of symmetric sections this is coincident with the centroid, **this is NOT always the case.**
- Consider the “C” section beam:



$$I_{22} \approx \frac{(h^3 + 6bh^2)t}{12} + \mathcal{O}(t^2)$$

$$q_{BA}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

$$q_{AC}(X_3) = -\frac{htV_3}{2I_{22}}\left(b + \frac{h}{4}\right) + \frac{tV_3}{2I_{22}}X_3^2$$

$$q_{CD}(X_2) = -\frac{htV_3}{2I_{22}}(b - X_2)$$

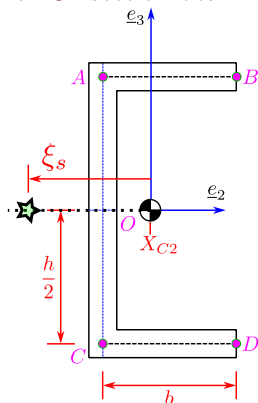
$$M_1 = \oint pqds = -\frac{b^2h^2tV_3}{4I_{22}} := V_3\xi_s$$

$$\xi_s \approx -\frac{3b^2}{h + 6b} + \mathcal{O}(t)$$

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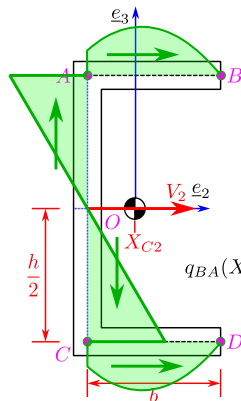
$$M_1 = \oint pqds = -\frac{b^2h^2tV_3}{4I_{22}} := -V_3\xi_s$$

$$\xi_s \approx \frac{3b^2}{h + 6b} + \mathcal{O}(t)$$

## 2.1. Shear Center

### Shear Stress and Flow in Sections

- Shear Center is the point of shear load such that  $\sum m_1 = 0$ . Although in a lot of symmetric sections this is coincident with the centroid, **this is NOT always the case.**
- Consider the “C” section beam:



$$XC_2 = \frac{(bt)\frac{b}{2} \times 2}{2bt + ht} = \frac{b}{2 + h/b}$$

$$q(s) - q(0) = -\frac{tV_2}{I_{33}} \int_0^s X_2 ds$$

### Shear Flow

$$q_{BA}(X_2) = \frac{tV_2}{2I_{33}} \frac{(X_2^2(h+2b)^2 - b^2(h+b)^2)}{(h+2b)^2}$$

$$q_{AC}(X_3) = -\frac{tb^2V_2}{(2b+h)I_{33}} X_3$$

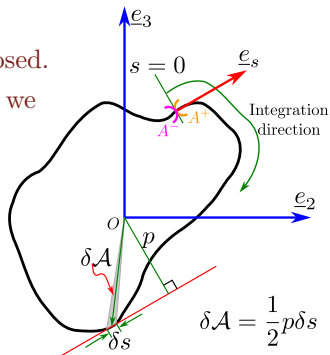
$$q_{CD}(X_2) = -q_{BA}(X_2)$$

## 2.2. Closed Sections

### Shear Stress and Flow in Sections

- The shear-flow integral formula makes no reference to whether the section is open or closed.
- Considering the generic closed section shown, we start the integral at some **arbitrary** point  $A$ , denoting the point right before it as  $A^-$ . The integral is then written as,

$$q(s) - q_{A^-} = - \underbrace{\int_0^s t \sigma_{11,1} ds}_{q_b(s)}$$



- When no twisting is expected at the section, the moment along  $\underline{e}_1$  about  $O$  has to be zero. This is computed as,

$$\int_{A^-}^{A^+} dM_1 = \oint p q(s) ds = q_{A^-} \underbrace{\oint p q(s) ds}_{2A} + \oint p q_b(s) ds \implies \boxed{q_{A^-} = -\frac{1}{2A} \oint p q_b(s) ds}$$

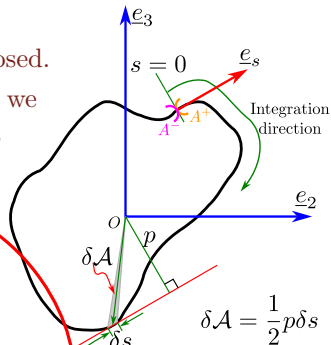
## 2.2. Closed Sections

### Shear Stress and Flow in Sections

- The shear-flow integral formula makes no reference to whether the section is open or closed.
- Considering the generic closed section shown, we start the integral at some arbitrary point  $A$ ,  $A^-$ . The

Are we already assuming that  $O$  is the shear center with this?!

$$q(s) - q_{A^-} = - \underbrace{\int_0^s t \sigma_{11,1} ds}_{q_b(s)}$$



$$\delta \mathcal{A} = \frac{1}{2} p \delta s$$

- When no twisting is expected at the section, the moment along  $\underline{e}_1$  about  $O$  has to be zero. This is computed as,

$$\int_{A^-}^{A^+} dM_1 = \oint p q(s) ds = q_{A^-} \overbrace{\oint p q(s) ds}^{2\mathcal{A}} + \oint p q_b(s) ds \implies q_{A^-} = -\frac{1}{2\mathcal{A}} \oint p q_b(s) ds$$

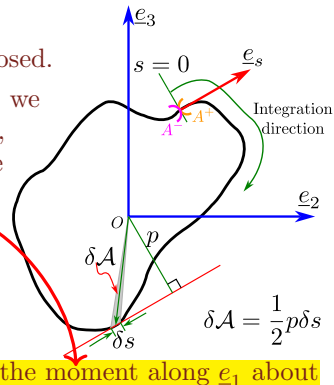
## 2.2. Closed Sections

### Shear Stress and Flow in Sections

- The shear-flow integral formula makes no reference to whether the section is open or closed.
- Considering the generic closed section shown, we start the integral at some arbitrary point  $A$ ,  $A^-$ . The

Are we already assuming that  $O$  is the shear center with this?!

Yes. We need to add an extra term when resultants  $V_2, V_3$  are acting with an offset.



When no twisting is expected at the section, the moment along  $\underline{e}_1$  about  $O$  has to be zero. This is computed as,

$$\int_{A^-}^{A^+} dM_1 = \oint pq(s) ds = q_{A^-} \overbrace{\oint pq(s) ds}^{2\mathcal{A}} + \oint pq_b(s) ds \implies q_{A^-} = -\frac{1}{2\mathcal{A}} \oint pq_b(s) ds.$$

## 2.2. Closed Sections

### Shear Stress and Flow in Sections

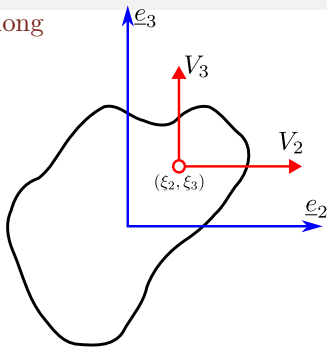
- If we suppose that the resultants are acting along some  $(\xi_2, \xi_3)$ , the applied moment is:  
 $(\xi_2 \underline{e}_2 + \xi_3 \underline{e}_3) \times (V_2 \underline{e}_2 + V_3 \underline{e}_3)$ :

$$M_1 = \xi_2 V_3 - \xi_3 V_2.$$

- Equating this to the moment developed from the shear flow distribution, we get:

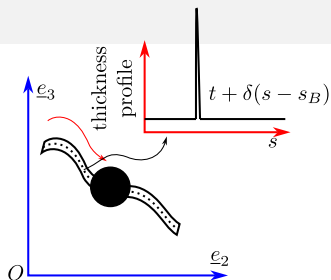
$$\xi_2 V_3 - \xi_3 V_2 = 2Aq_{A-} + \oint pq_b(s) ds$$

- For shear center determination (the point around which resultants act), this is not enough.
- We will additionally invoke an argument of **zero twist** ( $\theta_{1,1} = 0$ ) in the deflection field to get an additional relationship. **This will be covered in the next module (Torsion)**.
- For symmetric closed sections**, however, the shear center coincides with the centroid (through symmetry arguments).



### 3. Stringer-Web Idealization

- A stringer, when looked at from a section, looks like a feature with a sudden increase in thickness.
- Consider the  $r^{\text{th}}$  “Boom” to be located at  $(X_{r2}, X_{r3})$ , with area  $A_r$ .



- So the shear flow integral can be generalized to,

$$q(s) - q(0) = - \frac{\left[ t \int_0^s X_3 ds + \sum_r A_r X_{r3} \quad -t \int_0^s X_2 ds - \sum_r A_r X_{r2} \right]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

- When thickness  $t$  is negligible in comparison to the boom sections, this further simplifies to,

$$q(s) - q(0) = - \frac{\left[ \sum_r A_r X_{r3} \quad - \sum_r A_r X_{r2} \right]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}$$

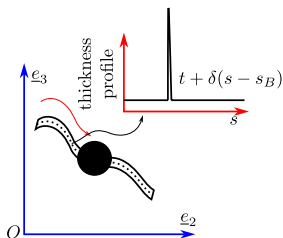


### 3. Stringer-Web Idealization

- Considering the integral right across the boom, we have,

$$q^+ - q^- = - \frac{[A_r X_{r3} \quad -A_r X_{r2}]}{I_{22}I_{33} - I_{23}^2} \begin{bmatrix} I_{33}V_3 - I_{23}V_2 \\ I_{23}V_3 - I_{22}V_2 \end{bmatrix}.$$

- For the sections without a boom, there is **no change in the shear flow**.
- Therefore, the shear flow is constant in the webs that join two booms.



#### General Design Principle

- As a general principle, the stringers/booms are added to support bending.
- The web thicknesses are chosen to ensure the shear stresses don't exceed failure threshold (we should like to have  $t = 0$  to minimize weight!).

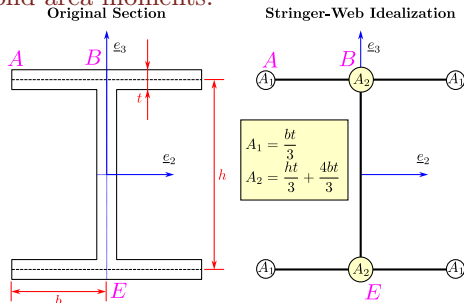
### 3. Stringer-Web Idealization

#### Idealization of the I-Section

- We idealize the I-section by lumping area elements  $A_1$  and  $A_2$ .  $A_1$  and  $A_2$  are estimated by matching the second area moments:

$$\underbrace{4 \times \left( \frac{A_1 h^2}{4} \right) + 2 \times \left( \frac{A_2 h^2}{4} \right)}_{I_{22,ideal}} = \underbrace{\left( \frac{h^3}{12} + bh^2 \right) t + \mathcal{O}(t^2)}_{I_{22,orig}}$$

$$\underbrace{4 \times (A_1 b^2)}_{I_{33,ideal}} = \underbrace{\frac{4b^3 t}{3} + \mathcal{O}(t^2)}_{I_{33,ideal}}$$



- The total sectional area of the original section is  $ht + 4bt$ .
- The total area of the new section (assuming the web thickness is drastically reduced) is  $4A_1 + 2A_2 = \frac{2ht}{3} + 4bt$ , which is a **slight reduction**.
- Looking at this from a manufacturing standpoint, this shows that a **web-stringer construction can achieve similar bending stiffness with lesser material expenditure**.

### 3. Stringer-Web Idealization

#### Idealization of the I-Section: Shear Flow Comparisons

- For the idealized section  $AB$ , the shear flow is given by,

$$q_{AB,ideal} = -\frac{V_3}{I_{22,ideal}} A_1 \frac{h}{2} = -\frac{V_3 b}{h^2 + 12bh}$$

The average flow for the original section is,

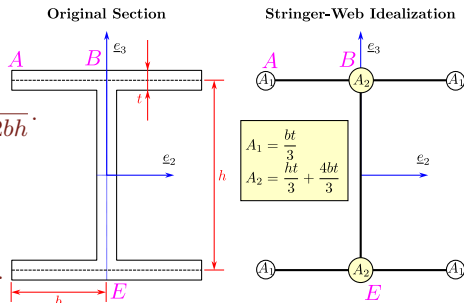
$$q_{AB,avg} = \frac{1}{b} \left( -\frac{b^2 ht V_3}{4I_{22}} \right) = \frac{3V_3 b}{h^2 + 12bh}$$

- On  $BE$ , the idealized flow is

$$q_{BE,ideal} = 2q_{AB,ideal} - \frac{V_3}{I_{22,ideal}} A_2 \frac{h}{2} = -\frac{V_3}{h},$$

which is the same for the original section also.

- In the stringer-web section, therefore, the flanges carry lesser average shear than the original section, and the web carries the same shear.



## 4. Shear Lag

Example from Sun [1] (Section 3.1)

- We already saw the St. Venant's principle. Can we say something more specific about “how fast” the end effects start getting smoothed out in the stress field? Consider:
- Applying equilibrium on the section at some  $X_1 = x$  ① leads to top and bottom booms

experiencing  $\int_{\mathcal{A}_1} \sigma_{11} dA = P(x)$ ,

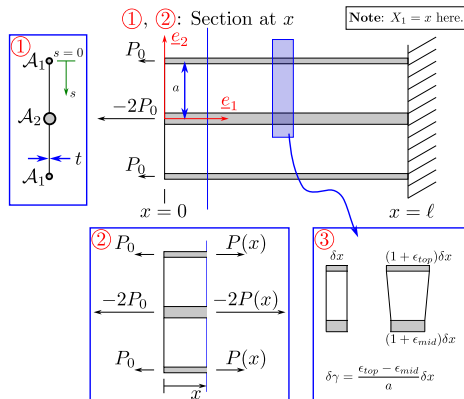
and the central boom,

$$\int_{\mathcal{A}_2} \sigma_{11} dA = -2P(x).$$

- Shear flow in top web is given as

$$q_{top} = -\sigma_{11,1} \mathcal{A}_1 = -P_{,1}$$

or,  $-\tau_{12} = -P_{,1}$ .



# 4. Shear Lag

Example from Sun [1] (Section 3.1)

- Considering the shear differential in an infinitesimal section (③) we have,

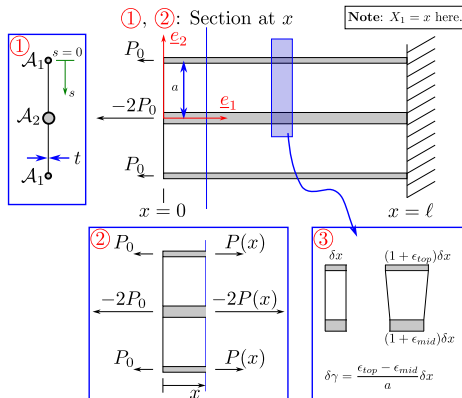
$$\begin{aligned}\frac{\partial \gamma_{12}}{\partial x} &= \frac{1}{a}(\epsilon_{top} - \epsilon_{mid}) \\ &= \frac{1}{E_y a} \left( \frac{P(x)}{A_1} - \frac{-2P(x)}{A_2} \right)\end{aligned}$$

- Since  $\gamma_{12} = \frac{1}{G}\sigma_{12}$ , this becomes,

$$\sigma_{12,1} = \frac{G}{E_y a} \left( \frac{1}{A_1} + \frac{2}{A_2} \right) P(x).$$

- Using the force balance

relationship  $\sigma_{12} = \frac{1}{t}P_{,1}$  also



$$\Rightarrow P_{,11} = \frac{Gt}{E_y a} \left( \frac{1}{A_1} + \frac{2}{A_2} \right) P$$

## 4. Shear Lag

Example from Sun [1] (Section 3.1)

- The equation governing the **boom restoring axial force** is of the form

$$P_{,11} - \lambda^2 P = 0, \quad \lambda = \sqrt{\frac{Gt}{E_y a} \left( \frac{1}{\mathcal{A}_1} + \frac{2}{\mathcal{A}_2} \right)}.$$

- This is solved by

$$P(X_1) = C_1 e^{-\lambda X_1} + C_2 e^{\lambda X_1}.$$

- Solving it over  $X_1 \in (0, \infty)$ , it is easy to see that  $C_2$  must be 0 for  $P(X_1)$  to be bounded. So we have an **exponentially decaying** restoring force on the booms:

$$P(X_1) = P_0 e^{-\lambda X_1}.$$

- Substituting for  $\sigma_{12}$  we have (for the top web),

$$\sigma_{12} = -\frac{\lambda P_0}{t} e^{-\lambda X_1},$$

which also decays exponentially in  $X_1$ .

## 4. Shear Lag: Decay Rate

Example from Sun [1] (Section 3.1)

- The shear lag factor  $\lambda$  controls how quickly the effects “diffuse out”.
  - Large  $\lambda$  implies “fast” diffusion and potentially high concentration around the ends..
  - Small  $\lambda$  implies “slow” diffusion and potentially low concentrations.
- In general for stringer-web structures,

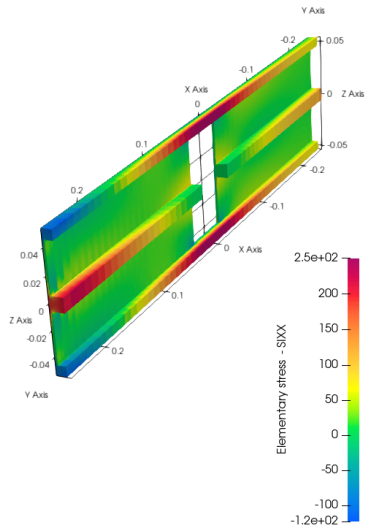
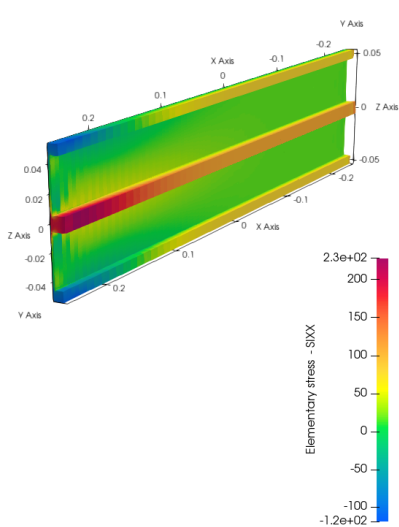
$$\lambda \propto \sqrt{\frac{G}{E_y}}$$

- $\uparrow G$  (stiffer web),  $\uparrow \lambda$  (“faster” diffusion).
- $\uparrow E_y$  (stiffer boom),  $\downarrow \lambda$  (“slower” diffusion).

The terms “faster” and “slower” are used in the sense that “slower” implies that the stress distribution needs a longer axial distance to get averaged out (vise versa for “faster”).

# 4. Shear Lag

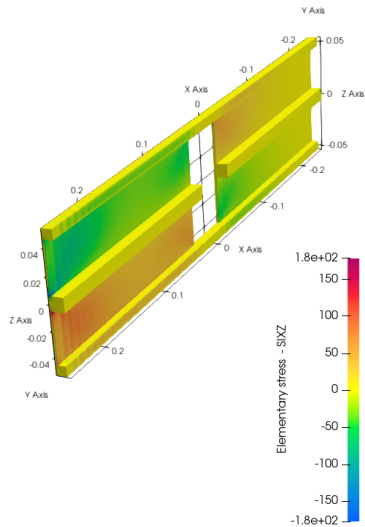
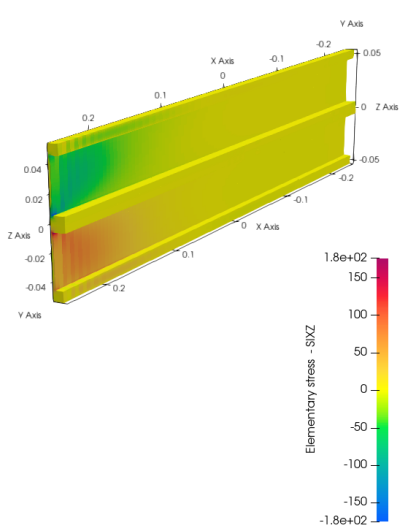
Example from Sun [1] (Section 3.1): FE Results





# 4. Shear Lag

Example from Sun [1] (Section 3.1): FE Results



# References I

- [1] C. T. Sun. *Mechanics of Aircraft Structures*, 2nd edition. Hoboken, N.J: Wiley, June 2006. ISBN: 978-0-471-69966-8 (cit. on pp. 2, 36–41).
- [2] T. H. G. Megson. *Aircraft Structures for Engineering Students*, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).