



AS3020: Aerospace Structures

Module 3: Introduction to Elasticity

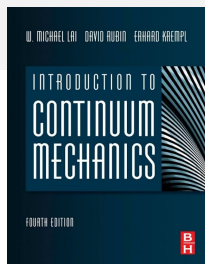
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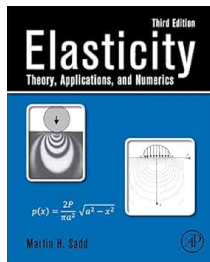
August 27, 2024

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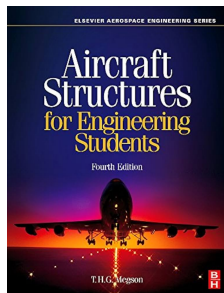
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Chapters 1-5 in Lai, Rubin, and Krempf [1]



Chapters 1-5 in Sadd [2]

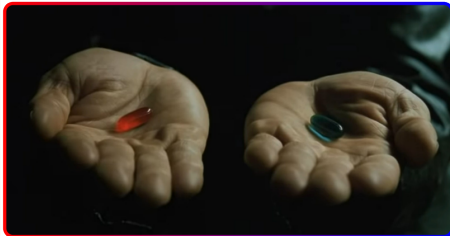
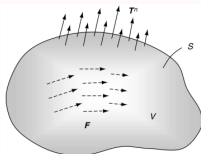
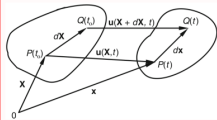


Chapters 1,2 in Megson [3]

We have to make a choice!

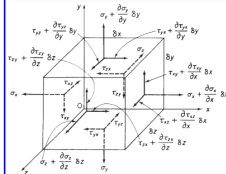
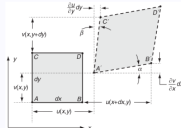
Red Pill

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$



Blue Pill

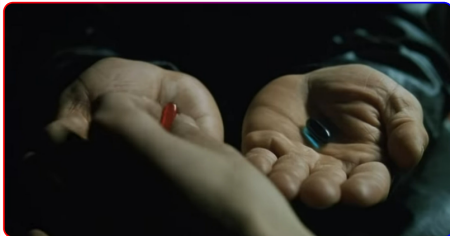
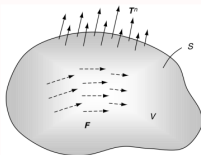
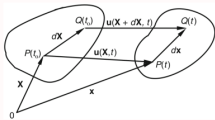
$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}(\sigma_y + \sigma_z)$$



~~We have to~~ ^I make a choice!

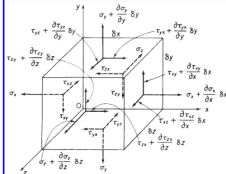
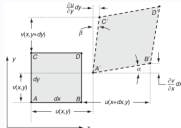
Red Pill

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$



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$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}(\sigma_y + \sigma_z)$$



1.1. Indicial Notation I

1. Mathematical Rudiments

Einstein's Summation Convention: Dummy Indices

$$s = a_1x_1 + a_2x_2 + \cdots = \sum_{i=1}^n a_i x_i \rightarrow a_i x_i = a_k x_k = a_m x_m$$

$$\text{Consider } \alpha = a_{ij}x_i x_j, \underline{v} = v_i \hat{e}_i, \underline{\underline{T}} = T_{ij} \hat{e}_i \hat{e}_j$$

Free Indices

$$\left. \begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \right\} \implies y_i = a_{ij}x_j$$

$$\text{Consider } T_{ij} = A_{im}A_{jm}.$$

1.1. Indicial Notation II

1. Mathematical Rudiments

The Kronecker Delta

$$\delta_{ij} := \hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Consider $C_{ijkl} = \delta_{ik}\delta_{jl}$, $C_{ijkl} = \delta_{il}\delta_{jk}$.

The Levi-Civita Symbol

$$\epsilon_{ijk} := \hat{e}_i \cdot \underbrace{(\hat{e}_j \times \hat{e}_k)}_{\epsilon_{ijk} \hat{e}_i} = \begin{cases} 1 & \text{if } \{(i, j, k)\} \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\} \\ -1 & \text{if } \{(i, j, k)\} \in \{(3, 2, 1), (2, 1, 3), (1, 3, 2)\} \\ 0 & \text{otherwise} \end{cases}$$

Consider $\underline{a} \cdot (\underline{b} \times \underline{c})$, $\Delta \underline{F}$.

1.1. Indicial Notation III

1. Mathematical Rudiments

Property: $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

$$\begin{aligned} \epsilon_{ijk}\epsilon_{mnk} &= (\epsilon_{ijk}\hat{e}_k) \cdot (\epsilon_{mnk}\hat{e}_k) = (\hat{e}_i \times \hat{e}_j) \cdot (\hat{e}_m \times \hat{e}_n) \\ (\hat{e}_i \times \hat{e}_j) \cdot (\hat{e}_m \times \hat{e}_n) &= \begin{cases} 1, & \hat{e}_i \times \hat{e}_j = \hat{e}_m \times \hat{e}_n \\ -1, & \hat{e}_i \times \hat{e}_j = -\hat{e}_m \times \hat{e}_n = \hat{e}_n \times \hat{e}_m \\ 0, & \text{otherwise} \end{cases} \\ &= \boxed{\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}} \end{aligned}$$

Consider $\underline{a} \times (\underline{b} \times \underline{c})$

1.1. Indicial Notation IV

1. Mathematical Rudiments

Derivative Notation

$$\frac{\partial u_i}{\partial x_j} := u_{i,j}$$

Consider $\nabla \underline{u}$, $\nabla \cdot \underline{u}$, $\nabla \times \underline{u}$, $\nabla \times \underline{Q}$

Exercise

$$\nabla \underline{u}, \underbrace{\nabla \cdot (\nabla \underline{u})}_{\nabla^2 \underline{u}}, \nabla \cdot (\nabla \times \underline{u}), \nabla \times \nabla \times \underline{u}, \nabla \cdot \underline{\underline{\sigma}}$$

1.1. Indicial Notation V

1. Mathematical Rudiments

Vectors, Tensors

$$\underline{u} = u^i \hat{e}_i, \quad \underline{\underline{T}} = T^{ij} \hat{e}_i \hat{e}_j$$

Consider:

- Order of a tensor
- Vector-components as first order tensors
- The tensor product and 2nd order tensors
- Tensors as defining an operation
- Identity **tensors**
- Coordinate transformation
- “Notational abuse”
- Symmetric, antisymmetric tensors
- Antisymmetry as a cross product
- Representation of Eigen-decomposition
- Calculus: Gradient, Divergence, Laplacian, Curl, curvilinear coordinates

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

Curvilinear Coordinates

- Scalar field ϕ gradient:

$$\begin{aligned}\delta\phi &= \frac{\partial\phi}{\partial x_1}\delta x_1 + \frac{\partial\phi}{\partial x_2}\delta x_2 \\ &= \frac{\partial\phi}{\partial r}\delta r + \frac{\partial\phi}{\partial\theta}\delta\theta\end{aligned}$$

- Polar bases

$$\begin{aligned}\underline{e}_r &= C_\theta\underline{e}_1 + S_\theta\underline{e}_2 \implies \delta\underline{e}_r = \delta\theta\underline{e}_\theta \\ \underline{e}_\theta &= -S_\theta\underline{e}_1 + C_\theta\underline{e}_2 \implies \delta\underline{e}_\theta = -\delta\theta\underline{e}_r\end{aligned}$$

- Position vector

$$\begin{aligned}\delta\underline{r} &= \delta r\underline{e}_r + r\delta\underline{e}_r \\ &= \delta r\underline{e}_r + r\delta\theta\underline{e}_\theta\end{aligned}$$

- For $\delta\phi = \nabla\phi \cdot \delta\underline{r}$,

$$\nabla\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta$$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

Integral Calculus

- The line integral: $\int \underline{F} \cdot d\underline{x}$

Potential theory:

$$\int_{\partial \mathcal{D}} F_i dx_i = 0 \implies$$

- $F_i = \phi_{,i}$ and $\epsilon_{ijk} F_{k,j}|_{\mathcal{D}} = 0$
- $\underline{F} = \nabla \phi$ and $\nabla \times \underline{F} = \underline{0}$
- **Gauss Divergence Theorem**

$$\int_{\mathcal{D}} P_{ijk\dots,i} d\mathcal{D} = \int_{\partial \mathcal{D}} P_{ijk\dots} dA_i$$
- **Stoke's Law:**

$$\int_A (\nabla \times \underline{F}) \cdot d\underline{A} = \int_{\partial A} \underline{F} \cdot d\underline{x}$$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differ

- Scalar, ve
- Gradients
- Divergen
- Curviline
divergenc
coordinat

Stoke's Law as a Special Case of Gauss Divergence in 2D

$$\begin{aligned}
 \int_S (\nabla \times \underline{v}) \cdot d\underline{S} &= \int_S \epsilon_{ijk} v_{k,j} \hat{n}_i d|S| \\
 &= \int_S (\epsilon_{ijk} \hat{n}_i v_{k,j}) d|S| \\
 &= \int_{\partial S} \epsilon_{ijk} \hat{n}_i v_k \hat{b}_j d|\ell| \\
 &= \int_{\partial S} \underbrace{(\epsilon_{ijk} \hat{n}_i \hat{b}_j)}_{\hat{n} \times \hat{b}} v_k d|\ell| \\
 &= \int_{\partial S} v_k t_k d|\ell| = \int_{\partial S} \underline{v} \cdot d\underline{\ell}
 \end{aligned}$$

ulus

$$\underline{F} \cdot d\underline{x}$$

$$\begin{aligned}
 & \epsilon_{ijk} F_{k,j} |D| = 0 \\
 & \nabla \times \underline{F} = \underline{0}
 \end{aligned}$$

e Theorem

$$P_{ijk} \dots dA_i$$

$$\int_{\partial A} \underline{F} \cdot d\underline{x}$$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

Integral Calculus

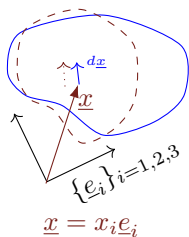
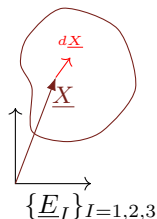
- The line integral: $\int \underline{F} \cdot d\underline{x}$
- **Potential theory:**
 $\int_{\partial \mathcal{D}} F_i dx_i = 0 \implies$
 - $F_i = \phi_{,i}$ and $\epsilon_{ijk} F_{k,j}|_{\mathcal{D}} = 0$
 - $\underline{F} = \nabla \phi$ and $\nabla \times \underline{F} = \underline{0}$
- **Gauss Divergence Theorem**
 $\int_{\mathcal{D}} P_{ijk\dots,i} d\mathcal{D} = \int_{\partial \mathcal{D}} P_{ijk\dots} dA_i$
- **Stoke's Law:**
 $\int_A (\nabla \times \underline{F}) \cdot d\underline{A} = \int_{\partial A} \underline{F} \cdot d\underline{x}$
- **Determinant of a Tensor**
 $\epsilon_{IJK} \Delta \underline{\underline{F}} = \epsilon_{ijk} F_{iI} F_{jJ} F_{kK}$
 - Related to volume change through transformation

2. Deformations and Strain

2.1. The Basic Premise

How to describe the change in shape **independently** of rigid body motions?

$$\underline{X} = X_I \underline{E}_I$$



- The deformations are mapped as
Lagrangian $x_i = x_i(\underline{X})$
Eulerian $X_i = X_i(\underline{x})$
- Under the **Lagrangian description** we have,

$$dx_i = \overbrace{\frac{\partial x_i}{\partial X_I}}^{F_{iI}} dX_I$$

Length $ds^2 = dx_i dx_i =$
 $dX_I \left[\frac{\partial x_i}{\partial X_I} \frac{\partial x_i}{\partial X_J} \right] dX_J$
 Angle $ds_1 ds_2 \cos \theta = dx_i dx_j =$
 $dX_I \left[\frac{\partial x_i}{\partial X_I} \frac{\partial x_j}{\partial X_J} \right] dX_J$

How does $d\underline{X}$
transform into $d\underline{x}$?

$$\underline{x} = \underline{X} + \underline{u}$$

2.2. Coordinate Transformation

2. Deformations and Strain

- A vector \underline{v} is written as

$$\underline{v} = v_i \underline{e}_i,$$

and is defined as a **linear combination of the bases of its vector-space**.

- Suppose I have another coordinate system spanning the same vector-space, this comes with its own set of basis vectors $\{\underline{b}_i\}_{i=1,\dots,n}$.
- If the vector represents a physical/geometrical measurement, it **can not change based on coordinate system**, i.e., it is coordinate invariant.
- So, the following equality must hold:

$$\underline{v} = v_i \underline{e}_i = \bar{v}_i \underline{b}_i,$$

with v_i and \bar{v}_i being the **components of the vector** under the different coordinate systems.

2.2. Coordinate Transformation

2. Deformations and Strain

- Assuming that both $\{\underline{e}_i\}$ and $\{\underline{b}_i\}$ represent **orthogonal coordinate systems** (inner products $\langle \underline{e}_i, \underline{e}_j \rangle \equiv \langle \underline{b}_i, \underline{b}_j \rangle = \delta_{ij}$), we write down:

$$v_i = \langle \underline{v}, \underline{e}_i \rangle; \quad \bar{v}_i = \langle \underline{v}, \underline{b}_i \rangle.$$

- Evaluating \bar{v}_i we obtain,

$$\bar{v}_i = \langle v_j \underline{e}_j, \underline{b}_i \rangle = \langle \underline{b}_i, \underline{e}_j \rangle v_j.$$

Denoting $\langle \underline{b}_i, \underline{e}_j \rangle = Q_{ij}$, we get our **component transformation law for a vector**:

$$\bar{v}_i = Q_{ij} v_j$$

What about the basis vectors themselves?

How can I **combine** the \underline{e}_i 's to obtain the \underline{b}_i 's ?

- How **should I combine them** so that my vector is invariant?

2.2. Coordinate Transformation

2. Deformations and Strain

- Given the $\bar{v}_i = Q_{ij}v_j$, and the requirement $v_i\underline{e}_i = \bar{v}_j\underline{b}_j$, we write (after swapping $i \leftrightarrow j$ in LHS),

$$Q_{ji}v_i\underline{b}_j = v_i\underline{e}_i \implies Q_{ji}\underline{b}_j = \underline{e}_i \quad (\text{multiply both sides by } (\mathbb{Q}^{-1})_{ik})$$

$$\begin{aligned} \overbrace{Q_{ji}(\mathbb{Q}^{-1})_{ik}}^{\delta_{jk}} \underline{b}_j &= (\mathbb{Q}^{-1})_{ik}\underline{e}_i \\ \implies \underline{b}_i &= (\mathbb{Q}^{-1})_{ji}\underline{e}_j \end{aligned}$$

- Comparing the two, we have

$$\boxed{\bar{v}_i = Q_{ij}v_j}$$

$$\boxed{\underline{b}_i = (\mathbb{Q}^{-1})_{ji}\underline{e}_j}$$

- This is a **necessary requirement** so that the vector **remains invariant**.

2.2. Coordinate Transformation: Array Notation

2. Deformations and Strain

- Now we introduce the Array Notation for vectors. **Let \underline{v} be a vector.** The array of its components with respect to the basis $\{\underline{e}_i\}$ is written as,

$$\underline{v} = \begin{bmatrix} \langle \underline{v}, \underline{e}_1 \rangle \\ \langle \underline{v}, \underline{e}_2 \rangle \\ \vdots \end{bmatrix} \quad (\text{similarly for } \bar{v}).$$

- We also define the **array of coordinate vectors** as

$$\underline{\underline{e}} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \vdots \end{bmatrix}; \quad \underline{\underline{b}} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \end{bmatrix}.$$

\bar{v} “contra-varies”
w.r.t. v , in
comparison with
how $\underline{\underline{b}}$ and $\underline{\underline{e}}$ are
related.

- Under this notation we have,

$$\boxed{\bar{v} = Q \underline{v}} \quad \text{and} \quad \boxed{\underline{\underline{b}} = Q^{-T} \underline{\underline{e}}}.$$

$\implies v$ are the
contravariant
components of \underline{v} .

2.2. Coordinate Transformation: Tensors

2. Deformations and Strain

- We will define a (2nd order) tensor as a **linear combination of basis-dyads**:

$$\underline{\underline{T}} = T_{ij} \underline{e}_i \underline{e}_j = \bar{T}_{ij} \underline{b}_i \underline{b}_j,$$

where we have required $\underline{\underline{T}}$ to be **invariant** under coordinate change.

- Using a **double-contraction** operation, we write down the components of \bar{T}_{ij} as,

$$\begin{aligned} \bar{T}_{ij} &= T_{mn} \underbrace{\langle \underline{b}_i, \underline{e}_m \rangle}_{Q_{im}} \underbrace{\langle \underline{b}_j, \underline{e}_n \rangle}_{Q^{jn}} \\ &= Q_{im} T_{mn} Q^{jn}. \end{aligned}$$

For a tensor to be invariant, its components have to transform in this fashion.

- In array notation we write the components as,

$$\bar{\mathbf{T}} = \mathbf{Q} \mathbf{T} \mathbf{Q}^T.$$

2.2. Coordinate Transformation: Summary

2. Deformations and Strain

Supposing I specify a basis change by

$$\underline{\underline{b}} = \mathbb{Q}^{-T} \underline{\underline{e}},$$

- for a vector $\underline{v} = v^T \underline{\underline{e}}$ to be invariant, its components have to transform as

$$\bar{v} = \mathbb{Q}v.$$

- for a tensor $\underline{\underline{T}} = T_{\underline{\underline{e}}} \otimes \underline{\underline{e}}$ to be invariant, its components have to transform as

$$\bar{\underline{\underline{T}}} = \mathbb{Q}T\mathbb{Q}^T$$

- If it transforms in any other fashion, **then invariance is not guaranteed.**

2.2. Coordinate Transformation: Relationship to Gradients

2. Deformations and Strain

We will now establish a relationship between coordinate transformation and **component-gradients**.

- Consider an infinitesimal line vector $d\underline{x} = dx_i \underline{e}_i = d\bar{x}_i \underline{b}_i$.
- It is obvious that the components $d\bar{x}$ have to be related to the components dx . So we write

$$d\bar{x}_i = \frac{\partial \bar{x}_i}{\partial x_j} dx_j \quad (1)$$

- By invariance requirements, we have

$$d\bar{x}_i = Q_{ij} dx_j. \quad (2)$$

- Comparing eq. (1) and eq. (2) we obtain,

$$Q_{ij} = \frac{\partial \bar{x}_i}{\partial x_j}$$

or

$$Q = \text{grad}(\bar{x})$$

$\text{grad}(\cdot)$ operator
 \implies gradient
 operation

2.2. Coordinate Transformation: The Deformation Gradient

2. Deformations and Strain

- The **components of the deformation gradient** are written as

$$F_{iI} = \frac{\partial x_i}{\partial X_I}.$$

- Under coordinate change we have,

$$\begin{aligned} \bar{F}_{iI} &= \frac{\partial \bar{x}_i}{\partial x_j} \frac{\partial x_j}{\partial X_J} \frac{\partial X_J}{\partial \bar{X}_I} \\ &= Q_{ij}^{(x)} F_{jJ} (Q^{(X)})^{-1}_{JI} \implies \boxed{\bar{\mathbb{F}} = \mathbb{Q}^{(x)} \mathbb{F} \mathbb{Q}^{(X)-1}}. \end{aligned}$$

This is transforming quite unlike a tensor for 2 reasons

- $\mathbb{Q}^{(x)}$ and $\mathbb{Q}^{(X)}$ **need not necessarily be the same** (we are free to choose measurement coordinates at each instant)
- $\mathbb{Q}^{-1} = \mathbb{Q}^T$ on for orthonormal coordinate systems (cartesian, for eg.). For non-orthonormal bases, this is not so.

2.2. Coordinate Transformation: The Cauchy Deformation Tensor

2. Deformations and Strain

- Now we consider $\mathbb{C} = \mathbb{F}^T \mathbb{F}$. Under coordinate change this becomes,

$$\begin{aligned}\bar{\mathbb{C}} &= \bar{\mathbb{F}}^T \bar{\mathbb{F}} = \left(\mathbb{Q}^{(x)} \mathbb{F} \mathbb{Q}^{(X)^{-1}} \right)^T \left(\mathbb{Q}^{(x)} \mathbb{F} \mathbb{Q}^{(X)^{-1}} \right) \\ &= \mathbb{Q}^{(X)^{-T}} \mathbb{F}^T \mathbb{Q}^{(x)T} \mathbb{Q}^{(x)} \mathbb{F} \mathbb{Q}^{(X)^{-1}}\end{aligned}$$

- Suppose we choose to stick with **coordinate systems with orthonormal bases**, $\mathbb{Q}^{-1} = \mathbb{Q}^T$ (for both (x) and (X)). Hereby the components matrix \mathbb{C} reduces to

$$\mathbb{C} = \mathbb{Q}^{(X)} \mathbb{F}^T \mathbb{F} \mathbb{Q}^{(X)T}$$

Unlike the deformation gradient...

...this is transforming like a tensor's components!

So I can define the **Cauchy deformation tensor** as: $\underline{\underline{C}} = C_{IJ} \underline{E}_I \underline{E}_J$

2.3. The Strain Tensor

2. Deformations and Strain

- We are now ready to define the strain tensor based on length change. We wrote,

$$\begin{aligned} ds^2 - dS^2 &= dX_I (F_{iI} F_{jJ} - \delta_{IJ}) dX_J \\ &= d\tilde{X}^T [\mathbb{F}^T \mathbb{F} - \mathbb{I}] d\tilde{X} = d\tilde{X}^T [\mathbb{C} - \mathbb{I}] d\tilde{X}. \end{aligned}$$

- For small changes in length,
 $ds^2 - dS^2 = (ds + dS)(ds - dS) \approx 2dS(ds - dS).$
- Representing the elongation as a fraction of the total length we write $(ds - dS) = \epsilon dS$. Using this we have,

$$2dS^2 \epsilon = d\tilde{X}^T [\mathbb{C} - \mathbb{I}] d\tilde{X} \implies 2d\tilde{X}^T d\tilde{X} \epsilon = d\tilde{X}^T [\mathbb{C} - \mathbb{I}] d\tilde{X}.$$

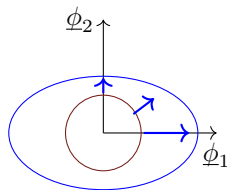
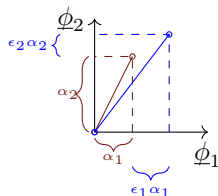
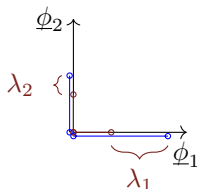
Here the single factor ϵ represents what the matrix $\mathbb{E} = \frac{1}{2} [\mathbb{C} - \mathbb{I}]$ is doing in the **bi-linear form** $d\tilde{X}^T \mathbb{E} d\tilde{X}$.

The matrix \mathbf{E} represents the components of the **Strain Tensor**.

2.3. The Strain Tensor: Infinitesimal Case

2. Deformations and Strain

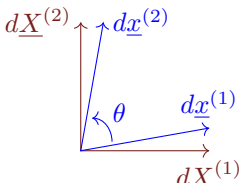
- Consider the operation $\mathbb{E}u$. Say, $v = \mathbb{E}u$.
 v represents the **components of a vector** which can be arbitrarily oriented w.r.t. u .
- Consider some unit vector ϕ such that $\mathbb{E}\phi = \lambda\phi$.
 The operation of the matrix \mathbb{E} leads to perfect stretching by a factor of λ .
- The pair (λ, ϕ) are known as an **eigenpair** of \mathbb{E} ϕ represents a principal direction.
- For 3D mechanics, we have 3 principal directions.
 Consider the 2D case below:



2.3. The Strain Tensor: Infinitesimal Case

2. Deformations and Strain

- $d\underline{X}^T \mathbb{E} d\underline{X}$ represents elongation/shortening of length **without regard to orientation changes**.
- For considering orientation change, it is **not enough just to look at a single line-segment**.
- Let us consider 2 line-vectors $d\underline{X}^{(1)}$, $d\underline{X}^{(2)}$ that are **perpendicular in the undeformed condition** $\implies \langle d\underline{X}^{(1)}, d\underline{X}^{(2)} \rangle = d\underline{X}^{(1)T} d\underline{X}^{(2)} = 0$.
- In the deformed condition, the inner product is $\langle d\underline{x}^{(1)}, d\underline{x}^{(2)} \rangle = d\underline{X}^{(1)T} \mathbb{C} d\underline{X}^{(2)} = d\underline{X}^{(1)T} 2\mathbb{E} d\underline{X}^{(2)}$.
- For small angle changes, the LHS simplifies as,

$$\begin{aligned} \langle d\underline{x}^{(1)}, d\underline{x}^{(2)} \rangle &= |d\underline{x}^{(1)}| |d\underline{x}^{(2)}| \cos \theta \approx |d\underline{x}^{(1)}| |d\underline{x}^{(2)}| \left(0 + \left(\theta - \frac{\pi}{2} \right) (-1) + \dots \right) \\ &= |d\underline{x}^{(1)}| |d\underline{x}^{(2)}| \underbrace{\left(\frac{\pi}{2} - \theta \right)}_{\gamma} \end{aligned}$$


2.3. The Strain Tensor: Shear Strain

2. Deformations and Strain

- Consider $d\underline{X}^{(1)} = |d\underline{X}^{(1)}| \underline{e}_1$, $d\underline{X}^{(2)} = |d\underline{X}^{(2)}| \underline{e}_2$.
Then we have,

$$d\underline{X}^{(1)T} \mathbb{E} d\underline{X}^{(2)} = |d\underline{X}^{(1)}| |d\underline{X}^{(2)}| E_{12},$$

— i.e., the off-diagonal component E_{12} .

- So the complete equality is written as,

$$|d\underline{x}^{(1)}| |d\underline{x}^{(2)}| \gamma = |d\underline{X}^{(1)}| |d\underline{X}^{(2)}| 2E_{12}.$$

- Under the condition of no elongation (pure shear), the off-diagonal components measure the angle-change.
- We will interpret it as being under the condition of small elongation.

2.3. The Strain Tensor: In terms of displacement

2. Deformations and Strain

Let us now express strain in terms of the displacement field $\underline{u}(\underline{X})$.

- We have $x_i = X_i + u_i$. So the deformation gradient is written as,

$$F_{iI} = \frac{\partial x_i}{\partial X_I} = \delta_{iI} + u_{i,I}.$$

- Cauchy deformation tensor is written as (with components $\mathbb{C} = \mathbb{F}^T \mathbb{F}$),

$$C_{IJ} = F_{iI} F_{iJ} = \delta_{IJ} + u_{I,J} + u_{J,I} + u_{i,I} u_{i,J}.$$

- From this, the strain tensor is written as (with components $\mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbb{I})$)

$$E_{IJ} = \frac{1}{2} \left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \underbrace{\frac{\partial u_i}{\partial X_I} \frac{\partial u_i}{\partial X_J}}_{\text{ignored for small strain}} \right)$$

- **Infinitesimal Strain Tensor:** $E_{IJ} = \frac{1}{2}(u_{I,J} + u_{J,I})$.

2.3. The Strain Tensor: Volume Change

2. Deformations and Strain

- Consider three arbitrarily oriented vectors $d\underline{X}^{(1)}$, $d\underline{X}^{(2)}$, $d\underline{X}^{(3)}$ in the undeformed configuration. The volume that they describe is given by

$$dV = \epsilon_{IJK} dX_I^{(1)} dX_J^{(2)} dX_K^{(3)}.$$

- Upon deformation, using the same notation as above, the volume **becomes**

$$dv = \epsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)}.$$

Using the deformation gradient to write this out ($d\underline{x} = \mathbb{F}d\underline{X}$), we have

$$dv = \underbrace{\epsilon_{ijk} F_{iI} F_{jJ} F_{kK}}_{\text{det}(\mathbb{F})} dX_I^{(1)} dX_J^{(2)} dX_K^{(3)}$$

- We have previously seen that $\epsilon_{ijk} F_{iI} F_{jJ} F_{kK} = \epsilon_{IJK} \det(\mathbb{F})$. Substituting this in the above we get,

$$dv = \epsilon_{IJK} \det(\mathbb{F}) dX_I^{(1)} dX_J^{(2)} dX_K^{(3)} = \det(\mathbb{F}) dV.$$

- $J := \det(\mathbb{F})$ is known as the *Jacobi determinant*. $\boxed{dv = JdV}$

2.3. The Strain Tensor: Infinitesimal Volume Change

2. Deformations and Strain

- For the infinitesimal case, the deformation gradient component matrix is expressed as

$$\mathbb{F} = \mathbb{I} + \epsilon \nabla u,$$

where $\epsilon > 0$ is **some small number** ($\epsilon \ll 1$).

- Since ϵ is small, we will try to expand out J as a Taylor series in ϵ about $\epsilon = 0$:

$$J(\epsilon) = J(\epsilon = 0) + \epsilon \left. \frac{dJ}{d\epsilon} \right|_{\epsilon=0} + \mathcal{O}(\epsilon^2).$$

Derivative of Determinant

$$\frac{d}{dp} (\det(\mathbb{M})) = \text{trace} \left(\text{Adj}(\mathbb{M}) \frac{d\mathbb{M}}{dp} \right)$$

For invertible \mathbb{M} , $\text{Adj}(\mathbb{M}) = J\mathbb{M}^{-1}$.

- This simplifies as,

$$J(\epsilon) = \det(\mathbb{I}) + \epsilon (J(\epsilon = 0) \text{trace} (\mathbb{I}^{-1} \nabla u)) + \mathcal{O}(\epsilon^2) \approx 1 + \epsilon \text{tr}(\nabla u)$$

2.3. The Strain Tensor: Infinitesimal Volume Change

2. Deformations and Strain

- Undeformed volume is dV , deformed volume is $dv = JdV$. So **relative change in volume** is

$$\frac{dv - dV}{dV} = J - 1.$$

- For the infinitesimal case $J \approx 1 + \text{tr}(\nabla u)$ (we have set $u \rightarrow \epsilon u$ here). Substituting, we get

$$\frac{dv - dV}{dV} = \text{tr}(\nabla u) = u_{I,I} = E_{II} = \text{tr}(\mathbb{E}).$$

- So the **trace of the strain tensor** is the relative volume change.

In Summary we have, for the strain tensor,

- Each diagonal element corresponds to **stretching/compressing**,
- Off-diagonal elements correspond to **shearing**,
- Trace (sum of diagonal elements) corresponds to **volume change**.

Summary

2. Deformations and Strain

- We have defined the deformation gradient \mathbb{F} and the strain tensor $\underline{\underline{E}}$.
- **Notice:** Under no deformation, if you just changed the coordinate frame of observation, \mathbb{F} will change, **but $\underline{\underline{E}}$ will not.**

Rigid Body Motion

$$\underline{x} = \underline{c} + \mathbb{R}(\underline{X} - \underline{X}_0)$$

- What is the deformation gradient here?
 - What is the **infinitesimal strain tensor** here?
 - What is the **finite strain tensor** here?
-
- What should the material **respond to**? What is the quantity that the **material wants to resist**?

2.4. Strain Compatibility

2. Deformations and Strain

Necessary Reading

Read Section 1.10 in Megson [3]

- Since strains are defined **based on the displacement field**, the different strain components are related.
- For the infinitesimal case we have: $2E_{ij} = u_{i,j} + u_{j,i}$.
We want to manipulate this such that we get an **equality fully expressed in the strains alone**.
- Differentiating by X_k and premultiplying by ϵ_{jkm} we have,

$$2\epsilon_{jkm}E_{ij,m} = \cancel{\epsilon_{jkm}u_{i,jk}} + \epsilon_{jkm}u_{j,ik} \rightarrow \boxed{\text{free indices: } i, m}$$

- We differentiate this by X_l and premultiply by ϵ_{iln} to get:

$$2\epsilon_{iln}\epsilon_{jkm}E_{ij,mn} = \cancel{\epsilon_{jkm}\epsilon_{iln}u_{j,ikl}} + \epsilon_{jkm}\epsilon_{iln}u_{j,ikl} \rightarrow \boxed{\text{free indices: } k, l}$$

2.4. Strain Compatibility

2. Deformations and Strain

- The compatibility equation $\epsilon_{mjk}\epsilon_{nil}E_{ij,mn} = 0$ represents a 3×3 system of 9 equations.
- We have **two symmetries**: $E_{ij} = E_{ji}$ (strain tensor symmetry), and $E_{ij,kl} = E_{ij,lk}$ (strain continuously differentiable). Applying this can convince us that the equation is also symmetric. So we have $\frac{3(3+1)}{2} = 6$ **unique equations**.
- In component notation, these can be written out as,

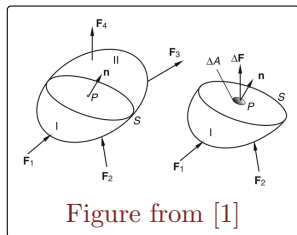
$$\begin{aligned}
 & \begin{matrix} (k, l) = (1, 1) & & (k, l) = (1, 3) \\ (k, l) = (2, 2) & & (k, l) = (1, 2) \\ (k, l) = (3, 3) & & (k, l) = (2, 3) \end{matrix} \\
 & E_{22,33} + E_{33,22} = 2E_{23,23}, \quad E_{22,13} + E_{13,22} = E_{12,23} + E_{23,12} \\
 & E_{33,11} + E_{11,33} = 2E_{13,13}, \quad E_{33,12} + E_{12,33} = E_{13,23} + E_{23,13} \\
 & E_{11,22} + E_{22,11} = 2E_{12,12}, \quad E_{11,23} + E_{23,11} = E_{12,13} + E_{13,12}
 \end{aligned}$$

The strains have to satisfy these conditions for them to “have been generated” by a continuously differentiable displacement field.

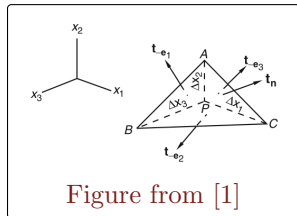
3. Stress and Equilibrium

Force is a vector. Area is a vector. What is **pressure** (F/A)?

- Consider a small area ΔA in a cut-section of an elastic body as shown. The **traction vector** \underline{t} is the limiting force



$$\underline{t} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F}{\Delta A}.$$



- By basic force-balance arguments, we can argue that the relationship between the traction vector and the normal vector to the chosen area **has to be linear**.

Cauchy Stress Tensor: $\sigma_{ij} \underline{e}_i \underline{e}_j$

$$t_i = \sigma_{ij} n_j.$$

Force Equilibrium

3. Stress and Equilibrium

- Consider the forces on a small volume dv in the **deformed configuration** (denoted Ω_d):

Body loads $\int_{\Omega_d} f_i(\underline{x})dv$

Surface tractions $\int_{\partial\Omega_d} t_i d|a|$

- Static equilibrium is written as

$$\int_{\partial\Omega_d} \sigma_{ij} da_j + \int_{\Omega_d} f_i dv = 0.$$

- Applying Gauss divergence, this simplifies to,

$$\int_{\Omega_d} \sigma_{ij,j} + f_i dv = 0 \implies \boxed{\sigma_{ij,j} + f_i = 0}.$$

- This is the **static equilibrium equation**.

Moment Equilibrium

3. Stress and Equilibrium

- We next consider the balance of the moments of forces on the same differential element.

$$\int_{\partial\Omega_d} \underbrace{\epsilon_{ijk} x_j \sigma_{kl}}_{\underline{x} \times \underline{t} d|a|} da_l + \int_{\Omega_d} \epsilon_{ijk} x_j f_k dv = 0.$$

- Applying Gauss divergence again we get,

$$\int_{\Omega_d} \epsilon_{ijk} ((x_j \sigma_{kl})_{,l} + x_j f_k) dv = \int_{\Omega_d} \epsilon_{ijk} (\delta_{jl} \sigma_{kl} + x_j (\cancel{\sigma_{kl,l}} + f_k)) dv = 0$$

$$\implies \boxed{\epsilon_{ijk} \sigma_{jk} = 0}$$

which is an assertion of **symmetry of the stress tensor**.

- Note that we have assumed the absence of body moments here.

3.1. Stress Work Done

3. Stress and Equilibrium

- Let us now consider the work done by the stress. For convenience, we start with the **rate of work done**: force \times velocity .
- On the infinitesimal element we have,

$$\frac{dU}{dt} = \int_{\partial\Omega_d} \sigma_{ij} \dot{u}_i da_j + \int_{\Omega_d} f_i \dot{u}_i dv.$$

- Application of Gauss divergence leads to,

$$\begin{aligned} \frac{dU}{dt} &= \int_{\Omega_d} (\sigma_{ij} \dot{u}_i)_{,j} + f_i \dot{u}_i dv = \int_{\Omega_d} \sigma_{ij} \dot{u}_{i,j} + \dot{u}_i (\cancel{\sigma_{ij,j}} + f_i) dv \\ &\implies \frac{dU}{dt} = \int_{\Omega_d} \underbrace{\sigma_{ij}}_{\dot{F}_{iI}} \underbrace{\frac{\partial \dot{u}_i}{\partial X_I} \frac{\partial X_I}{\partial x_j}}_{{(\mathbb{F}^{-1})_{Ij}}} dv \end{aligned}$$

- The power density is written as,

$$\boxed{\frac{d\mathcal{U}}{dt} = \sigma_{ij} (\mathbb{F}^{-1})_{Ij} \dot{F}_{iI} .}$$

3.1. Stress Work Done: Non-Dissipative Solid

3. Stress and Equilibrium

- For a general non-dissipative solid, the work done must be path-independent, i.e., the contents of the energy integral must be an **exact differential** of the conserved quantity (stress/strain energy).
- Here we have,

$$\int_{\Omega_d} \frac{d\mathcal{U}}{dt} dv = \int_{\Omega_d} \frac{\partial \mathcal{U}}{\partial F_{iI}} \dot{F}_{iI} dv = \int_{\Omega_d} \sigma_{ij}(\mathbb{F}^{-1})_{Ij} \dot{F}_{iI} dv.$$

- It must be noted that the domain of integration, Ω_d is also **deformation dependent**, making this inconvenient. So we **map everything back to the undeformed reference (denoted Ω)**:

$$\int_{\Omega} \frac{\partial \mathcal{U}}{\partial F_{iI}} \dot{F}_{iI} \underbrace{\det(\mathbb{F})}_{dv} dV = \int_{\Omega} \sigma_{ij}(\mathbb{F}^{-1})_{Ij} \dot{F}_{iI} \det(\mathbb{F}) dV$$

$$\implies \boxed{\sigma_{ij} = \frac{1}{\det(\mathbb{F})} \frac{\partial \mathcal{U}}{\partial F_{iI}} F_{jI}}, \quad \boxed{\underset{\approx}{\sigma} = \frac{1}{\det(\mathbb{F})} \frac{\partial \mathcal{U}}{\partial \mathbb{F}} \mathbb{F}^T}.$$

3.1. Stress Work Done: Non-Dissipative Solid under infinitesimal strain

3. Stress and Equilibrium

- For the infinitesimal strain case, it can be shown that the above expression simplifies to,

$$\sigma_{IJ} = \frac{\partial \mathcal{U}}{\partial E_{IJ}}.$$

- Intuitively, under this condition, the deformed and undeformed coordinates are **almost the same**. Mathematically, this can be worked out by using a perturbative formalism by setting $\underline{u} \rightarrow \epsilon \underline{u}$.

4. Constitutive Relationships

- We have developed tensor-representations of both the stress, $\underline{\underline{\sigma}} = \sigma_{ij}\underline{e}_i\underline{e}_j$ and strain, $\underline{\underline{E}} = E_{ij}\underline{e}_i\underline{e}_j$. **We are now interested in relating the components of the two.**
- The most general **linear** relationship that one can assume is

$$\sigma_{ij} = C_{ijkl}E_{kl}.$$

- If the system is **non-dissipative**, then the stress must be expressible as $\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial E_{ij}}$. So,

$$\frac{\partial^2 \mathcal{U}}{\partial E_{ij} \partial E_{kl}} = C_{ijkl}.$$

- Since we expect a smooth energy density, the indices (i, j) and (k, l) must be swappable. This represents the first symmetry property of C_{ijkl} ($i, j \leftrightarrow k, l$).
- Since stress and strain are also symmetric, the following index-swaps must be permissible: $i \leftrightarrow j, k \leftrightarrow l$.

Simplification Arguments

4. Constitutive Relationships

- In summary we have the following roadmap for simplification:

General Case	C_{ijkl}	$3 \times 3 \times 3 \times 3 = 81$ terms
Stress-Strain Symmetry	$i \leftrightarrow j, k \leftrightarrow l$	$\frac{3(3+1)}{2} \times \frac{3(3+1)}{2} = 36$ terms
Non-dissipativity, smoothness	$(i, j) \leftrightarrow (k, l)$	$\frac{6(6+1)}{2} = 21$ terms

- Suppose the **material is isotropic**, then the components C_{ijkl} are invariant under coordinate transformations. This means that it must be composed of $\delta_{..}$ symbols.
- Under symmetry, we have 3 unique combinations:

$$\delta_{ij}\delta_{kl}, \quad \delta_{ik}\delta_{jl}, \quad \delta_{il}\delta_{jk},$$

and we write:

$$C_{ijkl} = \alpha_1 \delta_{ij} \delta_{kl} + \alpha_2 \delta_{ik} \delta_{jl} + \alpha_3 \delta_{il} \delta_{jk}.$$

- Applying this to the stress-strain relationship, we get:

$$\sigma_{ij} = \alpha_1 \delta_{ij} E_{kk} + \alpha_2 E_{ij} + \alpha_3 E_{ji} \quad \Longrightarrow \quad \boxed{\sigma_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}}.$$

4.1. Mohr's Circles

4. Constitutive Relationships

- Consider a 2D case with $\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$.
- Consider a plane section with normal $\hat{n} = [\cos \theta \quad \sin \theta]^T$. The perpendicular is denoted $\hat{s} = [-\sin \theta \quad \cos \theta]^T$.
- The traction vector is given by $\underline{t} = \underline{\sigma} \hat{n}$:

$$\underline{t} = \begin{bmatrix} \sigma_{11} \cos \theta + \sigma_{12} \sin \theta \\ \sigma_{12} \cos \theta + \sigma_{22} \sin \theta \end{bmatrix}.$$

- This is resolved along the (\hat{n}, \hat{s}) directions by the coordinate transformation,

$$\begin{bmatrix} \sigma_n \\ \tau_s \end{bmatrix} = \begin{bmatrix} \hat{n}^T \\ \hat{s}^T \end{bmatrix} \underline{t} = \begin{bmatrix} \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \sigma_{12} \sin 2\theta \\ -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \sigma_{12} \cos 2\theta \end{bmatrix}$$

4.1. Mohr's Circles

4. Constitutive Relationships

- Now we consider two infinitesimal lines initially oriented along \hat{n} and \hat{t} ($dS\hat{n}$, $dS\hat{t}$).
- $dS\hat{n}$ experiences the elongation,

$$\frac{ds - dS}{dS} = \hat{n}^T \mathbb{E} \hat{n} = \epsilon_\ell.$$

- The shear strain between them is,

$$\gamma_s = 2\hat{t}^T \mathbb{E} \hat{n}.$$

- Simplifying, we get

$$\begin{bmatrix} \epsilon_\ell \\ \gamma_s \end{bmatrix} = \begin{bmatrix} \frac{E_{11} + E_{22}}{2} + \frac{E_{11} - E_{22}}{2} \cos 2\theta + E_{12} \sin 2\theta \\ -(E_{11} - E_{22}) \sin 2\theta + 2E_{12} \cos 2\theta \end{bmatrix}$$

4.1. Mohr's Circles

4. Constitutive Relationships

- When no shear load/response is observed, these reduce to,

$$\begin{bmatrix} \sigma_n \\ \tau_s \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta \\ -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta \end{bmatrix}, \quad \begin{bmatrix} \epsilon_\ell \\ \gamma_s \end{bmatrix} = \begin{bmatrix} \frac{E_{11} + E_{22}}{2} + \frac{E_{11} - E_{22}}{2} \cos 2\theta \\ -(E_{11} - E_{22}) \sin 2\theta \end{bmatrix}.$$

- For a linear-elastic material, **causal links** may be made between $\sigma_n \leftrightarrow \epsilon_\ell$ and $\tau_s \leftrightarrow \gamma_s$.

4.2. Linear Isotropic Elasticity

4. Constitutive Relationships

- From basic arguments one can motivate

$$E_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}).$$

- For the 2D case under pure tension,

$$E_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22}, \quad E_{22} = -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22}.$$

- For some section oriented by angle θ we have,

$$\gamma_s(\theta) = -(E_{11} - E_{22}) \sin 2\theta = -\frac{1 + \nu}{E} \underbrace{(\sigma_{11} - \sigma_{22}) \sin 2\theta}_{2\tau_s},$$

which implies,
$$\boxed{\gamma_s(\theta) = 2\frac{1 + \nu}{E}\tau_s.}$$

- E : Young's Modulus, ν : Poisson's Ratio, and $G = \frac{E}{2(1+\nu)}$: Shear Modulus.

4.2. Linear Isotropic Elasticity

4. Constitutive Relationships

- We have also spoken about volume change. In terms of strains this is,

$$\begin{aligned}\frac{dv - dV}{dv} &= E_{11} + E_{22} + E_{33} \\ &= \frac{1 - 2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}).\end{aligned}$$

- In other words we have $E_{ii} = \kappa\sigma_{ii}$, where $\kappa = \frac{1 - 2\nu}{E}$, the bulk modulus.
- From physical arguments, it is clear that $\kappa > 0$, which implies $\nu < 0.5$, which presents an **upper bound for the Poisson's ratio**.
- The shear modulus must also be positive. So we have $\frac{E}{2(1+\nu)} > 0$, which implies $\nu > -1$, which presents a **lower bound for the Poisson's ratio**.
- In summary we have, $\nu \in (-1, 0.5)$, $E > 0$.

4.2. Linear Isotropic Elasticity

4. Constitutive Relationships

- In tensor notation, this can be written as,

$$E_{ij} = \frac{1}{E} [(1 + \nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}].$$

5. 2D Problems

- In 2D, the governing equations can be written as,

$$\sigma_{11,1} + \sigma_{12,2} + f_1 = 0$$

$$\sigma_{12,1} + \sigma_{22,2} + f_2 = 0.$$

- Differentiation the first by X_1 and the second by X_2 leads to

$$\sigma_{11,11} + \sigma_{22,22} + 2\sigma_{12,12} + f_{1,1} + f_{2,2} = 0.$$

- Strain Compatibility equations in 2D reads:

$$2E_{12,12} = E_{11,22} + E_{22,11}$$

- We, however, need compatibility in terms of stresses, not strains. Now we formalize the notion of two dimensions:

Plane Stress $\sigma_{33} = 0$

Plane Strain $E_{33} = 0$

The “Plane Stress” Case

5. 2D Problems

- Here, we assume $\sigma_{33} = 0$ (but $E_{33} \neq 0$ in general). So the stress-strain relationships are,

$$E_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22}, \quad E_{22} = \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{11}$$

$$2E_{12} = 2\frac{1+\nu}{E}\sigma_{12}, \quad E_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$$

- Substituting this into the compatibility equations we get,

$$\begin{aligned} \implies \frac{2(1+\nu)}{E}\sigma_{12,12} &= \frac{1}{E} \left((\sigma_{11} - \nu\sigma_{22})_{,22} + (-\nu\sigma_{11} + \sigma_{22})_{,11} \right) \\ &= \frac{1}{E} \left((\sigma_{11,22} + \sigma_{22,11}) - \nu(\sigma_{11,11} + \sigma_{22,22}) \right) \end{aligned}$$

- Combining the two we get,

$$\frac{\sigma_{11,11} + \sigma_{11,22} + \sigma_{22,11} + \sigma_{22,22}}{1 + \nu} + f_{1,1} + f_{2,2} = 0 \implies \boxed{\sigma_{ii,jj} + (1 + \nu)f_{i,i} = 0}.$$

The “Plane Strain” Case

5. 2D Problems

- Here, we assume $E_{33} = 0$ ($\sigma_{33} \neq 0$ in general). So the stress-strain relationships are simplified as,

$$E_{33} = \frac{\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})}{E} = 0 \implies \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}),$$

$$\implies E_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}) = \frac{1 - \nu^2}{E}\sigma_{11} - \frac{\nu(1 + \nu)}{E}\sigma_{22}$$

$$\implies E_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E}(\sigma_{11} + \sigma_{33}) = \frac{1 - \nu^2}{E}\sigma_{22} - \frac{\nu(1 + \nu)}{E}\sigma_{11}$$

- Substituting this into the compatibility equations we get,

$$\begin{aligned} \implies \frac{2(1 + \nu)}{E}\sigma_{12,12} &= \frac{1 + \nu}{E} \left(((1 - \nu)\sigma_{11} - \nu\sigma_{22})_{,22} + (-\nu\sigma_{11} + (1 - \nu)\sigma_{22})_{,11} \right) \\ &= \frac{1 + \nu}{E} \left((1 - \nu)(\sigma_{11,22} + \sigma_{22,11}) - \nu(\sigma_{11,11} + \sigma_{22,22}) \right) \end{aligned}$$

- Combining the two we get,

$$(1 - \nu)(\sigma_{11,11} + \sigma_{11,22} + \sigma_{22,11} + \sigma_{22,22}) + f_{1,1} + f_{2,2} = 0 \implies \sigma_{ii,jj} + \frac{1}{1 - \nu} f_{i,i} = 0.$$

5.1. The Airy's Stress Function

5. 2D Problems

- We can now combine both the **governing equations** and the **compatibility equations**, so we can write out the solution **fully in terms of stress only**.
- For the homogeneous case ($f_i = 0$), we have (for both plane stress and plane strain),

$$\left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} \right) (\sigma_{11} + \sigma_{22}) = 0. \quad (3)$$

- We introduce the Airy's Stress function ϕ that simplifies the system of two PDE's into a scalar PDE by the substitutions:

$$\sigma_{11} := \frac{\partial^2 \phi}{\partial X_2^2}, \quad \sigma_{22} := \frac{\partial^2 \phi}{\partial X_1^2}, \quad \sigma_{12} := -\frac{\partial \phi}{\partial X_1 \partial X_2}.$$

(it is easily verified that this satisfies the governing equations $\sigma_{i,j,j} = 0$ by definition)

- Substitution into eq. (3) leads to

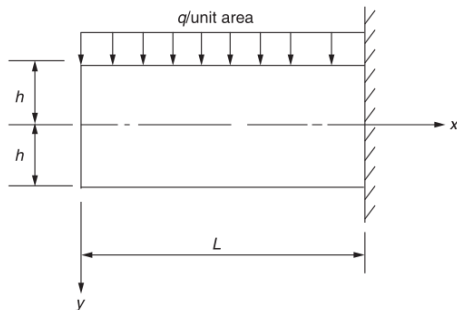
$$\phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = \left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} \right)^2 \phi = 0, \quad \boxed{\nabla^4 \phi = 0},$$

sometimes known as the **Biharmonic Equation**.

5.1. The Airy's Stress Function: Tutorial

5. 2D Problems

- The Airy stress function can be used to solve problems with boundary loads. Consider this example from your text book (example 2.3):



Boundary Conditions

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0, \quad y = h$$

$$\sigma_{11} = \sigma_{12} = 0, \quad y = h$$

$$\sigma_{22} = -q, \quad y = -h$$

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0, \quad x = 0$$

$$\int_{-h}^h y \sigma_{11} dy = 0, \quad x = 0$$

with a candidate Airy stress function

$$\phi(x, y) = Ax^2 + Bx^2y + Cy^3 + D(5x^2y^3 - y^5).$$

5.1. The Airy's Stress Function

5. 2D Problems

- It may be the case that the Airy stress function doesn't meet all the boundary conditions. In this case we find a stress function that **approximately satisfies the BCs in some sense.**
- So is this completely useless? **No.**

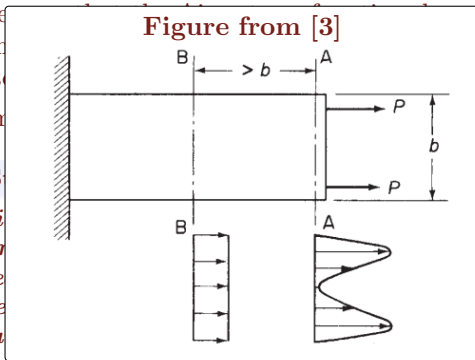
St. Venant's Principle (rephrased as in [1])

If some distribution of forces acting on a portion of the surface of a body is replaced by a different distribution of forces acting on the same portion of the body, then the effects of the two different distributions on the parts of the body sufficiently far removed from the region of application of the forces are essentially the same, provided that the two distribution of forces have the same resultant force and the same resultant couple.

5.1. The Airy's Stress Function

5. 2D Problems

- It may be the boundary conditions are not met all the conditions that
- So is this condition that



S

If some distributed load is replaced by a different load on some portion of the body, then the effect is the same as long as the resultant force and the resultant couple are essentially the same.

[1])

surface of a body is replaced by some portion of the body, then the forces are the same as long as the forces have the same resultant force and the same resultant couple.

References I

- [1] W. M. Lai, D. Rubin, and E. Kreml. *Introduction to Continuum Mechanics*, 4th ed. Amsterdam Boston: Butterworth-Heinemann/Elsevier, 2010. ISBN: 978-0-7506-8560-3 (cit. on pp. 2, 36, 55, 56).
- [2] M. H. Sadd. *Elasticity: Theory, Applications, and Numerics*, 2nd ed. Amsterdam ; Boston: Elsevier/AP, 2009. ISBN: 978-0-12-374446-3 (cit. on p. 2).
- [3] T. H. G. Megson. *Aircraft Structures for Engineering Students*, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on pp. 2, 34, 55, 56).