AS3020: Aerospace Structures

Module 3: Introduction to Elasticity

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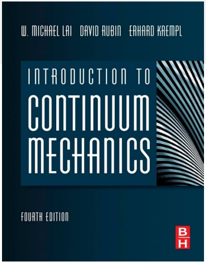
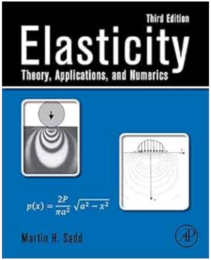
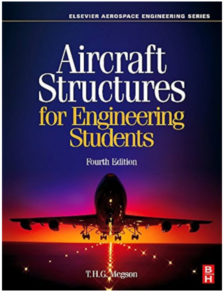


Chapters 1-5 in Lai, Rubin, and Krempl [1]



Chapters 1-5 in Sadd [2]

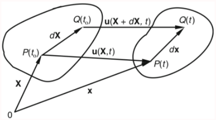


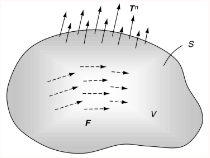
Chapters 1 in Megson [3]

We have to make a choice!

Red Pill

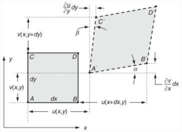
$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

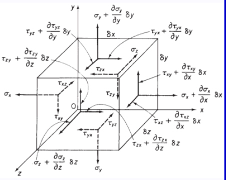




Blue Pill

$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}(\sigma_y + \sigma_y)$$

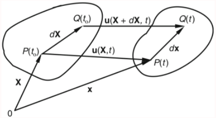


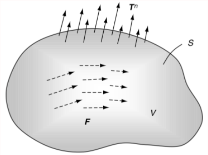


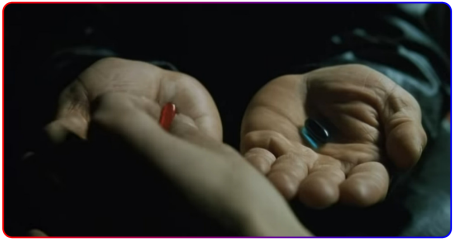
~~We have to~~ I make a choice!

Red Pill

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$







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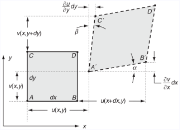

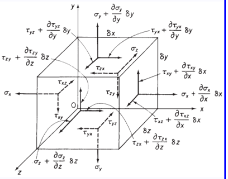

1.1. Indicial Notation I

1. Mathematical Rudiments

Einstein's Summation Convention: Dummy Indices

$$s = a_1 x_1 + a_2 x_2 + \cdots = \sum_{i=1}^{n} a_i x_i \rightarrow a_i x_i = a_k x_k = a_m x_m$$

Consider $\alpha = a_{ij} x_i x_j$, $\underline{v} = v_i \hat{e}_i$, $\underline{\underline{T}} = T_{ij} \hat{e}_i \hat{e}_j$

Free Indices

$$\left.\begin{aligned} y_1 &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\ y_2 &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \\ y_3 &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3 \end{aligned}\right\} \implies y_i = a_{ij} x_j$$

Consider $T_{ij} = A_{im} A_{jm}$.

1.1. Indicial Notation II

1. Mathematical Rudiments

The Kronecker Delta

$$\delta_{ij} := \hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Consider $C_{ijkl} = \delta_{ik}\delta_{jl}$, $C_{ijkl} = \delta_{il}\delta_{jk}$.

The Levi-Civita Symbol

$$\epsilon_{ijk} := \hat{e}_i \cdot \underbrace{(\hat{e}_j \times \hat{e}_k)}_{\epsilon_{ijk}\hat{e}_i} = \begin{cases} 1 & \text{if } \{(i,j,k)\} \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{if } \{(i,j,k)\} \in \{(3,2,1),(2,1,3),(1,3,2)\} \\ 0 & \text{otherwise} \end{cases}$$

Consider $\underline{a} \cdot (\underline{b} \times \underline{c})$, $\Delta \underline{\underline{F}}$.

1.1. Indicial Notation III

1. Mathematical Rudiments

Property: $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

$$\epsilon_{ijk}\epsilon_{mnk} = (\epsilon_{ijk}\hat{e}_k)\cdot(\epsilon_{mnk}\hat{e}_k) = (\hat{e}_i \times \hat{e}_j)\cdot(\hat{e}_m \times \hat{e}_n)$$

$$(\hat{e}_i \times \hat{e}_j)\cdot(\hat{e}_m \times \hat{e}_n) = \begin{cases} 1, & \hat{e}_i \times \hat{e}_j = \hat{e}_m \times \hat{e}_n \\ -1, & \hat{e}_i \times \hat{e}_j = -\hat{e}_m \times \hat{e}_n = \hat{e}_n \times \hat{e}_m \\ 0, & \text{otherwise} \end{cases}$$

$$= \boxed{\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$

Consider $\underline{a} \times (\underline{b} \times \underline{c})$

1.1. Indicial Notation IV

1. Mathematical Rudiments

Derivative Notation

$$\frac{\partial u_i}{\partial x_j} := u_{i,j}$$

Consider $\nabla \underline{u}$, $\nabla \cdot \underline{u}$, $\nabla \times \underline{u}$, $\nabla \times \underline{\underline{Q}}$

Exercise

$$\nabla \underline{u}, \underbrace{\nabla \cdot (\nabla \underline{u})}_{\nabla^2 \underline{u}}, \nabla \cdot (\nabla \times \underline{u}), \nabla \times \nabla \times \underline{u}, \nabla \cdot \underline{\underline{\sigma}}$$

1.1. Indicial Notation V

1. Mathematical Rudiments

Vectors, Tensors

$$\underline{u} = u^i \hat{e}_i, \quad \underline{\underline{T}} = T^{ij} \hat{e}_i \hat{e}_j$$

Consider:

- Order of a tensor
- Vector-components as first order tensors
- The tensor product and 2nd order tensors
- Tensors as defining an operation
- Identity **tensors**
- Coordinate transformation
- "Notational abuse"
- Symmetric, antisymmetric tensors
- Antisymmetry as a cross product
- Representation of Eigen-decomposition
- Calculus: Gradient, Divergence, Laplacian, Curl, curvilinear coordinates

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
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Curvilinear Coordinates

- Scalar field ϕ gradient:
$$\delta\phi = \frac{\partial\phi}{\partial x_1}\delta x_1 + \frac{\partial\phi}{\partial x_2}\delta x_2$$
$$= \frac{\partial\phi}{\partial r}\delta r + \frac{\partial\phi}{\partial\theta}\delta\theta$$
- Polar bases
$$\underline{e}_r = C_\theta \underline{e}_1 + S_\theta \underline{e}_2 \implies \delta\underline{e}_r = \delta\theta\underline{e}_\theta$$
$$\underline{e}_\theta = -S_\theta \underline{e}_1 + C_\theta \underline{e}_2 \implies \delta\underline{e}_\theta = -\delta\theta\underline{e}_r$$
- Position vector
$$\delta\underline{r} = \delta r\underline{e}_r + r\delta\underline{e}_r$$
$$= \delta r\underline{e}_r + r\delta\theta\underline{e}_\theta$$
- For $\delta\phi = \nabla\phi\cdot\delta\underline{r}$,
$$\nabla\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta$$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
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Integral Calculus

- The line integral: $\int \underline{F} \cdot d\underline{x}$
  Potential theory:
  $\int_{\partial \mathcal{D}} F_i dx_i = 0 \implies$
  - $F_i = \phi_{,i}$ and $\epsilon_{ijk} F_{k,j}|_{\mathcal{D}} = 0$
  - $\underline{F} = \nabla \phi$ and $\nabla \times \underline{F} = \underline{0}$
- **Gauss Divergence Theorem**
  $\int_{\mathcal{D}} P_{ijk\dots,i} d\mathcal{D} = \int_{\partial \mathcal{D}} P_{ijk\dots} dA_i$
- **Stoke's Law**:
  $\int_A (\nabla \times \underline{F}) \cdot d\underline{A} = \int_{\partial A} \underline{F} \cdot d\underline{x}$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Stoke's Law as a Special Case of Gauss Divergence in 2D

$$\int_{\mathcal{S}} (\nabla \times \underline{v}) \cdot d\underline{S} = \int_{\mathcal{S}} \epsilon_{ijk} v_{k,j} \hat{n}_i d|S|$$

$$= \int_{\mathcal{S}} \left(\epsilon_{ijk} \hat{n}_i v_k\right)_{,j} d|S|$$

$$= \int_{\partial\mathcal{S}} \epsilon_{ijk} \hat{n}_i v_k \hat{b}_j d|\ell|$$

$$= \int_{\partial\mathcal{S}} \underbrace{\left(\epsilon_{ijk} \hat{n}_i \hat{b}_j\right)}_{\underline{\hat{n}} \times \underline{\hat{b}}} v_k d|\ell|$$

$$= \int_{\partial\mathcal{S}} v_k t_k d|\ell| = \int_{\partial\mathcal{S}} \underline{v} \cdot d\underline{\ell}$$

Differ

- Scalar, ve
- Gradient
- Divergen
- Curviline divergenc *coordinat*

ulus

$\underline{F} \cdot d\underline{x}$

$_{ijk} F_{k,j}|_{\mathcal{D}} = 0$

$\times \underline{F} = \underline{0}$

e Theorem

$P_{ijk\ldots} dA_i$

$_{\partial A} \underline{F} \cdot d\underline{x}$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

Integral Calculus

- The line integral: $\int \underline{F} \cdot d\underline{x}$
  Potential theory:
  $\int_{\partial\mathcal{D}} F_i dx_i = 0 \implies$
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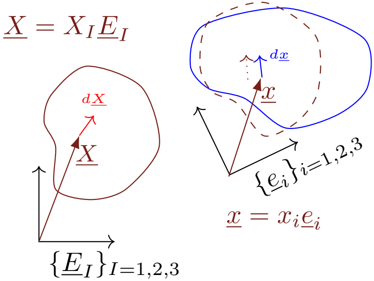
- **Gauss Divergence Theorem**
  $\int_{\mathcal{D}} P_{ijk\dots,i} d\mathcal{D} = \int_{\partial\mathcal{D}} P_{ijk\dots} dA_i$
- **Stoke's Law**:
  $\int_A (\nabla \times \underline{F}) \cdot d\underline{A} = \int_{\partial A} \underline{F} \cdot d\underline{x}$
- **Determinant of a Tensor**
  $\epsilon_{IJK} \Delta \underline{\underline{F}} = \epsilon_{ijk} F_{iI} F_{jJ} F_{kK}$
  - Related to volume change through transformation

2. Deformations and Strain

2.1. The Basic Premise

How to describe the change in shape **independently** of rigid body motions?



How does $d\underline{X}$ transform into $d\underline{x}$?

$$\underline{x} = \underline{X} + \underline{u}$$

- The deformations are mapped as

$$\begin{aligned} \text{Lagrangian} \quad & x_i = x_i(\underline{X}) \\ \text{Eulerian} \quad & X_i = X_i(\underline{x}) \end{aligned}$$

- Under the **Lagrangian description** we have,

$$dx_i = \overbrace{\frac{\partial x_i}{\partial X_I}}^{F_{iI}} dX_I$$

$$\begin{aligned} \text{Length} \quad & ds^2 = dx_i dx_i = \\ & dX_I \left[\frac{\partial x_i}{\partial X_I}\frac{\partial x_i}{\partial X_J}\right] dX_J \\ \text{Angle} \quad & ds_1 ds_2 \cos\theta = dx_i dx_j = \\ & dX_I \left[\frac{\partial x_i}{\partial X_I}\frac{\partial x_j}{\partial X_J}\right] dX_J \end{aligned}$$

2.2. Coordinate Transformation

2. Deformations and Strain

- A vector $\underline{v}$ is written as

$$\underline{v} = v_i \underline{e}_i,$$

  and is defined as **a linear combination of the bases of its vector-space**.
- Suppose I have another coordinate system spanning the same vector-space, this comes with its own set of basis vectors $\{\underline{b}_i\}_{i=1,\dots,n}$.
- If the vector represents a physical/geometrical measurement, it **can not change based on coordinate system**, i.e., it is coordinate invariant.
- So, the following equality must hold:

$$\underline{v} = v_i \underline{e}_i = \overline{v_i} \underline{b}_i,$$

  with v_i and $\overline{v_i}$ being the **components of the vector** under the different coordinate systems.

2.2. Coordinate Transformation

2. Deformations and Strain

- Assuming that both $\{\underline{e}_i\}$ and $\{\underline{b}_i\}$ represent **orthogonal coordinate systems** (inner products $\langle \underline{e}_i, \underline{e}_j \rangle \equiv \langle \underline{b}_i, \underline{b}_j \rangle = \delta_{ij}$), we write down:

$$v_i = \langle \underline{v}, \underline{e}_i \rangle; \quad \overline{v_i} = \langle \underline{v}, \underline{b}_i \rangle.$$

- Evaluating $\overline{v_i}$ we obtain,

$$\overline{v_i} = \langle v_j \underline{e}_j, \underline{b}_i \rangle = \langle \underline{b}_i, \underline{e}_j \rangle v_j.$$

Denoting $\langle \underline{b}_i, \underline{e}_j \rangle = Q_{ij}$, we get our **component tranformation law for a vector**:

$$\boxed{\overline{v_i} = Q_{ij} v_j}$$

What about the basis vectors themselves?

How can I **combine** the $\underline{e}_i$'s to obtain the $\underline{b}_i$'s ?

- How **should I combine them** so that my vector is invariant?

2.2. Coordinate Transformation

2. Deformations and Strain

- Given the $\overline{v_i} = Q_{ij}v_j$, and the requirement $v_i\underline{e}_i = \overline{v_j}\underline{b}_j$, we write (after swapping $i \leftrightarrow j$ in LHS),

$$Q_{ji}v_i\underline{b}_j = v_i\underline{e}_i \implies Q_{ji}\underline{b}_j = \underline{e}_i \qquad \text{(multiply both sides by } (\mathbb{Q}^{-1})_{ik})$$

$$\overbrace{Q_{ji}(\mathbb{Q}^{-1})_{ik}}^{\delta_{jk}}\,\underline{b}_j = (\mathbb{Q}^{-1})_{ik}\underline{e}_i$$

$$\implies \underline{b}_i = (\mathbb{Q}^{-1})_{ji}\underline{e}_j$$

- Comparing the two, we have

$$\boxed{\overline{v_i} = Q_{ij}v_j} \qquad \boxed{\underline{b}_i = (\mathbb{Q}^{-1})_{ji}\underline{e}_j}$$

- This is a **necessary requirement** so that the vector **remains invariant**.

2.2. Coordinate Transformation: Array Notation

- Now we introduce the Array Notation for vectors. **Let $\underline{v}$ be a vector.** The array of its components with respect to the basis $\{\underline{e}_i\}$ is written as,

$$\underset{\sim}{v} = \begin{bmatrix} \langle \underline{v}, \underline{e}_1 \rangle \\ \langle \underline{v}, \underline{e}_2 \rangle \\ \vdots \end{bmatrix} \qquad \text{(similarly for } \underset{\sim}{\overline{v}}\text{)}.$$

- We also define the **array of coordinate vectors** as

$$\underset{\sim}{e} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \vdots \end{bmatrix}; \qquad \underset{\sim}{b} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \end{bmatrix}.$$

- Under this notation we have,

$$\boxed{\underset{\sim}{\overline{v}} = \mathbb{Q}\underset{\sim}{v}} \quad \text{and} \quad \boxed{\underset{\sim}{b} = \mathbb{Q}^{-T}\underset{\sim}{e}}.$$

$\underset{\sim}{\overline{v}}$ "contra-varies" w.r.t. $\underset{\sim}{v}$, **in comparison with** how $\underset{\sim}{b}$ and $\underset{\sim}{e}$ are related.
$\implies \underset{\sim}{v}$ are the **contravariant components** of $\underline{v}$.

2.2. Coordinate Transformation: Tensors

2. Deformations and Strain

- We will define a (2nd order) tensor are a **linear combination of basis-dyads**:

$$\underline{\underline{T}} = T_{ij}\underline{e}_i\underline{e}_j = \overline{T}_{ij}\underline{b}_i\underline{b}_j,$$

where **we have required $\underline{\underline{T}}$ to be invariant** under coordinate change.

- Using a **double-contraction** operation, we write down the components of $\overline{T}_{ij}$ as,

$$\overline{T}_{ij} = T_{mn} \underbrace{\langle \underline{b}_i, \underline{e}_m \rangle}_{Q_{im}} \overbrace{\langle \underline{b}_j, \underline{e}_n \rangle}^{Q^{jn}}$$
$$= Q_{im}T_{mn}Q_{jn}.$$

- In array notation we write the components as,

$$\boxed{\overline{\mathbb{T}} = \mathbb{Q}\mathbb{T}\mathbb{Q}^T}.$$

For a tensor to be invariant, its components have to transform in this fashion.

2.2. Coordinate Transformation: Summary

2. Deformations and Strain

Supposing I specify a basis change by

$$\underset{\sim}{\underline{b}} = \mathbb{Q}^{-T} \underset{\sim}{\underline{e}},$$

- for a vector $\underline{v} = \underset{\sim}{v}^T \underset{\sim}{\underline{e}}$ to be invariant, its components have to transform as

$$\overline{\underset{\sim}{v}} = \mathbb{Q}\underset{\sim}{v}.$$

- for a tensor $\underline{\underline{T}} = \mathbb{T}\underset{\sim}{\underline{e}} \otimes \underset{\sim}{\underline{e}}$ to be invariant, its components have to transform as

$$\overline{\mathbb{T}} = \mathbb{Q}\mathbb{T}\mathbb{Q}^T$$

- If it transforms in any other fashion, **then invariance is not guaranteed**.

2.2. Coordinate Transformation: Relationship to Gradients

2. Deformations and Strain

We will now establish a relationship between coordinate transformation and **component-gradients**.

- Consider an infinitesimal line vector $d\underline{x} = dx_i \underline{e}_i = d\overline{x}_i \underline{b}_i$.
- It is obvious that the components $d\underset{\sim}{\overline{x}}$ have to be related to the components $d\underset{\sim}{x}$. So we write

$$d\overline{x}_i = \frac{\partial \overline{x}_i}{\partial x_j} dx_j \tag{1}$$

- By invariance requirements, we have

$$d\overline{x}_i = Q_{ij} dx_j. \tag{2}$$

- Comparing eq. (1) and eq. (2) we obtain,

$$\boxed{Q_{ij} = \frac{\partial \overline{x}_i}{\partial x_j}} \quad \text{or} \quad \boxed{\mathbb{Q} = grad\left(\underset{\sim}{\overline{x}}\right)}$$

$grad(\cdot)$ operator $\implies$ gradient operation

2.2. Coordinate Transformation: The Deformation Gradient

2. Deformations and Strain

- The **components of the deformation gradient** are written as

$$F_{iI} = \frac{\partial x_i}{\partial X_I}.$$

- Under coordinate change we have,

$$\overline{F}_{iI} = \frac{\partial \overline{x}_i}{\partial x_j} \frac{\partial x_j}{\partial X_J} \frac{\partial X_J}{\partial \overline{X}_I}$$
$$= Q_{ij}^{(x)} F_{jJ} (\mathbb{Q}^{(X)^{-1}})_{JI} \implies \boxed{\overline{\mathbb{F}} = \mathbb{Q}^{(x)} \mathbb{F} \mathbb{Q}^{(X)^{-1}}}.$$

This is transforming quite unlike a tensor for 2 reasons

1. $\mathbb{Q}^{(x)}$ and $\mathbb{Q}^{(X)}$ **need not necessarily be the same** (we are free to choose measurement coordinates at each instant)
2. $\mathbb{Q}^{-1} = \mathbb{Q}^T$ on for orthonormal coordinate systems (cartesian, for eg.). For non-orthonormal bases, this is not so.

2.2. Coordinate Transformation: The Cauchy Deformation Tensor

2. Deformations and Strain

- Now we consider $\mathbb{C} = \mathbb{F}^T\mathbb{F}$. Under coordinate change this becomes,

$$\begin{aligned}\overline{\mathbb{C}} = \overline{\mathbb{F}}^T\overline{\mathbb{F}} &= \left(\mathbb{Q}^{(x)}\mathbb{F}{\mathbb{Q}^{(X)}}^{-1}\right)^T \left(\mathbb{Q}^{(x)}\mathbb{F}{\mathbb{Q}^{(X)}}^{-1}\right) \\ &= {\mathbb{Q}^{(X)}}^{-T}\mathbb{F}^T{\mathbb{Q}^{(x)}}^T\mathbb{Q}^{(x)}\mathbb{F}{\mathbb{Q}^{(X)}}^{-1}\end{aligned}$$

- Suppose we choose to stick with **coordinate systems with orthonormal bases**, $\mathbb{Q}^{-1} = \mathbb{Q}^T$ (for both (x) and (X)). Hereby the components matrix $\mathbb{C}$ reduces to

$$\boxed{\mathbb{C} = \mathbb{Q}^{(X)}\mathbb{F}^T\mathbb{F}{\mathbb{Q}^{(X)}}^T}$$

Unlike the deformation gradient...

...this is transforming like a tensor's components!
So I can define the **Cauchy deformation tensor** as: $\underline{\underline{C}} = C_{IJ}\underline{E}_I\underline{E}_J$

2.3. The Strain Tensor

- We are now ready to define the strain tensor based on length change. We wrote,

$$\begin{aligned} ds^2 - dS^2 &= dX_I \left(F_{iI} F_{jJ} - \delta_{IJ}\right) dX_J \\ &= d\underset{\sim}{X}^T \left[\mathbb{F}^T \mathbb{F} - \mathbb{I}\right] d\underset{\sim}{X} = d\underset{\sim}{X}^T \left[\mathbb{C} - \mathbb{I}\right] d\underset{\sim}{X}. \end{aligned}$$

- For small changes in length,
  $ds^2 - dS^2 = (ds + dS)(ds - dS) \approx 2dS(ds - dS)$.
- Representing the elongation as a fraction of the total length we write $(ds - dS) = \epsilon dS$. Using this we have,

$$2dS^2\epsilon = d\underset{\sim}{X}^T[\mathbb{C} - \mathbb{I}]d\underset{\sim}{X} \implies 2d\underset{\sim}{X}^T d\underset{\sim}{X}\epsilon = d\underset{\sim}{X}^T[\mathbb{C} - \mathbb{I}]d\underset{\sim}{X}.$$

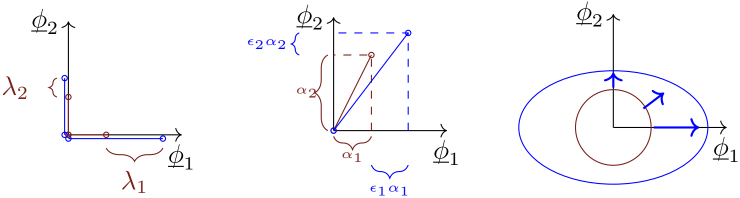
Here the single factor ϵ represents what the matrix $\mathbb{E} = \frac{1}{2}\left[\mathbb{C} - \mathbb{I}\right]$ is doing in the **bi-linear form** $d\underset{\sim}{X}^T \mathbb{E} d\underset{\sim}{X}$.

The matrix **E** represents the components of the **Strain Tensor**.

2.3. The Strain Tensor: Infinitesimal Case

2. Deformations and Strain

- Consider the operation $\mathbb{E}\underset{\sim}{u}$. Say, $\underset{\sim}{v} = \mathbb{E}\underset{\sim}{u}$.
  $\underset{\sim}{v}$ represents the **components of a vector** which can be arbitrarily oriented w.r.t. $\underset{\sim}{u}$.
- Consider some unit vector $\underset{\sim}{\phi}$ such that $\mathbb{E}\underset{\sim}{\phi} = \lambda\underset{\sim}{\phi}$.
  The operation of the matrix $\mathbb{E}$ leads to perfect stretching by a factor of λ.
- The pair $(\lambda, \underset{\sim}{\phi})$ are known as an **eigenpair** of $\mathbb{E}$ $\underset{\sim}{\phi}$ represents a principal direction.
- For 3D mechanics, we have 3 prinicpal directions.
  Consider the 2D case below:

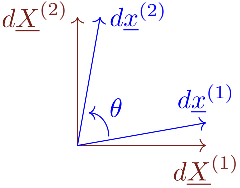


2.3. The Strain Tensor: Infinitesimal Case

2. Deformations and Strain

- $d\underset{\sim}{X}^T \mathbb{E} d\underset{\sim}{X}$ represents elongation/shortening of length **without regard to orientation changes**.
- For considering orientation change, it is **not enough just to look at a single line-segment**.
- Let us consider 2 line-vectors $d\underline{X}^{(1)}$, $d\underline{X}^{(2)}$ that are **perpendicular in the undeformed condition** $\implies \langle d\underline{X}^{(1)}, d\underline{X}^{(2)} \rangle = d\underset{\sim}{X}^{(1)^T} d\underset{\sim}{X}^{(2)} = 0$.
- In the deformed condition, the inner product is $\langle d\underline{x}^{(1)}, d\underline{x}^{(2)} \rangle = d\underset{\sim}{X}^{(1)^T} \mathbb{C} d\underset{\sim}{X}^{(2)} = d\underset{\sim}{X}^{(1)^T} 2\mathbb{E} d\underset{\sim}{X}^{(2)}$.
- For small angle changes, the LHS simplifies as,

$$\begin{aligned}\langle d\underline{x}^{(1)}, d\underline{x}^{(2)} \rangle = |d\underline{x}^{(1)}||d\underline{x}^{(2)}| \cos\theta &\approx |d\underline{x}^{(1)}||d\underline{x}^{(2)}| \left(0 + (\theta - \frac{\pi}{2})(-1) + \dots\right) \\ &= |d\underline{x}^{(1)}||d\underline{x}^{(2)}| \underbrace{(\frac{\pi}{2} - \theta)}_{\gamma}\end{aligned}$$



2.3. The Strain Tensor: Shear Strain

2. Deformations and Strain

- Consider $d\underline{X}^{(1)} = |d\underline{X}^{(1)}|\underline{e}_1$, $d\underline{X}^{(2)} = |d\underline{X}^{(2)}|\underline{e}_2$.
  Then we have,
  $$d\underset{\sim}{X}^{(1)^T}\mathbb{E}d\underset{\sim}{X}^{(2)} = |d\underline{X}^{(1)}||d\underline{X}^{(2)}|E_{12},$$
  — i.e., the off-diagonal component E_{12}.
- So the complete equality is written as,
  $$|d\underline{x}^{(1)}||d\underline{x}^{(2)}|\gamma = |d\underline{X}^{(1)}||d\underline{X}^{(2)}|2E_{12}.$$
- Under the condition of no elongation (pure shear), the off-diagonal components measure the angle-change.
- We will interpret it as being under the condition of small elongation.

2.3. The Strain Tensor: In terms of displacement

2. Deformations and Strain

Let us now express strain in terms of the displacement field $\underline{u}(\underline{X})$.

- We have $x_i = X_i + u_i$. So the deformation gradient is written as,

$$F_{iI} = \frac{\partial x_i}{\partial X_I} = \delta_{iI} + u_{i,I}.$$

- Cauchy deformation tensor is written as (with components $\mathbb{C} = \mathbb{F}^T\mathbb{F}$),

$$C_{IJ} = F_{iI}F_{iJ} = \delta_{IJ} + u_{I,J} + u_{J,I} + u_{i,I}u_{i,J}.$$

- From this, the strain tensor is written as (with components $\mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbb{I})$)

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \underbrace{\frac{\partial u_i}{\partial X_I}\frac{\partial u_i}{\partial X_J}}_{\text{ignored for small strain}}\right)$$

- **Infinitesimal Strain Tensor:** $E_{IJ} = \frac{1}{2}(u_{I,J} + u_{J,I})$.

2.3. The Strain Tensor: Volume Change

- Consider three arbitrarily oriented vectors $d\underline{X}^{(1)}$, $d\underline{X}^{(2)}$, $d\underline{X}^{(3)}$ in the undeformed configuration. The volume that they describe is given by

$$dV = \epsilon_{IJK} dX_I^{(1)} dX_J^{(2)} dX_K^{(3)}.$$

- Upon deformation, using the same notation as above, the volume **becomes**

$$dv = \epsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)}.$$

  Using the deformation gradient to write this out ($d\underset{\sim}{x} = \mathbb{F} d\underset{\sim}{X}$), we have

$$dv = \underbrace{\epsilon_{ijk} F_{iI} F_{jJ} F_{kK}} dX_I^{(1)} dX_J^{(2)} dX_K^{(3)}$$

- We have previously seen that $\epsilon_{ijk} F_{iI} F_{jJ} F_{kK} = \epsilon_{IJK} det(\mathbb{F})$. Substituting this in the above we get,

$$dv = \epsilon_{IJK} det(\mathbb{F}) dX_I^{(1)} dX_J^{(2)} dX_K^{(3)} = det(\mathbb{F}) dV.$$

- $J := det(\mathbb{F})$ is known as the *Jacobi determinant*. $\boxed{dv = J dV}$

2.3. The Strain Tensor: Infinitesimal Volume Change

2. Deformations and Strain

- For the infinitesimal case, the deformation gradient component matrix is expressed as
$$\mathbb{F} = \mathbb{I} + \epsilon \nabla u,$$
where $\epsilon > 0$ is **some small number** ($\epsilon \ll 1$).
- Since ϵ is small, we will try to expand out J as a Taylor series in ϵ about $\epsilon = 0$:
$$J(\epsilon) = J(\epsilon = 0) + \epsilon \left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} + \mathcal{O}(\epsilon^2).$$

Derivative of Determinant

$$\frac{d}{dp}\left(det(\mathbb{M})\right) = trace\left(Adj(\mathbb{M})\frac{d\mathbb{M}}{dp}\right)$$

For invertible $\mathbb{M}$, $Adj(\mathbb{M}) = J\mathbb{M}^{-1}$.

- This simplifies as,
$$J(\epsilon) = det(\mathbb{I}) + \epsilon\left(J(\epsilon = 0) trace\left(\mathbb{I}^{-1}\nabla u\right)\right) + \mathcal{O}(\epsilon^2) \approx 1 + \epsilon tr(\nabla u)$$

2.3. The Strain Tensor: Infinitesimal Volume Change

2. Deformations and Strain

- Undeformed volume is $\boxed{dV}$, deformed volume is $\boxed{dv = JdV}$. So **relative change in volume is**

$$\frac{dv - dV}{dV} = J - 1.$$

- For the infinitesmial case $J \approx 1 + tr(\nabla u)$ (we have set $u \to \epsilon u$ here). Substituting, we get

$$\frac{dv - dV}{dV} = tr(\nabla u) = u_{I,I} = E_{II} = tr(\mathbb{E}).$$

- So the **trace of the strain tensor** is the relative volume change.

In Summary we have, for the strain tensor,

- Each diagonal element corresponds to **stretching/compressing**,
- Off-diagonal elements correspond to **shearing**,
- Trace (sum of diagonal elements) corresponds to **volume change**.

Summary

2. Deformations and Strain

- We have defined the deformation gradient $\mathbb{F}$ and the strain tensor $\underline{\underline{E}}$.
- **Notice:** Under no deformation, if you just changed the coordinate frame of observation, $\mathbb{F}$ will change, **but $\underline{\underline{E}}$ will not.**

Rigid Body Motion

$$\underset{\sim}{x} = \underset{\sim}{c} + \mathbb{R}(\underset{\sim}{X} - \underset{\sim}{X}_0)$$

- What is the deformation gradient here?
- What is the **infinitesimal strain tensor** here?
- What is the **finite strain tensor** here?

- What should the material **respond to**? What is the quantity that the **material wants to resist**?

2.4. Strain Compatibility

2. Deformations and Strain

Necessary Reading

Read Section 1.10 in Megson [3]

- Since strains are defined **based on the displacement field**, the different strain components are related.
- For the infinitesimal case, this relationship can be summarized as (see Appendix 3.1 in Lai, Rubin, and Krempl [1]),

$$E_{IJ,KM} + E_{KM,IJ} - E_{IK,JM} - E_{JM,IK} = 0.$$

- This gives rise to **six independent equations**,

$$\begin{aligned} E_{11,22} + E_{22,11} &= 2E_{12,12}, & E_{11,23} + E_{23,11} &= E_{12,13} + E_{13,12} \\ E_{22,33} + E_{33,22} &= 2E_{23,23}, & E_{22,13} + E_{13,22} &= E_{12,23} + E_{23,12} \\ E_{33,11} + E_{11,33} &= 2E_{13,13}, & E_{33,12} + E_{12,33} &= E_{13,23} + E_{23,13} \end{aligned}$$

The strains have to satisfy these conditions for them to “have been generated” by a continuously differentiable displacement field.

3. Stress and Equilibrium

Force is a vector. Area is a vector. What is **pressure** (F/A)?

4. Constitutive Relationships

References I

[1] W. M. Lai, D. Rubin, and E. Krempl. Introduction to Continuum Mechanics, 4th ed. Amsterdam Boston: Butterworth-Heinemann/Elsevier, 2010. ISBN: 978-0-7506-8560-3 (cit. on pp. 2, 34).

[2] M. H. Sadd. Elasticity: Theory, Applications, and Numerics, 2nd ed. Amsterdam ; Boston: Elsevier/AP, 2009. ISBN: 978-0-12-374446-3 (cit. on p. 2).

[3] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on pp. 2, 34).