AS3020: Aerospace Structures

Module 3: Introduction to Elasticity

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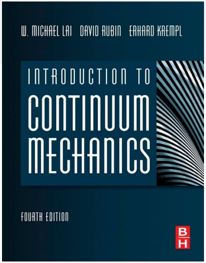
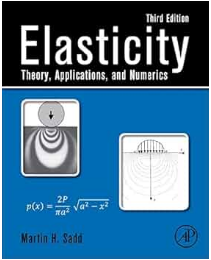
Table of Contents



Chapters 1-5 in Lai, Rubin, and Krempl [1]



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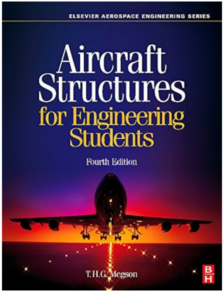
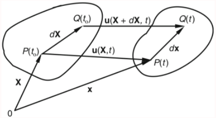


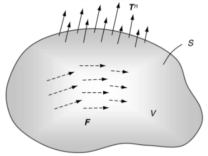
Chapters 1 in Megson [3]

We have to make a choice!

Red Pill

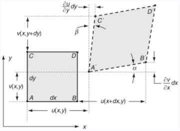
$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

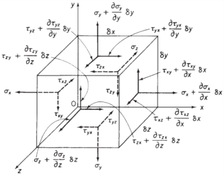




Blue Pill

$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}(\sigma_y + \sigma_y)$$





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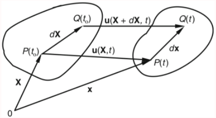

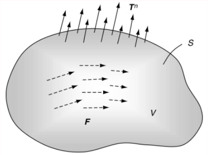

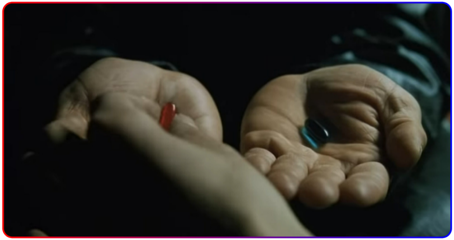

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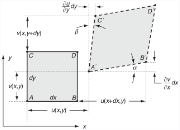

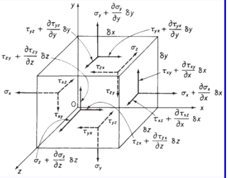

1.1. Indicial Notation I

1. Mathematical Rudiments

Einstein's Summation Convention: Dummy Indices

$$s = a_1 x_1 + a_2 x_2 + \cdots = \sum_{i=1}^{n} a_i x_i \rightarrow a_i x_i = a_k x_k = a_m x_m$$

Consider $\alpha = a_{ij} x_i x_j$, $\underline{v} = v_i \hat{e}_i$, $\underline{\underline{T}} = T_{ij} \hat{e}_i \hat{e}_j$

Free Indices

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}\right\} \implies y_i = a_{ij} x_j$$

Consider $T_{ij} = A_{im} A_{jm}$.

1.1. Indicial Notation II

1. Mathematical Rudiments

The Kronecker Delta

$$\delta_{ij} := \hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Consider $C_{ijkl} = \delta_{ik}\delta_{jl}$, $C_{ijkl} = \delta_{il}\delta_{jk}$.

The Levi-Civita Symbol

$$\epsilon_{ijk} := \hat{e}_i \cdot \underbrace{(\hat{e}_j \times \hat{e}_k)}_{\epsilon_{ijk}\hat{e}_i} = \begin{cases} 1 & \text{if } \{(i,j,k)\} \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{if } \{(i,j,k)\} \in \{(3,2,1),(2,1,3),(1,3,2)\} \\ 0 & \text{otherwise} \end{cases}$$

Consider $\underline{a} \cdot (\underline{b} \times \underline{c})$, $\Delta \underline{\underline{F}}$.

1.1. Indicial Notation III

1. Mathematical Rudiments

Property: $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

$$\epsilon_{ijk}\epsilon_{mnk} = (\epsilon_{ijk}\hat{e}_k) \cdot (\epsilon_{mnk}\hat{e}_k) = (\hat{e}_i \times \hat{e}_j) \cdot (\hat{e}_m \times \hat{e}_n)$$

$$(\hat{e}_i \times \hat{e}_j) \cdot (\hat{e}_m \times \hat{e}_n) = \begin{cases} 1, & \hat{e}_i \times \hat{e}_j = \hat{e}_m \times \hat{e}_n \\ -1, & \hat{e}_i \times \hat{e}_j = -\hat{e}_m \times \hat{e}_n = \hat{e}_n \times \hat{e}_m \\ 0, & \text{otherwise} \end{cases}$$

$$= \boxed{\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$

Consider $\underline{a} \times (\underline{b} \times \underline{c})$

1.1. Indicial Notation IV

1. Mathematical Rudiments

Derivative Notation

$$\frac{\partial u_i}{\partial x_j} := u_{i,j}$$

Consider $\nabla \underline{u}$, $\nabla \cdot \underline{u}$, $\nabla \times \underline{u}$, $\nabla \times \underline{\underline{Q}}$

Exercise

$$\nabla \underline{u}, \underbrace{\nabla \cdot (\nabla \underline{u})}_{\nabla^2 \underline{u}}, \nabla \cdot (\nabla \times \underline{u}), \nabla \times \nabla \times \underline{u}, \nabla \cdot \underline{\underline{\sigma}}$$

1.1. Indicial Notation V

1. Mathematical Rudiments

Vectors, Tensors

$$\underline{u} = u^i \hat{e}_i, \quad \underline{\underline{T}} = T^{ij} \hat{e}_i \hat{e}_j$$

Consider:

- Order of a tensor
- Vector-components as first order tensors
- The tensor product and 2nd order tensors
- Tensors as defining an operation
- Identity **tensors**
- Coordinate transformation
- "Notational abuse"
- Symmetric, antisymmetric tensors
- Antisymmetry as a cross product
- Representation of Eigen-decomposition
- Calculus: Gradient, Divergence, Laplacian, Curl, curvilinear coordinates

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

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Curvilinear Coordinates

- Scalar field ϕ gradient:
$$\delta\phi = \frac{\partial\phi}{\partial x_1}\delta x_1 + \frac{\partial\phi}{\partial x_2}\delta x_2 = \frac{\partial\phi}{\partial r}\delta r + \frac{\partial\phi}{\partial\theta}\delta\theta$$
- Polar bases
$$\underline{e}_r = C_\theta \underline{e}_1 + S_\theta \underline{e}_2 \implies \delta\underline{e}_r = \delta\theta\underline{e}_\theta$$
$$\underline{e}_\theta = -S_\theta \underline{e}_1 + C_\theta \underline{e}_2 \implies \delta\underline{e}_\theta = -\delta\theta\underline{e}_r$$
- Position vector
$$\delta\underline{r} = \delta r\underline{e}_r + r\delta\underline{e}_r = \delta r\underline{e}_r + r\delta\theta\underline{e}_\theta$$
- For $\delta\phi = \nabla\phi\cdot\delta\underline{r}$,
$$\nabla\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta$$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
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Integral Calculus

- The line integral: $\int \underline{F} \cdot d\underline{x}$
  Potential theory:
  $\int_{\partial \mathcal{D}} F_i dx_i = 0 \implies$
  - $F_i = \phi_{,i}$ and $\epsilon_{ijk} F_{k,j}|_{\mathcal{D}} = 0$
  - $\underline{F} = \nabla \phi$ and $\nabla \times \underline{F} = \underline{0}$
- **Gauss Divergence Theorem**
  $\int_{\mathcal{D}} P_{ijk\dots,i} d\mathcal{D} = \int_{\partial \mathcal{D}} P_{ijk\dots} dA_i$
- **Stoke's Law**:
  $\int_A (\nabla \times \underline{F}) \cdot d\underline{A} = \int_{\partial A} \underline{F} \cdot d\underline{x}$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Stoke's Law as a Special Case of Gauss Divergence in 2D

$$\int_{\mathcal{S}} (\nabla \times \underline{v}) \cdot d\underline{S} = \int_{\mathcal{S}} \epsilon_{ijk} v_{k,j} \hat{n}_i d|S|$$
$$= \int_{\mathcal{S}} \left(\epsilon_{ijk} \hat{n}_i v_k\right)_{,j} d|S|$$
$$= \int_{\partial\mathcal{S}} \epsilon_{ijk} \hat{n}_i v_k \hat{b}_j d|\ell|$$
$$= \int_{\partial\mathcal{S}} \underbrace{\left(\epsilon_{ijk} \hat{n}_i \hat{b}_j\right)}_{\underline{\hat{n}} \times \underline{\hat{b}}} v_k d|\ell|$$
$$= \int_{\partial\mathcal{S}} v_k t_k d|\ell| = \int_{\partial\mathcal{S}} \underline{v} \cdot d\underline{\ell}$$

Differ

- Scalar, ve
- Gradient
- Divergen
- Curviline divergenc *coordinat*

ulus

$\underline{F} \cdot d\underline{x}$

$_{ijk} F_{k,j}|_{\mathcal{D}} = 0$

$\times \underline{F} = \underline{0}$

e Theorem

$P_{ijk\ldots} dA_i$

$_{\partial A} \underline{F} \cdot d\underline{x}$

1.2. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

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- Gradients, directional derivative
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Integral Calculus

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  - $\underline{F} = \nabla\phi$ and $\nabla \times \underline{F} = \underline{0}$
- **Gauss Divergence Theorem**
  $\int_{\mathcal{D}} P_{ijk\dots,i} d\mathcal{D} = \int_{\partial\mathcal{D}} P_{ijk\dots} dA_i$
- **Stoke's Law**:
  $\int_A (\nabla \times \underline{F}) \cdot d\underline{A} = \int_{\partial A} \underline{F} \cdot d\underline{x}$
- **Determinant of a Tensor**
  $\epsilon_{IJK} \Delta \underline{\underline{F}} = \epsilon_{ijk} F_{iI} F_{jJ} F_{kK}$
  - Related to volume change through transformation

2. Deformations and Strain

How to describe the change in shape **independently** of rigid body motions?

3. Stress and Equilibrium

Force is a vector. Area is a vector. What is **pressure** (F/A)?

4. Constitutive Relationships

References I

[1] W. M. Lai, D. Rubin, and E. Krempl. Introduction to Continuum Mechanics, 4th ed. Amsterdam Boston: Butterworth-Heinemann/Elsevier, 2010. ISBN: 978-0-7506-8560-3 (cit. on p. 2).

[2] M. H. Sadd. Elasticity: Theory, Applications, and Numerics, 2nd ed. Amsterdam ; Boston: Elsevier/AP, 2009. ISBN: 978-0-12-374446-3 (cit. on p. 2).

[3] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).