AS2070: Aerospace Structural Mechanics

Module 1: Elastic Stability

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Dept. of Aerospace Engg., IIT Madras, Chennai

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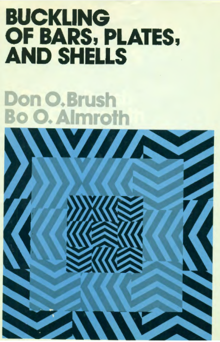
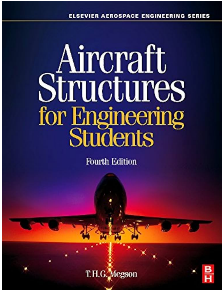


Chapters 1-3 in Brush and Almroth (1975).

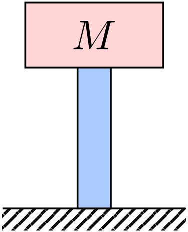


Chapters 7-9 in Megson (2013)

1. Introduction

Structural Stability: What?

- Consider supporting a mass M on the top of a rod.



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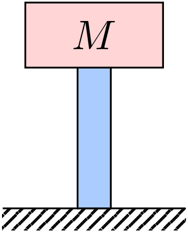
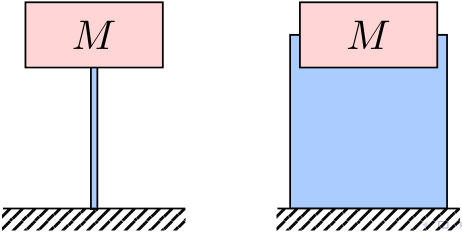


Two Extreme Cases:



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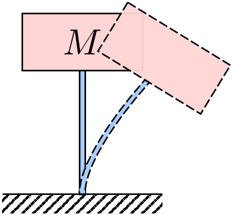

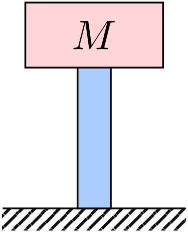

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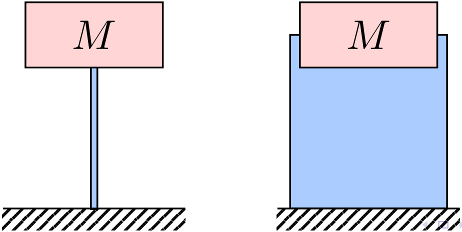

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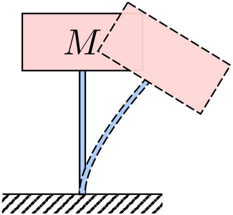
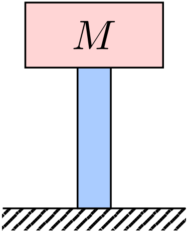


How can we mathematically describe this?



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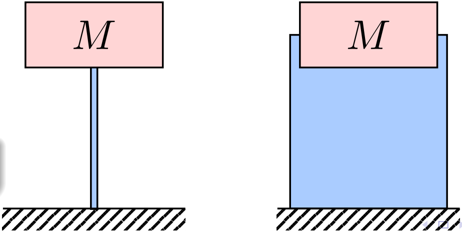


1. Introduction

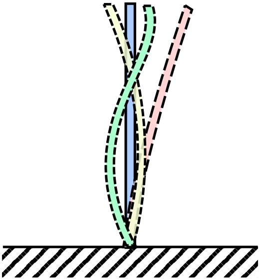
Structural Stability: Perturbation Behavior

Perturbation Behavior

Key insight we will invoke is behavior under **perturbation**:

How would the system respond if I slightly perturb it?

- Mathematically, by perturbation we mean *any change to the system's configuration.*
- In this case, this could be different deflection shapes.



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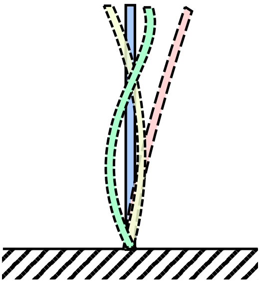


Question (Slightly more specific)

What will the system tend to do if an arbitrarily small magnitude of perturbation is introduced?

- Will it tend to **return to its original configuration**?
- Will it **blow up**?
- Will it do **something else entirely**?

1.1. Elastic Stability

Introduction

What do these words mean?

Elastic → Reversible → Conservative

Conservative System

- The restoring force of a conservative system can be written using a gradient of a **potential function**:

$$\underline{F} = -\nabla U.$$

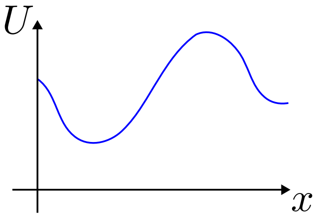
Equilibrium

- System achieves equilibrium when $\underline{F} = \underline{0}$, i.e.,

$$\nabla U = 0.$$

1D Example

Consider a system whose configuration is expressed by the scalar x and the potential is as shown.



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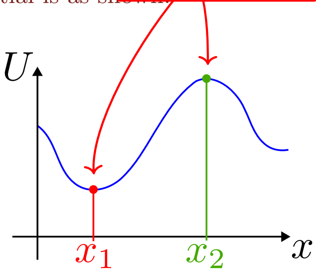

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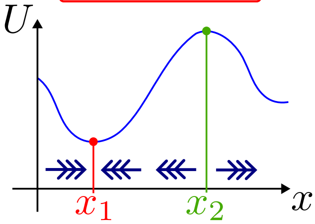
Remember, $F = -\frac{dU}{dx}$.



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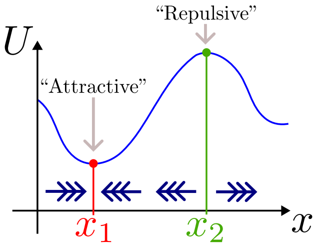

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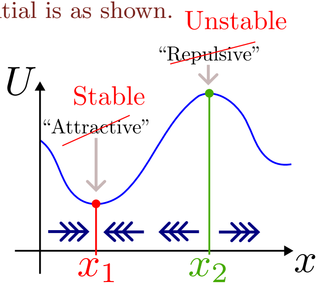
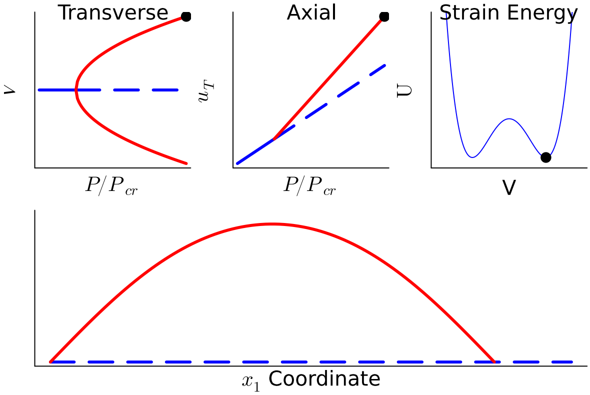


1.2. Bifurcation

Introduction

A system is said to have **undergone a bifurcation** if its state of stability has changed due to the variation of some parameter.

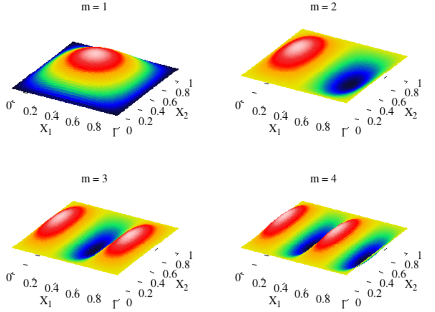


Example: *A pinned-pinned beam undergoing axial loading.*

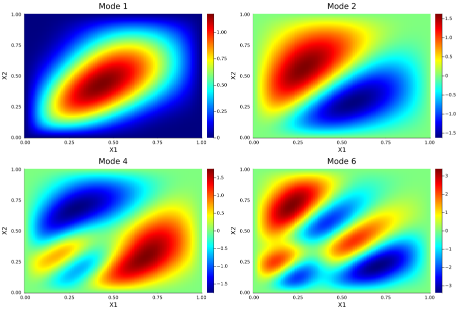
1.3. Modes of Stability Loss

Introduction

The **configuration** that a system can assume as it undergoes a bifurcation is the *mode* of the stability loss.



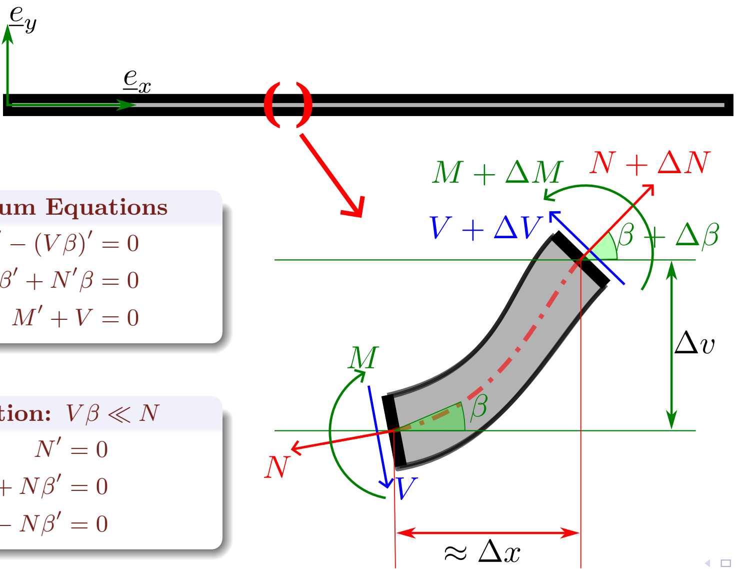
Example: *Thin plate (pinned) under axial loading*



Example: *Thin plate (pinned) under shear loading*

2.1. Equilibrium Equations

Euler Buckling of Columns



Equilibrium Equations

$$N' - (V\beta)' = 0$$

$$V' + N\beta' + N'\beta = 0$$

$$M' + V = 0$$

Assumption: $V\beta \ll N$

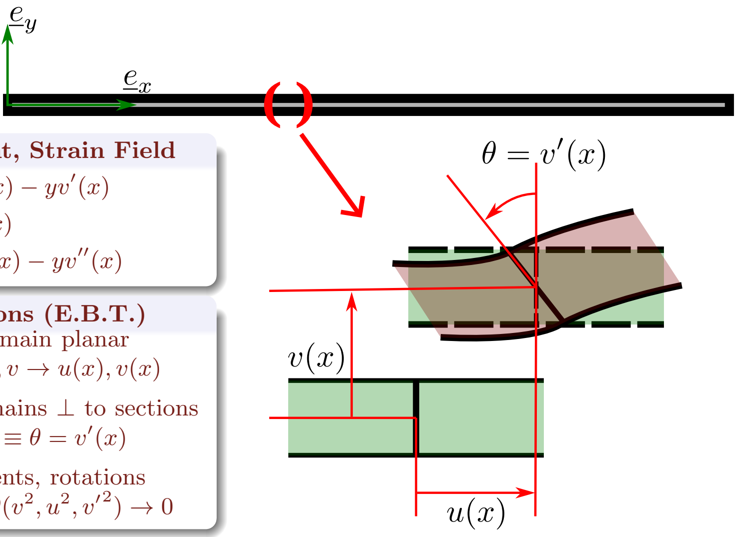
$$N' = 0$$

$$V' + N\beta' = 0$$

$$M'' - N\beta' = 0$$

2.2. Kinematic Description

Euler Buckling of Columns



Displacement, Strain Field

$$u_x = u(x) - y v'(x)$$
$$u_y = v(x)$$
$$\varepsilon_{xx} = u'(x) - y v''(x)$$

Assumptions (E.B.T.)

Plane sections remain planar

$$u, v \to u(x), v(x)$$

Neutral Axis remains $\perp$ to sections

$$\beta \equiv \theta = v'(x)$$

Small displacements, rotations

$$\mathcal{O}(v^2, u^2, v'^2) \to 0$$

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Small displacements, rotations

$$\mathcal{O}(v^2, u^2, v'^2) \rightarrow 0$$

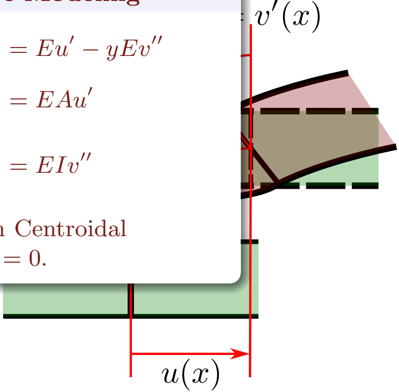
Constitutive Modeling

$$\sigma_{xx} = E\varepsilon_{xx} = Eu' - yEv''$$

$$N = \int_{\mathcal{A}} \sigma_{xx} = EAu'$$

$$M = \int_{\mathcal{A}} -y\sigma_{xx} = EIv''$$

Note: y measured in Centroidal coordinates s.t. $\int_{\mathcal{A}} y = 0$.



2.3. The Linear Buckling Problem

Euler Buckling of Columns

- Substituting, we are left with,

$$N' = \boxed{EAu'' = 0}, \quad M'' - N\beta' = \boxed{EIv'''' - Nv'' = 0}.$$

Axial Problem

- Boundary conditions representing axial compression:

$$u(x=0) = 0, \quad EAu'(x=\ell) = -P$$

- Solution:

$$\boxed{u(x) = -\frac{P}{EA}x}$$

Transverse Problem

- Substituting $N = -P$ we have,

$$v'''' + k^2 v'' = 0, \quad k^2 = \frac{P}{EI}.$$

- The general solution to this **Homogeneous ODE** are

$$\boxed{v(x) = A_0 + A_1 x + A_2 \cos kx + A_3 \sin kx}$$

- Boundary conditions on the transverse displacement function $v(x)$ are necessary to fix A_0, A_1, A_2, A_3.

2.3.1. The Pinned-Pinned Column

The Linear Buckling Problem

- For a Pinned-pinned beam we have $v = 0$ on the ends and zero reaction moments at the supports:

$$v = 0, \quad x = \{0, \ell\}$$
$$v'' = 0, \quad x = \{0, \ell\}$$

- So the general solution reduces to

$$v(x) = A_3 \sin kx,$$

with the boundary condition

$$\boxed{A_3 \sin k\ell = 0.}$$

- Apart from the trivial solution $(A_3 = 0)$ we have

$$k_{(n)}\ell = n\pi \implies k_n = n\frac{\pi}{\ell}$$

or in terms of the compressive load P,

$$\boxed{P_{cr,n} = n^2 \frac{\pi^2 EI}{\ell^2}}$$

- **Interpretation**: If $P \neq P_{cr,n}$, $A_3 = 0$ to satisfy boundary conditions. But for $P = P_{cr,n}$, A_3 **CAN BE ANYTHING!**.

2.3.1. The Pinned-Pinned Column

The Linear Buckling Problem

- For a Pinn … solution
  have $v = 0$ … $_n = n\frac{\pi}{\ell}$
  reaction m

v …

v'' … pressive

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$v($

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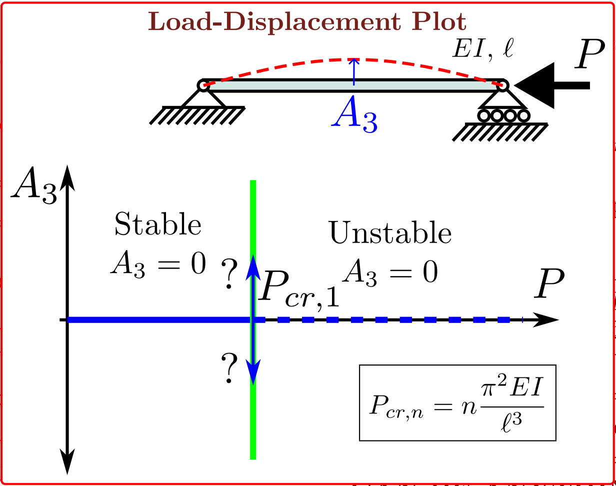


2.3.1. The Pinned-Pinned Column: The Imperfect Case I

The Linear Buckling Problem

- Suppose there are initial imperfections in the beam's neutral axis such that the neutral axis can be written as $v_0(x)$.
- Noting that strains are accumulated only on the *relative displacement* $v(x) - v_0(x)$, we write

$$EI(v - v_0)'''' + Pv'' = 0.$$

Note that the axial load P acts on the **net rotation** of the deflected beam, so we do not need to use $(v - v_0)''$ here.

- The governing equations become

$$EIv'''' + Pv'' = EIv_0'''',$$

or, in more convenient notation,

$$\boxed{v'''' + k^2 v'' = v_0''''}.$$

2.3.1. The Pinned-Pinned Column: The Imperfect Case II

The Linear Buckling Problem

- Describing the imperfect neutral axis using an infinite series,

$$v_0 = \sum_n C_n \sin(n\frac{\pi x}{\ell}) \quad \left(\implies v_0'''' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell}) \right),$$

the governing equations become

$$v'''' + k^2 v'' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell}).$$

2.3.1. The Pinned-Pinned Column: The Imperfect Case III

The Linear Buckling Problem

- This is solved by,

$$v(x) = \sum_n \frac{\left(n\frac{\pi}{\ell}\right)^2}{\left(n\frac{\pi}{\ell}\right)^2 - k^2} C_n \sin(n\frac{\pi x}{\ell})$$

$$= \sum_n \frac{\frac{n^2\pi^2 EI}{\ell^2}}{\frac{n^2\pi^2 EI}{\ell^2} - P} C_n \sin(n\frac{\pi x}{\ell}) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell})$$

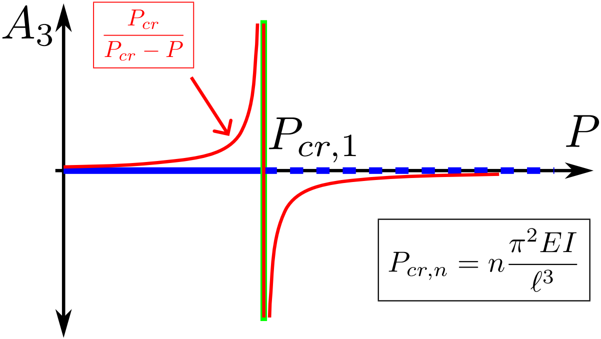
2.3.1. The Pinned-Pinned Column: The Imperfect Case

The Linear Buckling Problem

- Look carefully at the solution

$$v(x) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell}).$$

- Clearly $P \to P_{cr,n}$ are **singularities**. Even for very small C_n, the "blow-up" is huge.



2.3.2. The Southwell Plot

The Linear Buckling Problem

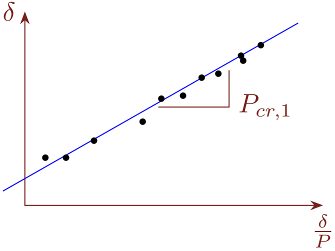
- The relative deformation amplitude at the mid-point is given as (for $P < P_{cr,1}$),

$$\delta \approx \frac{P_{cr,1}}{P_{cr,1} - P} C_1 - C_1 = \frac{C_1}{\frac{P_{cr,1}}{P} - 1}$$

$$\implies \boxed{\delta = P_{cr,1} \frac{\delta}{P} - C_1}$$

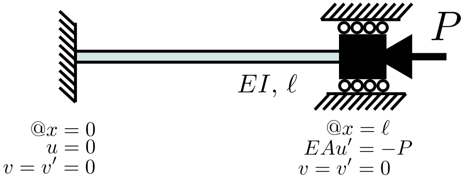
The Southwell Plot

- Plotting δ vs $\frac{\delta}{P}$ allows **Non-Destructive Evaluation of the critical load**
- $P_{cr,1}$ is estimated without having to buckle the column



2.3.3. The Clamped-Clamped Column

The Linear Buckling Problem



- The axial solution is the same as before: $u(x) = -\frac{P}{EA}x$.
- The transverse general solution also has the same form but boundary conditions are different.

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 1 & x & \cos(kx) & \sin(kx) \\ 0 & 1 & -k\sin(kx) & k\cos(kx) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

- The boundary conditions may be expressed as

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & \ell & \cos(k\ell) & \sin(k\ell) \\ 0 & 1 & -k\sin(k\ell) & k\cos(k\ell) \end{bmatrix}}_{\underline{\underline{M}}} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- There can be non-trivial solutions only when $\underline{\underline{M}}$ is singular, i.e., **for choices of k such that $\Delta(\underline{\underline{M}}) = 0$.**

The Eigenvalue Problem

This problem setting of finding k such that $\Delta(\underline{\underline{M}}(k)) = 0$ is known as an **eigenvalue problem**.

2.3.3. The Clamped-Clamped Column

The Linear Buckling Problem

$@x = 0$
$u = 0$
$v = v' = 0$

- The boundary conditions may be

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- The axia ... is before: v ... **es of k**
- The tran ... also has ... boundar ...

$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} =$ ... $\begin{bmatrix} A_2 \\ A_3 \end{bmatrix}$ **eigenvalue problem**.

... k such ... an

Aside: Eigenvalue Problems ($\underline{\underline{M}} \in \mathbf{R}^{d\times d}$)

Linear Eigenvalue Problem (d eigenvalues)

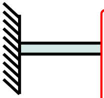
$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}}$$

Quadratic Eigenvalue Problem ($2d$ eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}} + k^2\underline{\underline{M_2}}$$

2.3.3. The Clamped-Clamped Column

The Linear Buckling Problem



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Quadratic Eigenvalue Problem ($2d$ eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}} + k^2\underline{\underline{M_2}}$$

Our matrix $\underline{\underline{M}}(k)$ has k-dependency in terms of k, $\sin(k\ell)$, $\cos(k\ell)$, making this a **Nonlinear Eigenvalue Problem**.

- $\implies \infty$ eigenvalues in general

2.3.3. The Clamped-Clamped Column I

The Linear Buckling Problem

- We proceed to solve this as,

$$\Delta\left(\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & k\\ 1 & \ell & \cos(k\ell) & \sin(k\ell)\\ 0 & 1 & -k\sin(k\ell) & k\cos(k\ell)\end{bmatrix}\right) = -k\left(k\ell\sin(k\ell) + 2\cos(k\ell) - 2\right)$$

- We set it to zero through the following factorizations:

$$\begin{aligned}\Delta(\underline{\underline{M}}(k)) &= -k\left(2k\ell\sin(\frac{k\ell}{2})\cos(\frac{k\ell}{2}) - 4\sin^2(\frac{k\ell}{2})\right)\\ &= -2k\sin(\frac{k\ell}{2})\left(k\ell\cos(\frac{k\ell}{2}) - 2\sin(\frac{k\ell}{2})\right) = 0\end{aligned}$$

$$\implies \boxed{\sin(\frac{k\ell}{2}) = 0}, \quad \text{(or)} \quad \boxed{\tan(\frac{k\ell}{2}) = \frac{k\ell}{2}}.$$

2.3.3. The Clamped-Clamped Column II

The Linear Buckling Problem

- Two "classes" of solutions emerge:
  1. $\sin(\frac{k\ell}{2}) = 0 \implies \frac{k_n \ell}{2} = n\pi \implies \boxed{P_n^{(1)} = 4n^2 \frac{\pi^2 EI}{\ell^2}}$
  2. $\tan(\frac{k\ell}{2}) = \frac{k\ell}{2} \implies \frac{k_n \ell}{2} \approx 0, 4.49, 7.72, \cdots \implies P_1^{(2)} \approx 8.98 \frac{\pi^2 EI}{\ell^2}$
- **The smallest critical load** is $P_n^{(1)} = 4\frac{\pi^2 EI}{\ell^2} = \frac{\pi^2 EI}{(\frac{\ell}{2})^2}$.

Concept of "Effective Length"

- **Question**: If the beam were simply supported, what would be the length such that it also has the same first critical load?
- Here it comes out to be $\ell_{eff} = \frac{\ell}{2}$.
- The column clamped on both ends can take the same buckling load as a column that is pinned on both ends with half the length.

2.3.3. The Clamped-Clamped Column III

The Linear Buckling Problem

Boundary conditions	Critical load P_{cr}	Deflection mode shape	Effective length KL
Simple support–simple support	$\frac{\pi^2 EI}{L^2}$		L
Clamped-clamped	$4\frac{\pi^2 EI}{L^2}$		$\frac{1}{2}L$
Clamped–simple support	$2.04\frac{\pi^2 EI}{L^2}$		$0.70L$
Clamped-free	$\frac{1}{4}\frac{\pi^2 EI}{L^2}$		$2L$

Effective lengths of beams with different boundary conditions *(Figure from Brush and Almroth 1975)*

Self-Study

- Derive the effective length for the clamped-simply supported and clamped-free columns.

2.3.3. The Clamped-Clamped Column: The Mode-shape

The Linear Buckling Problem

- Let us substitute $k_1 = \frac{2\pi}{\ell}$ into the matrix $\underline{\underline{M}}(k_1)$ so that the boundary conditions now read as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \\ 1 & \ell & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- This implies the following:

$$A_1 = 0, \quad A_3 = 0, \quad A_2 = -A_0.$$

- So, if $k = k_1$, the solution has to be the following to satisfy the boundary conditions:

$$v = A_0 \left(1 - \cos(\frac{2\pi x}{\ell})\right) \equiv A_0 \sin^2(\frac{\pi x}{\ell})$$

2.3.3. The Clamped-Clamped Column: The Mode-shape

The Linear Buckling Problem

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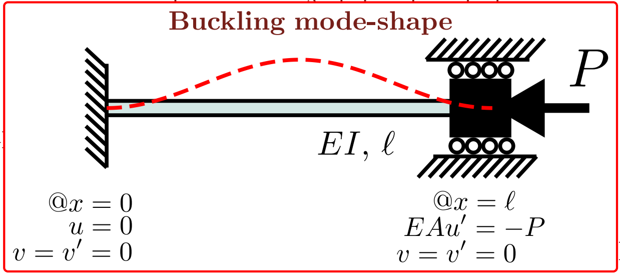

- This im

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3. Energy Perspectives

3.1. Post-Buckling Behavior

Energy Perspectives

4. Plate Buckling

References I

[1] D. O. Brush and B. O. Almroth. Buckling of Bars, Plates, and Shells, McGraw-Hill, 1975. ISBN: 978-0-07-008593-0 (cit. on pp. 2, 32).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).

6. Class Discussions (Outside of Slides)

- Ball on a hill. 2D, 3D cases.
- Assumptions behind compression of a bar.