AS2070: Aerospace Structural Mechanics

Module 1: Elastic Stability

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Dept. of Aerospace Engg., IIT Madras, Chennai

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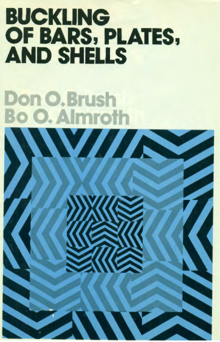
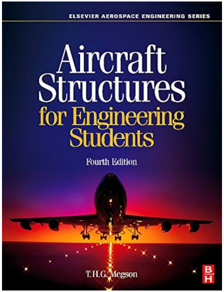


Chapters 1-3 in Brush and Almroth (1975).

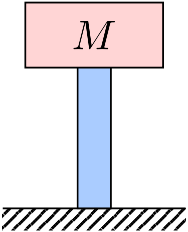


Chapters 7-9 in Megson (2013)

1. Introduction

Structural Stability: What?

- Consider supporting a mass M on the top of a rod.



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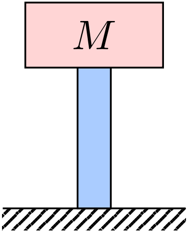
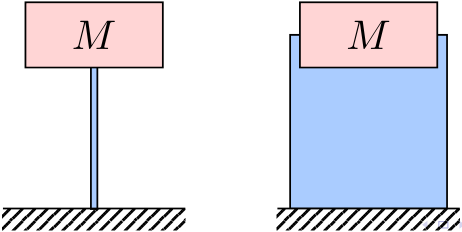


Two Extreme Cases:



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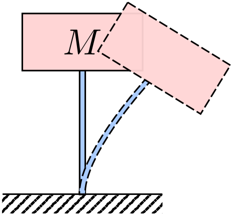

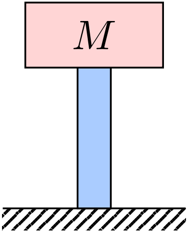

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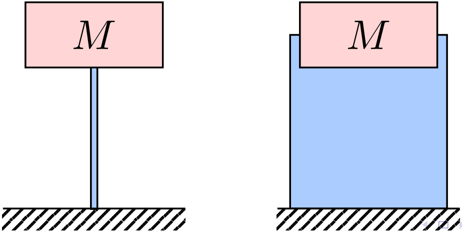

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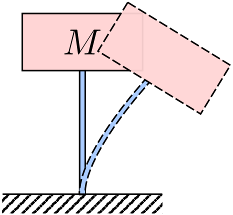
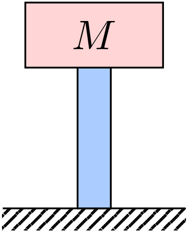


How can we mathematically describe this?



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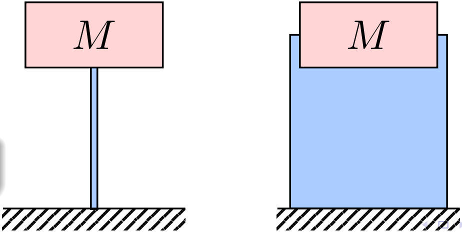


1. Introduction

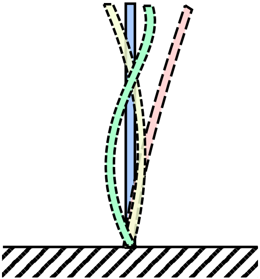
Structural Stability: Perturbation Behavior

Perturbation Behavior

Key insight we will invoke is behavior under **perturbation**:

How would the system respond if I slightly perturb it?

- Mathematically, by perturbation we mean *any change to the system's configuration.*
- In this case, this could be different deflection shapes.



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Structural Stability: Perturbation Behavior

Perturbation Behavior

Key insight we will invoke is behavior under **perturbation**:

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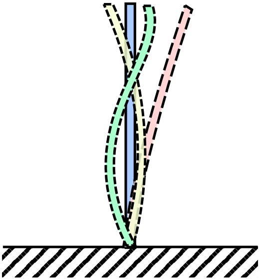
Question (Slightly more specific)

What will the system tend to do if an arbitrarily small magnitude of perturbation is introduced?

- Will it tend to **return to its original configuration**?
- Will it **blow up**?
- Will it do **something else entirely**?



1.1. Elastic Stability

Introduction

What do these words mean?

Elastic → Reversible → Conservative

Conservative System

- The restoring force of a conservative system can be written using a gradient of a **potential function**:

$$\underline{F} = -\nabla U.$$

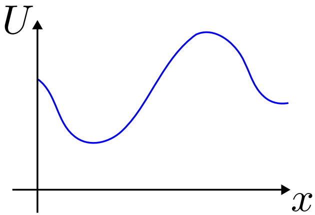
Equilibrium

- System achieves equilibrium when $\underline{F} = \underline{0}$, i.e.,

$$\nabla U = 0.$$

1D Example

Consider a system whose configuration is expressed by the scalar x and the potential is as shown.



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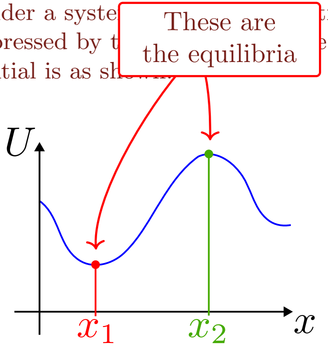

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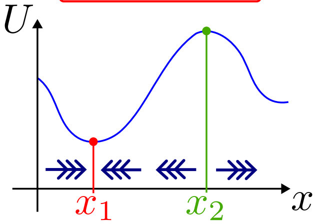
Remember, $F = -\frac{dU}{dx}$.



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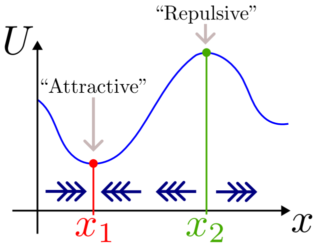

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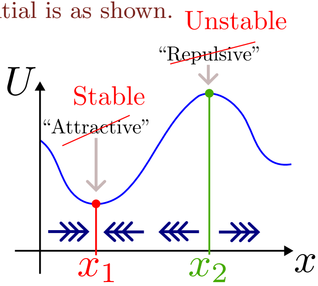
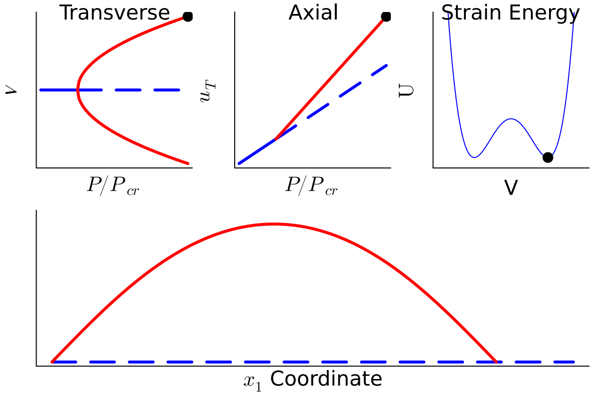


1.2. Bifurcation

Introduction

A system is said to have **undergone a bifurcation** if its state of stability has changed due to the variation of some parameter.

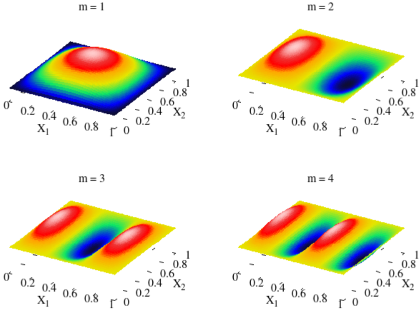


Example: *A pinned-pinned beam undergoing axial loading.*

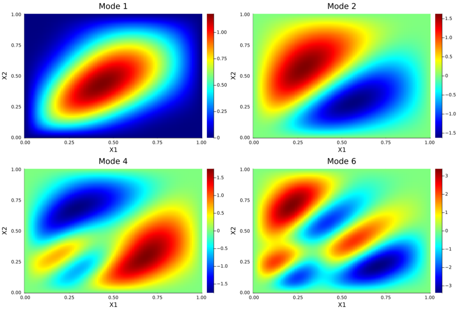
1.3. Modes of Stability Loss

Introduction

The **configuration** that a system can assume as it undergoes a bifurcation is the *mode* of the stability loss.



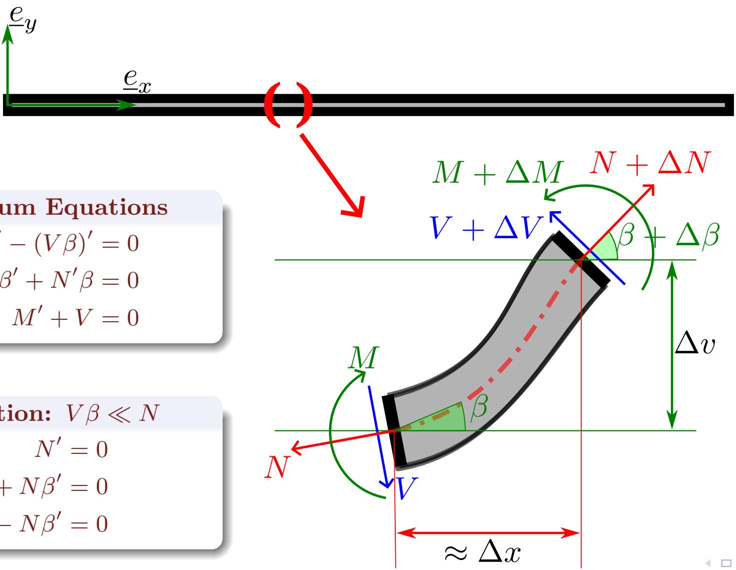
Example: *Thin plate (pinned) under axial loading*



Example: *Thin plate (pinned) under shear loading*

2.1. Equilibrium Equations

Euler Buckling of Columns



Equilibrium Equations

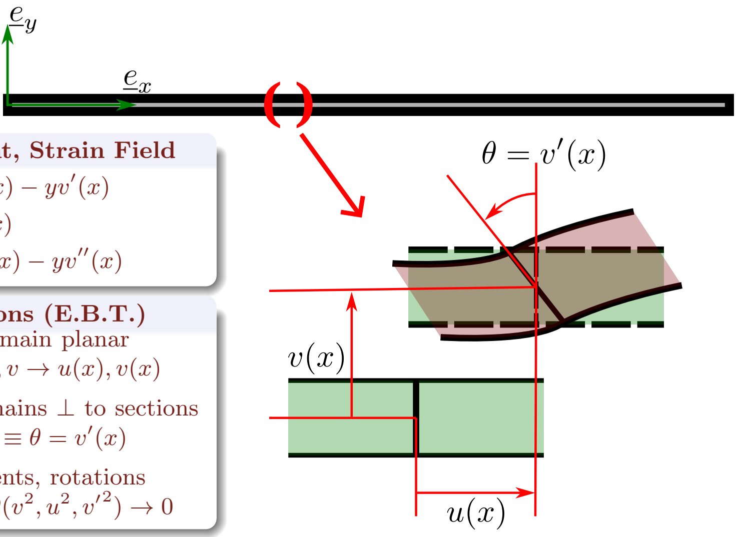
$$N' - (V\beta)' = 0$$
$$V' + N\beta' + N'\beta = 0$$
$$M' + V = 0$$

Assumption: $V\beta \ll N$

$$N' = 0$$
$$V' + N\beta' = 0$$
$$M'' - N\beta' = 0$$

2.2. Kinematic Description

Euler Buckling of Columns



Displacement, Strain Field

$$u_x = u(x) - yv'(x)$$
$$u_y = v(x)$$
$$\varepsilon_{xx} = u'(x) - yv''(x)$$

Assumptions (E.B.T.)

Plane sections remain planar

$$u, v \to u(x), v(x)$$

Neutral Axis remains $\perp$ to sections

$$\beta \equiv \theta = v'(x)$$

Small displacements, rotations

$$\mathcal{O}(v^2, u^2, v'^2) \to 0$$

2.2. Kinematic Description

Euler Buckling of Columns

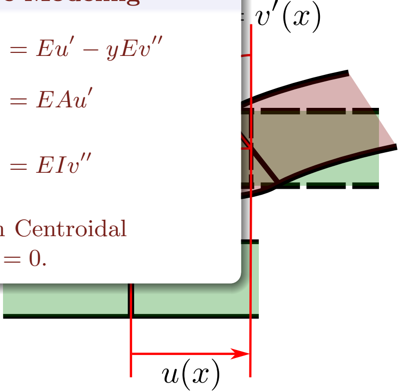
Displacement, S

$$u_x = u(x) - $$
$$u_y = v(x)$$
$$\varepsilon_{xx} = u'(x) - $$

Constitutive Modeling

$$\sigma_{xx} = E\varepsilon_{xx} = Eu' - yEv''$$
$$N = \int_{\mathcal{A}} \sigma_{xx} = EAu'$$
$$M = \int_{\mathcal{A}} -y\sigma_{xx} = EIv''$$

Note: y measured in Centroidal coordinates s.t. $\int_{\mathcal{A}} y = 0$.



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Plane sections remai

$u, v \rightarrow$

Neutral Axis remains

$$\beta \equiv \theta = v'(x)$$

Small displacements, rotations

$$\mathcal{O}(v^2, u^2, v'^2) \rightarrow 0$$

2.3. The Linear Buckling Problem

Euler Buckling of Columns

- Substituting, we are left with,

$$N' = \boxed{EAu'' = 0}, \quad M'' - N\beta' = \boxed{EIv'''' - Nv'' = 0}.$$

Axial Problem

- Boundary conditions representing axial compression:

$$u(x=0) = 0, \quad EAu'(x=\ell) = -P$$

- Solution:

$$\boxed{u(x) = -\frac{P}{EA}x}$$

Transverse Problem

- Substituting $N = -P$ we have,

$$v'''' + k^2 v'' = 0, \quad k^2 = \frac{P}{EI}.$$

- The general solution to this **Homogeneous ODE** are

$$\boxed{v(x) = A_0 + A_1 x + A_2 \cos kx + A_3 \sin kx}$$

- Boundary conditions on the transverse displacement function $v(x)$ are necessary to fix A_0, A_1, A_2, A_3.

2.3.1. The Pinned-Pinned Beam

The Linear Buckling Problem

- For a Pinned-pinned beam we have $v = 0$ on the ends and zero reaction moments at the supports:

$$v = 0, \quad x = \{0, \ell\}$$
$$v'' = 0, \quad x = \{0, \ell\}$$

- So the general solution reduces to

$$v(x) = A_3 \sin kx,$$

with the boundary condition

$$\boxed{A_3 \sin k\ell = 0.}$$

- Apart from the trivial solution ($A_3 = 0$) we have

$$k_{(n)}\ell = n\pi \implies k_n = n\frac{\pi}{\ell}$$

or in terms of the compressive load P,

$$\boxed{P_{cr,n} = n^2 \frac{\pi^2 EI}{\ell^2}}$$

- **Interpretation**: If $P \neq P_{cr,n}$, $A_3 = 0$ to satisfy boundary conditions. But for $P = P_{cr,n}$, A_3 **CAN BE ANYTHING!**.

2.3.1. The Pinned-Pinned Beam

The Linear Buckling Problem

- For a Pinn ... solution
  have $v = 0$ ...
  reaction m ... $_n = n\frac{\pi}{\ell}$

  v ...
  v'' ... pressive

- So the gen ... EI / ^{2}

  $v($ ...

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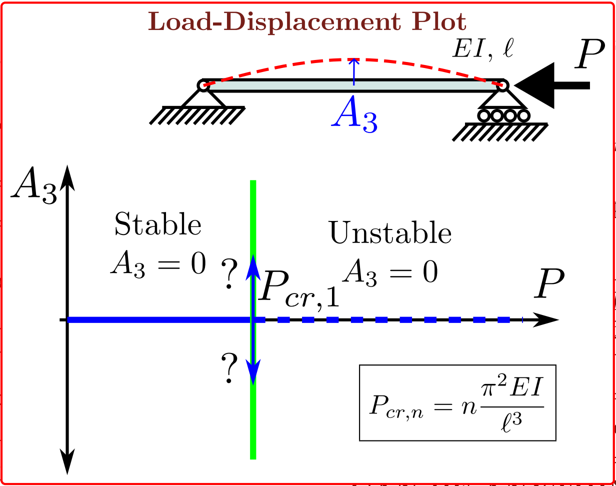


2.3.1. The Pinned-Pinned Beam: The Imperfect Case I

The Linear Buckling Problem

- Suppose there are initial imperfections in the beam's neutral axis such that the neutral axis can be written as $v_0(x)$.
- Noting that strains are accumulated only on the *relative displacement* $v(x) - v_0(x)$, we write

$$EI(v - v_0)'''' + Pv'' = 0.$$

  Note that the axial load P acts on the **net rotation** of the deflected beam, so we do not need to use $(v - v_0)''$ here.
- The governing equations become

$$EIv'''' + Pv'' = EIv_0'''',$$

  or, in more convenient notation,

$$\boxed{v'''' + k^2 v'' = v_0''''}.$$

2.3.1. The Pinned-Pinned Beam: The Imperfect Case II

- Describing the imperfect neutral axis using an infinite series,

$$v_0 = \sum_n C_n \sin(n\frac{\pi x}{\ell}) \quad \left(\implies v_0'''' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell})\right),$$

  the governing equations become

$$v'''' + k^2 v'' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell}).$$

- This is solved by,

$$\begin{aligned} v(x) &= \sum_n \frac{\left(n\frac{\pi}{\ell}\right)^2}{\left(n\frac{\pi}{\ell}\right)^2 - k^2} C_n \sin(n\frac{\pi x}{\ell}) \\ &= \sum_n \frac{\frac{n^2\pi^2 EI}{\ell^2}}{\frac{n^2\pi^2 EI}{\ell^2} - P} C_n \sin(n\frac{\pi x}{\ell}) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell}) \end{aligned}$$

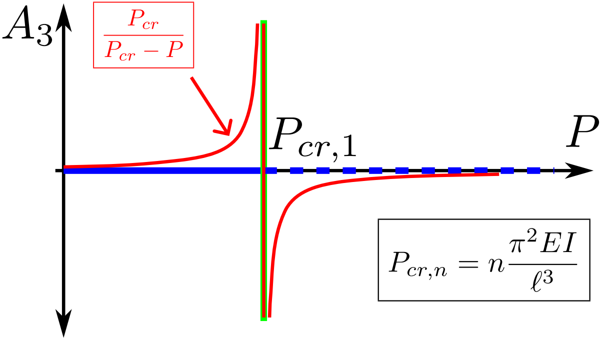
2.3.1. The Pinned-Pinned Beam: The Imperfect Case

The Linear Buckling Problem

- Look carefully at the solution

$$v(x) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell}).$$

- Clearly $P \to P_{cr,n}$ are **singularities**. Even for very small C_n, the "blow-up" is huge.



References I

[1] D. O. Brush and B. O. Almroth. Buckling of Bars, Plates, and Shells, McGraw-Hill, 1975. ISBN: 978-0-07-008593-0 (cit. on p. 2).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).

4. Class Discussions (Outside of Slides)

- Ball on a hill. 2D, 3D cases.
- Assumptions behind compression of a bar.