# AS2070: Aerospace Structural Mechanics

## Module 1: Elastic Stability

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**Dept. of Aerospace Engg., IIT Madras, Chennai**

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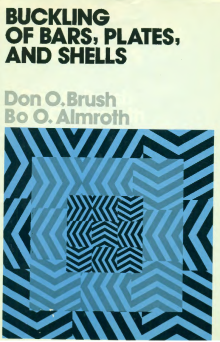
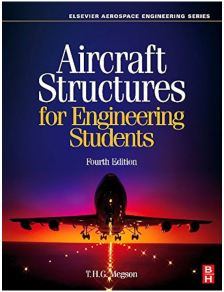
# Table of Contents






*Chapters 1-3 in Brush and Almroth (1975).*

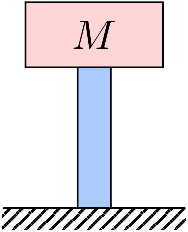



*Chapters 7-9 in Megson (2013)*

# 1. Introduction

Structural Stability: What?

- Consider supporting a mass $M$ on the top of a rod.



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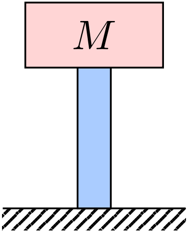
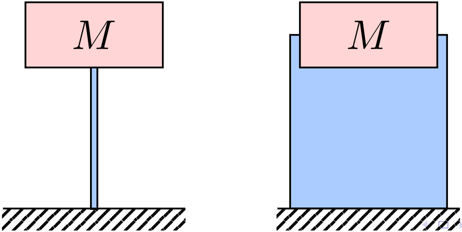



**Two Extreme Cases:**



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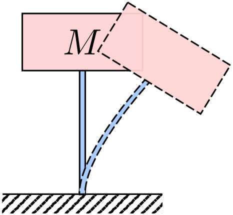


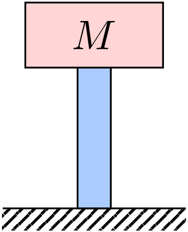


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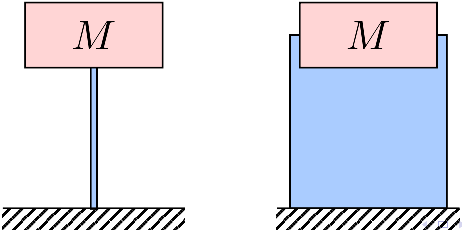

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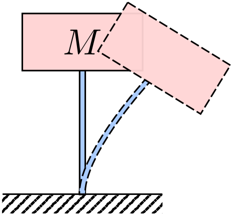
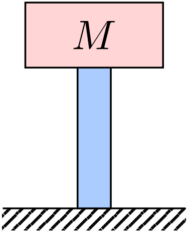



**How can we mathematically describe this?**

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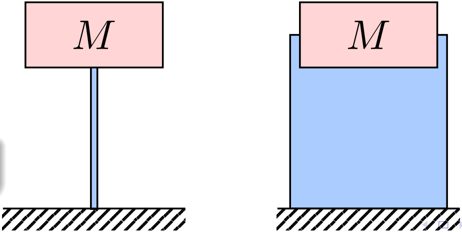


# 1. Introduction

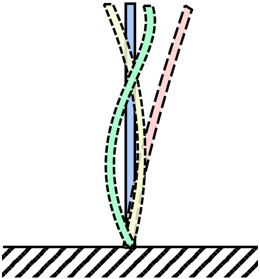
Structural Stability: Perturbation Behavior

## Perturbation Behavior

Key insight we will invoke is behavior under **perturbation**:

*How would the system respond if I slightly perturb it?*

- Mathematically, by perturbation we mean *any change to the system's configuration.*
- In this case, this could be different deflection shapes.



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**Perturbation Behavior**

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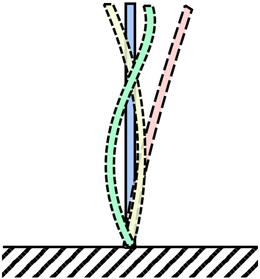


**Question (Slightly more specific)**

What will the system tend to do if an arbitrarily small magnitude of perturbation is introduced?

- Will it tend to **return to its original configuration**?
- Will it **blow up**?
- Will it do **something else entirely**?

# 1.1. Elastic Stability

Introduction

**What do these words mean?**

**Elastic → Reversible → Conservative**

### Conservative System

- The restoring force of a conservative system can be written using a gradient of a **potential function**:

$$\underline{F} = -\nabla U.$$

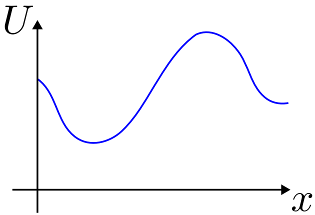
### Equilibrium

- System achieves equilibrium when $\underline{F} = \underline{0}$, i.e.,

$$\nabla U = 0.$$

### 1D Example

Consider a system whose configuration is expressed by the scalar $x$ and the potential is as shown.



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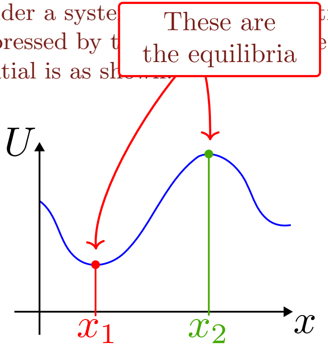

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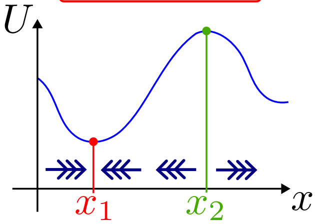
Remember, $F = -\frac{dU}{dx}$.



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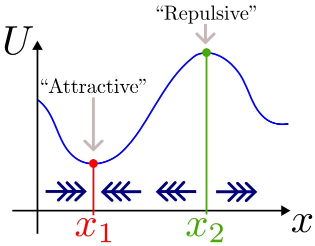

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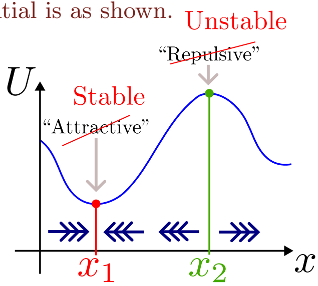
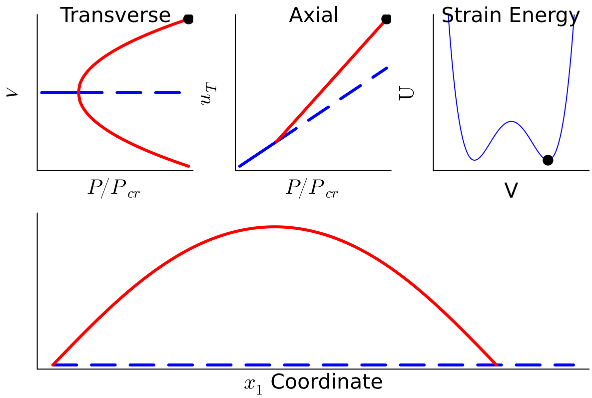


# 1.2. Bifurcation

Introduction

A system is said to have **undergone a bifurcation** if its state of stability has changed due to the variation of some parameter.

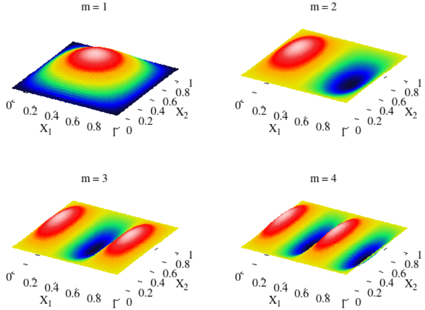



***Example:*** *A pinned-pinned beam undergoing axial loading.*

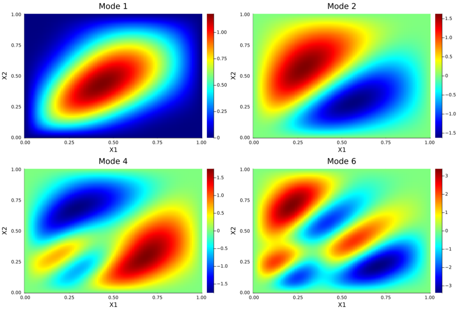
# 1.3. Modes of Stability Loss

Introduction

The **configuration** that a system can assume as it undergoes a bifurcation is the *mode* of the stability loss.



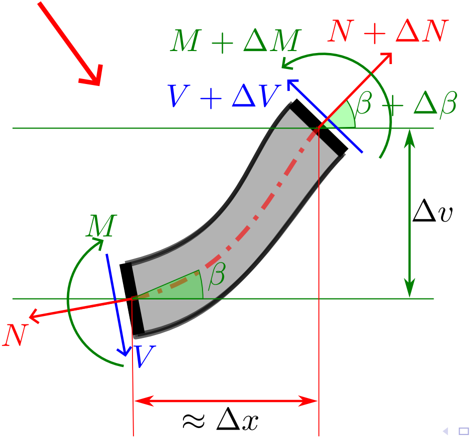

***Example:*** *Thin plate (pinned) under axial loading*




***Example:*** *Thin plate (pinned) under shear loading*

# 2.1. Equilibrium Equations

Euler Buckling of Columns


**Equilibrium Equations**

$$N' - (V\beta)' = 0$$

$$V' + N\beta' + N'\beta = 0$$

$$M' + V = 0$$

**Assumption:** $V\beta \ll N$

$$N' = 0$$

$$V' + N\beta' = 0$$

$$M'' - N\beta' = 0$$

# 2.2. Kinematic Description

Euler Buckling of Columns


**Displacement, Strain Field**

$$u_x = u(x) - yv'(x)$$
$$u_y = v(x)$$
$$\varepsilon_{xx} = u'(x) - yv''(x)$$

**Assumptions (E.B.T.)**

Plane sections remain planar

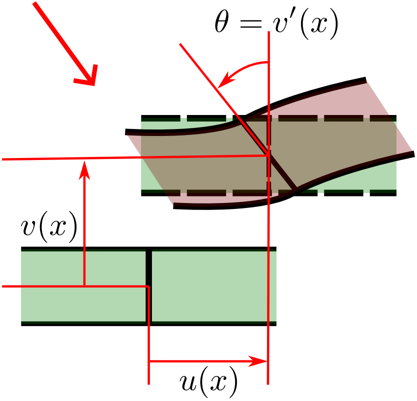
$$u, v \rightarrow u(x), v(x)$$

Neutral Axis remains $\perp$ to sections

$$\beta \equiv \theta = v'(x)$$

Small displacements, rotations

$$\mathcal{O}(v^2, u^2, v'^2) \rightarrow 0$$



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Plane sections remain

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Neutral Axis remains

$\beta \equiv \theta = v'(x)$

Small displacements, rotations

$\mathcal{O}(v^2, u^2, v'^2) \rightarrow 0$

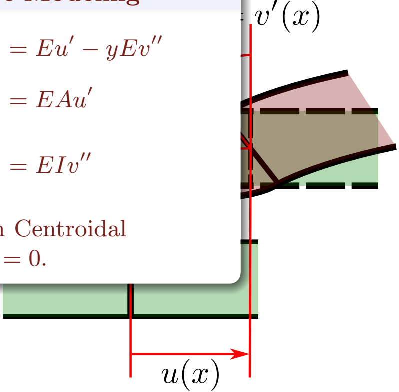
**Constitutive Modeling**

$$\sigma_{xx} = E\varepsilon_{xx} = Eu' - yEv''$$

$$N = \int_{\mathcal{A}} \sigma_{xx} = EAu'$$

$$M = \int_{\mathcal{A}} -y\sigma_{xx} = EIv''$$

**Note:** $y$ measured in Centroidal coordinates s.t. $\int_{\mathcal{A}} y = 0$.



# 2.3. The Linear Buckling Problem

Euler Buckling of Columns

- Substituting, we are left with,

$$N' = \boxed{EAu'' = 0}, \quad M'' - N\beta' = \boxed{EIv'''' - Nv'' = 0}.$$

### Axial Problem

- Boundary conditions representing axial compression:

$$u(x=0) = 0, \quad EAu'(x=\ell) = -P$$

- Solution:

$$\boxed{u(x) = -\frac{P}{EA}x}$$

### Transverse Problem

- Substituting $N = -P$ we have,

$$v'''' + k^2 v'' = 0, \quad k^2 = \frac{P}{EI}.$$

- The general solution to this **Homogeneous ODE** are

$$\boxed{v(x) = A_0 + A_1 x + A_2 \cos kx + A_3 \sin kx}$$

- Boundary conditions on the transverse displacement function $v(x)$ are necessary to fix $A_0, A_1, A_2, A_3$.

# 2.3.1. The Pinned-Pinned Column

The Linear Buckling Problem

- For a Pinned-pinned beam we have $v = 0$ on the ends and zero reaction moments at the supports:

$$v = 0, \quad x = \{0, \ell\}$$
$$v'' = 0, \quad x = \{0, \ell\}$$

- So the general solution reduces to

$$v(x) = A_3 \sin kx,$$

with the boundary condition

$$\boxed{A_3 \sin k\ell = 0.}$$

- Apart from the trivial solution ($A_3 = 0$) we have

$$k_{(n)}\ell = n\pi \implies k_n = n\frac{\pi}{\ell}$$

or in terms of the compressive load $P$,

$$\boxed{P_{cr,n} = n^2 \frac{\pi^2 EI}{\ell^2}}$$

- **Interpretation**: If $P \neq P_{cr,n}$, $A_3 = 0$ to satisfy boundary conditions. But for $P = P_{cr,n}$, $A_3$ **CAN BE ANYTHING!**.

# 2.3.1. The Pinned-Pinned Column

The Linear Buckling Problem

- For a Pinn … solution
  have $v = 0$ …
  reaction m … $_n = n\frac{\pi}{\ell}$

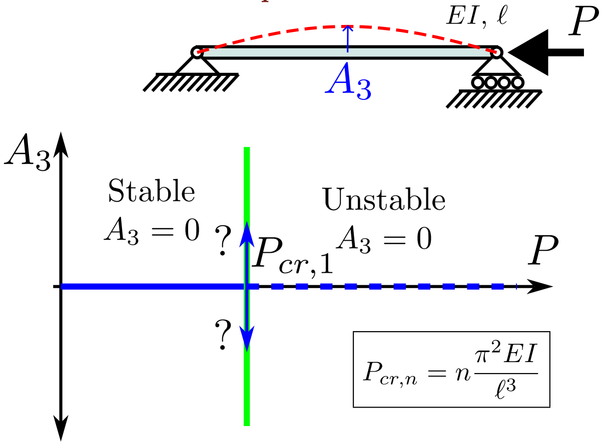
  $v$ …
  $v''$ … pressive

- So the gen … $EI$ … $2$

  $v($ …

  with the bo … $\neq P_{cr,n}$,
  … dary
  … $= P_{cr,n}$, $A_3$
  **CAN BE ANYTHING!**.

**Load-Displacement Plot**



# 2.3.1. The Pinned-Pinned Column: The Imperfect Case I

The Linear Buckling Problem

- Suppose there are initial imperfections in the beam's neutral axis such that the neutral axis can be written as $v_0(x)$.
- Noting that strains are accumulated only on the *relative displacement* $v(x) - v_0(x)$, we write

$$EI(v - v_0)'''' + Pv'' = 0.$$

Note that the axial load $P$ acts on the **net rotation** of the deflected beam, so we do not need to use $(v - v_0)''$ here.

- The governing equations become

$$EIv'''' + Pv'' = EIv_0'''',$$

or, in more convenient notation,

$$\boxed{v'''' + k^2 v'' = v_0''''}.$$

# 2.3.1. The Pinned-Pinned Column: The Imperfect Case II

The Linear Buckling Problem

- Describing the imperfect neutral axis using an infinite series,

$$v_0 = \sum_n C_n \sin(n\frac{\pi x}{\ell}) \quad \left(\implies v_0'''' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell})\right),$$

the governing equations become

$$v'''' + k^2 v'' = \sum_n \left(n\frac{\pi}{\ell}\right)^4 C_n \sin(n\frac{\pi x}{\ell}).$$

# 2.3.1. The Pinned-Pinned Column: The Imperfect Case III

The Linear Buckling Problem

- This is solved by,

$$v(x) = \sum_n \frac{\left(n\frac{\pi}{\ell}\right)^2}{\left(n\frac{\pi}{\ell}\right)^2 - k^2} C_n \sin(n\frac{\pi x}{\ell})$$

$$= \sum_n \frac{\frac{n^2\pi^2 EI}{\ell^2}}{\frac{n^2\pi^2 EI}{\ell^2} - P} C_n \sin(n\frac{\pi x}{\ell}) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell})$$

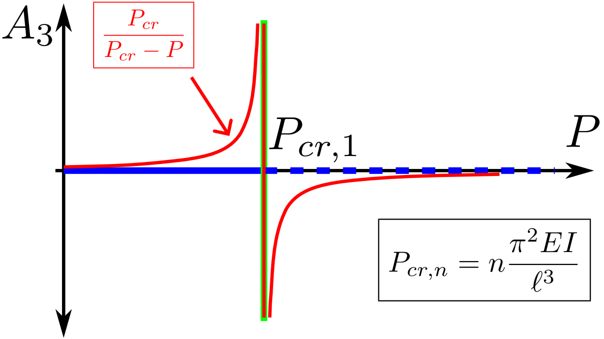
# 2.3.1. The Pinned-Pinned Column: The Imperfect Case

The Linear Buckling Problem

- Look carefully at the solution

$$v(x) = \sum_n \frac{P_{cr,n}}{P_{cr,n} - P} C_n \sin(n\frac{\pi x}{\ell}).$$

- Clearly $P \to P_{cr,n}$ are **singularities**. Even for very small $C_n$, the "blow-up" is huge.



# 2.3.2. The Southwell Plot

The Linear Buckling Problem

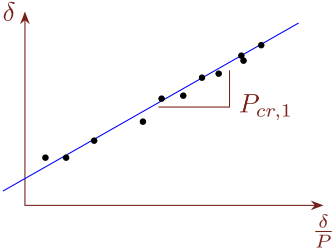
- The relative deformation amplitude at the mid-point is given as (for $P < P_{cr,1}$),

$$\delta \approx \frac{P_{cr,1}}{P_{cr,1} - P} C_1 - C_1 = \frac{C_1}{\frac{P_{cr,1}}{P} - 1}$$

$$\implies \boxed{\delta = P_{cr,1} \frac{\delta}{P} - C_1}$$

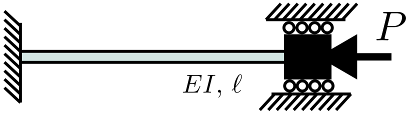
**The Southwell Plot**

- Plotting $\delta$ vs $\frac{\delta}{P}$ allows **Non-Destructive Evaluation of the critical load**
- $P_{cr,1}$ is estimated without having to buckle the column



# 2.3.3. The Clamped-Clamped Column

The Linear Buckling Problem




@$x = 0$
$u = 0$
$v = v' = 0$

@$x = \ell$
$EAu' = -P$
$v = v' = 0$

- The axial solution is the same as before: $u(x) = -\frac{P}{EA}x$.
- The transverse general solution also has the same form but boundary conditions are different.

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \begin{bmatrix} 1 & x & \cos(kx) & \sin(kx) \\ 0 & 1 & -k\sin(kx) & k\cos(kx) \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

- The boundary conditions may be expressed as

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & k \\ 1 & \ell & \cos(k\ell) & \sin(k\ell) \\ 0 & 1 & -k\sin(k\ell) & k\cos(k\ell) \end{bmatrix}}_{\underline{\underline{M}}} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- There can be non-trivial solutions only when $\underline{\underline{M}}$ is singular, i.e., **for choices of $k$ such that $\Delta(\underline{\underline{M}}) = 0$.**

**The Eigenvalue Problem**

This problem setting of finding $k$ such that $\Delta(\underline{\underline{M}}(k)) = 0$ is known as an **eigenvalue problem**.

# 2.3.3. The Clamped-Clamped Column

The Linear Buckling Problem


- The boundary conditions may be

$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

**Aside:** Eigenvalue Problems ($\underline{\underline{M}} \in \mathbf{R}^{d\times d}$)

**Linear Eigenvalue Problem** ($d$ eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}}$$

**Quadratic Eigenvalue Problem** ($2d$ eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}} + k^2\underline{\underline{M_2}}$$

- The axia ... is before: $v$ ... **es of $k$**
- The tran ... also has ... boundar ...

$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = $ ... $A_3$ ... $k$ such ... an **eigenvalue problem**.

# 2.3.3. The Clamped-Clamped Column

The Linear Buckling Problem

@$x = 0$
$u = 0$
$v = v' = 0$

- The boundary conditions may be

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- The axia... is before: ...s of $k$
- The tran... also has ... boundar...

$$\begin{bmatrix} v(x) \\ v'(x) \end{bmatrix} = \dots \begin{bmatrix} \dots \\ A_3 \end{bmatrix}$$

... $k$ such ... an **eigenvalue problem**.

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**Quadratic Eigenvalue Problem** ($2d$ eigenvalues)

$$\underline{\underline{M}}(k) = \underline{\underline{M_0}} + k\underline{\underline{M_1}} + k^2\underline{\underline{M_2}}$$

Our matrix $\underline{\underline{M}}(k)$ has $k$-dependency in terms of $k$, $\sin(k\ell)$, $\cos(k\ell)$, making this a **Nonlinear Eigenvalue Problem**.

- $\implies \infty$ eigenvalues here (not always though!)

# 2.3.3. The Clamped-Clamped Column I

The Linear Buckling Problem

- We proceed to solve this as,

$$\Delta\left(\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & k\\ 1 & \ell & \cos(k\ell) & \sin(k\ell)\\ 0 & 1 & -k\sin(k\ell) & k\cos(k\ell)\end{bmatrix}\right) = -k\left(k\ell\sin(k\ell)+2\cos(k\ell)-2\right)$$

- We set it to zero through the following factorizations:

$$\begin{aligned}\Delta(\underline{\underline{M}}(k)) &= -k\left(2k\ell\sin(\frac{k\ell}{2})\cos(\frac{k\ell}{2})-4\sin^2(\frac{k\ell}{2})\right)\\ &= -2k\sin(\frac{k\ell}{2})\left(k\ell\cos(\frac{k\ell}{2})-2\sin(\frac{k\ell}{2})\right)=0\end{aligned}$$

$$\implies \boxed{\sin(\frac{k\ell}{2})=0}, \quad \text{(or)} \quad \boxed{\tan(\frac{k\ell}{2})=\frac{k\ell}{2}}.$$

# 2.3.3. The Clamped-Clamped Column II

The Linear Buckling Problem

- Two "classes" of solutions emerge:
  1. $\sin(\frac{k\ell}{2}) = 0 \implies \frac{k_n\ell}{2} = n\pi \implies \boxed{P_n^{(1)} = 4n^2\frac{\pi^2 EI}{\ell^2}}$
  2. $\tan(\frac{k\ell}{2}) = \frac{k\ell}{2} \implies \frac{k_n\ell}{2} \approx 0, 4.49, 7.72, \cdots \implies P_1^{(2)} \approx 8.98\frac{\pi^2 EI}{\ell^2}$
- **The smallest critical load** is $P_n^{(1)} = 4\frac{\pi^2 EI}{\ell^2} = \frac{\pi^2 EI}{(\frac{\ell}{2})^2}$.

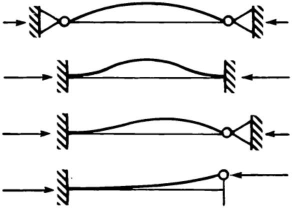
**Concept of "Effective Length"**

- **Question**: If the beam were simply supported, what would be the length such that it also has the same first critical load?
- Here it comes out to be $\ell_{eff} = \frac{\ell}{2}$.
- The column clamped on both ends can take the same buckling load as a column that is pinned on both ends with half the length.

# 2.3.3. The Clamped-Clamped Column III

The Linear Buckling Problem

| Boundary conditions | Critical load $P_{cr}$ | Deflection mode shape | Effective length $KL$ |
|---|---|---|---|
| Simple support–simple support | $\frac{\pi^2 EI}{L^2}$ | | $L$ |
| Clamped-clamped | $4\frac{\pi^2 EI}{L^2}$ | | $\frac{1}{2}L$ |
| Clamped–simple support | $2.04\frac{\pi^2 EI}{L^2}$ | | $0.70L$ |
| Clamped-free | $\frac{1}{4}\frac{\pi^2 EI}{L^2}$ | | $2L$ |



*Effective lengths of beams with different boundary conditions* (*Figure from Brush and Almroth 1975*)

**Self-Study**

- Derive the effective length for the clamped-simply supported and clamped-free columns.

# 2.3.3. The Clamped-Clamped Column: The Mode-shape

The Linear Buckling Problem

- Let us substitute $k_1 = \frac{2\pi}{\ell}$ into the matrix $\underline{\underline{M}}(k_1)$ so that the boundary conditions now read as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \\ 1 & \ell & 1 & 0 \\ 0 & 1 & 0 & \frac{2\pi}{\ell} \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- This implies the following:

$$A_1 = 0, \quad A_3 = 0, \quad A_2 = -A_0.$$

- So, if $k = k_1$, the solution has to be the following to satisfy the boundary conditions:

$$v = A_0 \left(1 - \cos(\frac{2\pi x}{\ell})\right) \equiv A_0 \sin^2(\frac{\pi x}{\ell})$$

# 2.3.3. The Clamped-Clamped Column: The Mode-shape

The Linear Buckling Problem

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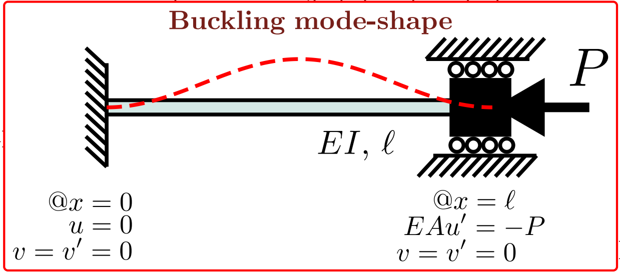
$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

- This im




- So, if $k$ ... he boundary conditions:

$$v = A_0\left(1 - \cos(\frac{2\pi x}{\ell})\right) \equiv A_0 \sin^2(\frac{\pi x}{\ell})$$

# 3. Energy Perspectives

- Concept of conservative force field.
- Work done by a force field:

$$W(\underline{x})\Big|_{\underline{x}_1}^{\underline{x}_2} = \int_{\underline{x}_1}^{\underline{x}_2} \underline{f}(\underline{x}) \cdot d\underline{x}.$$

- Introduction to work done.

$$W(\underline{x}) = \underbrace{\Pi(\underline{x})}_{\text{External Work}} - \underbrace{V(\underline{x})}_{\text{Internal Work/Potential Energy}}$$

**Example**

- Force balance reads: $F = kx$
- Work done expression: $W(x) = Fx - \frac{k}{2}x^2$

# 3. Energy Perspectives

- Expanding $W$ about some $\underline{x}_s$ we have,

$$W(\underline{x}_s + \delta\underline{x}) = W(\underline{x}_s) + \underline{\nabla} W\Big|_{\underline{x}_s} \delta\underline{x} + \mathcal{O}(\delta\underline{x}^2).$$

- Stationarity of work: $\boxed{\delta W = \underline{\nabla} W(\underline{x}_s)\delta\underline{x} = 0, \quad \forall \quad \underline{x} \in \Omega}$, where $\Omega$ is the configuration-space.

**Example**

- For the SDoF system above, we have $W = Fx - \frac{k}{2}x^2$ and

$$\nabla W(x_s) = \frac{dW}{dx} = F - kx_s = 0 \implies x_s = \frac{F}{k}.$$

- Work-stationarity hereby gives a convenient definition for equilibrium.
- What about higher order effects?

# 3. Energy Perspectives

- Continuing the Taylor expansion (SDoF case) for $W(x)$ we have,

$$W(x) = W(x_s) + \frac{dW}{dx}(x_s)\delta x + \frac{1}{2}\frac{d^2W}{dx^2}(x_s)\delta x^2 + \mathcal{O}(\delta x^3).$$

- At equilibrium, $\frac{dW}{dx}$ is zero. The sign of $\frac{d^2W}{dx^2}$ governs the local tendency of the work around equilibrium.

**Example**

- For the SDoF example, $\frac{d^2W}{dx^2} = -k$, implying $W$ is maximized.

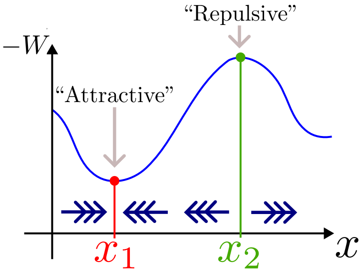
- If $\frac{d^2W}{dx^2} < 0$, then the second order effect of virtual displacements is to reduce the work scalar: **Stable Equilibrium**.
- The opposite case is **Unstable Equilibrium**.

# 3. Energy Perspectives

- Continuing the Taylor expansion (SDoF case) for $W(x)$ we have,

$$W(x) \ldots \frac{dW}{\ldots} \ldots \frac{1}{\ldots}\frac{d^2W}{\ldots} \ldots + \mathcal{O}(\delta x^3).$$

**Hypothetical Example**




- At equilibriun … he local tendency of the work arou…

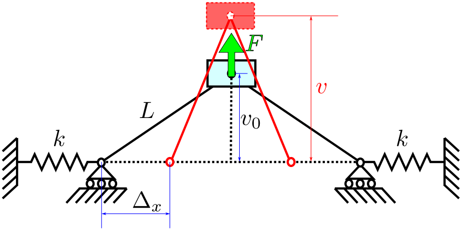
**Example**

- For the SDoF … ed.

- If $\frac{d^2W}{dx^2} < 0$, th … lacements is to reduce the wo…
- The opposite case is **Unstable Equilibrium**.

# 3.1. Snap-Through Buckling

Energy Perspectives




- We will consider the SDoF model to the right (from Wiebe et al. 2011).
- The strain energy on the springs (two) is

$$U(v) = 2\times\frac{k}{2}\Delta_x^2 = k\left(\sqrt{L^2 - v_0^2} - \sqrt{L^2 - v^2}\right)^2.$$

- The work done by the load (to take the mid-point from $v_0$ to $v$) is given by,

$$\Pi(v) = F(v - v_0).$$

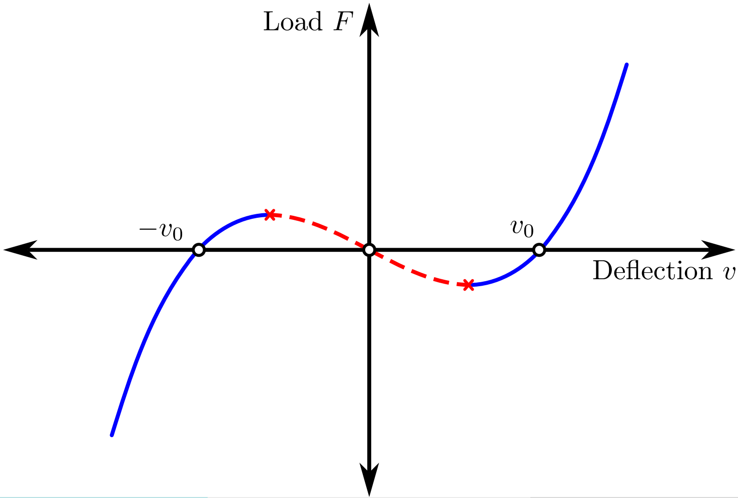
**Setting $\frac{dW}{dv} = 0$ we get**

$$F = -2kv\left(1 - \sqrt{\frac{L^2 - v_0^2}{L^2 - v^2}}\right).$$

# 3.1. Snap-Through Buckling

Energy Perspectives

- Instead of an analytical treatment, we will use **Graphical Inspection** to understand this function.



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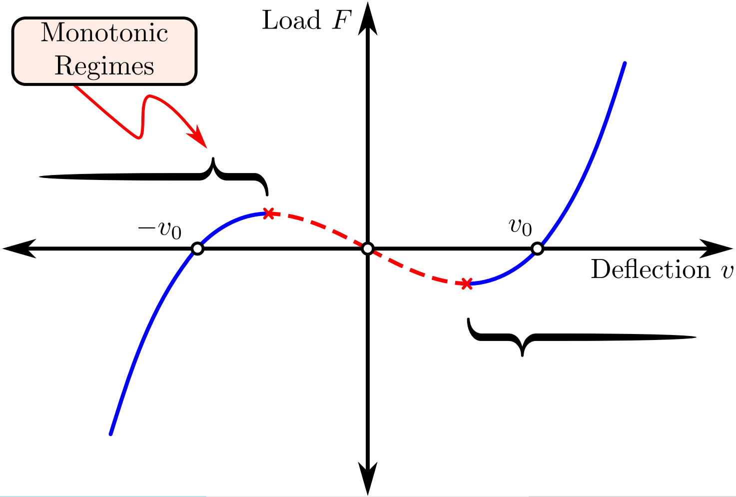

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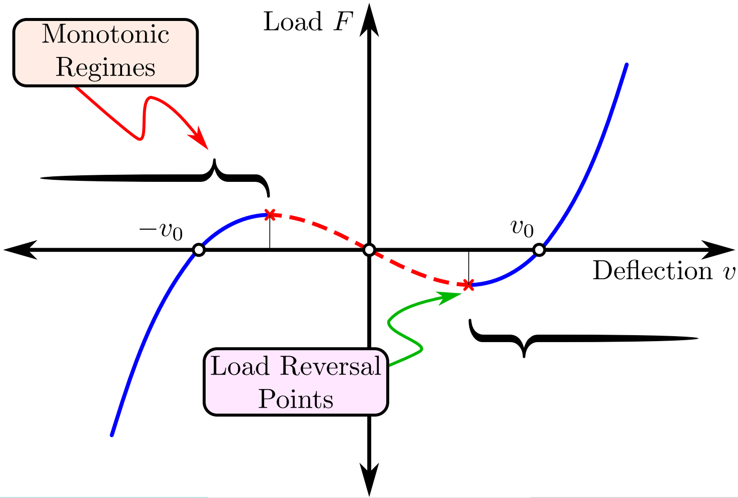

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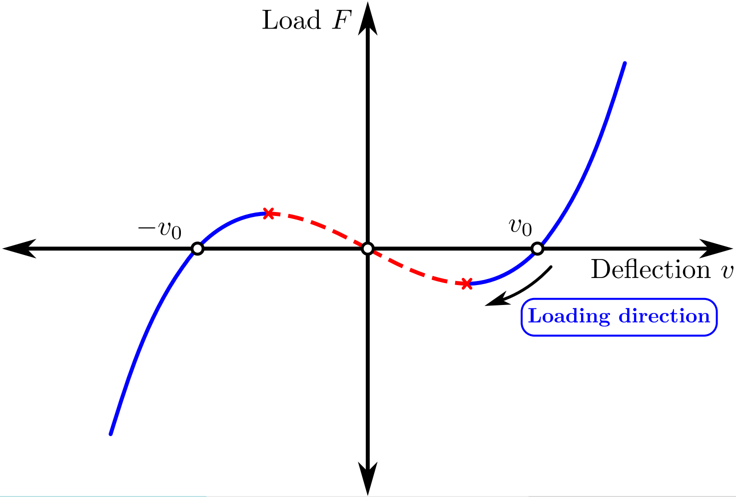

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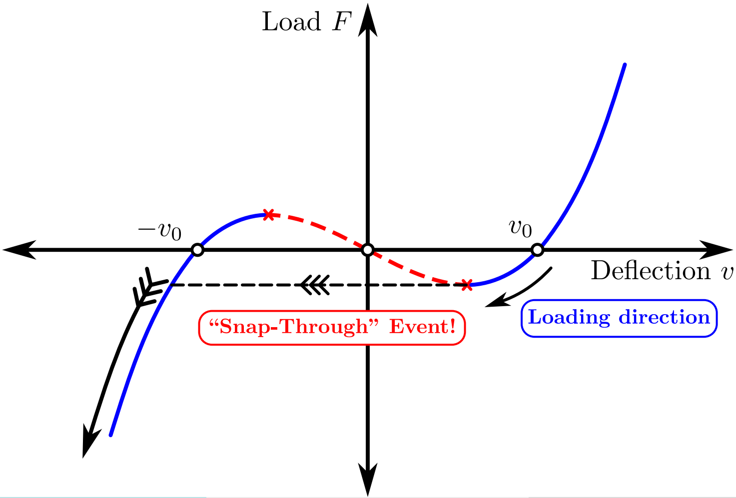

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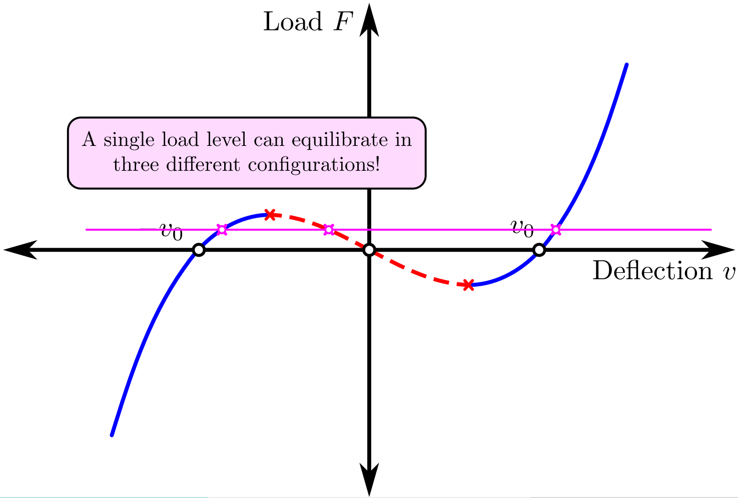
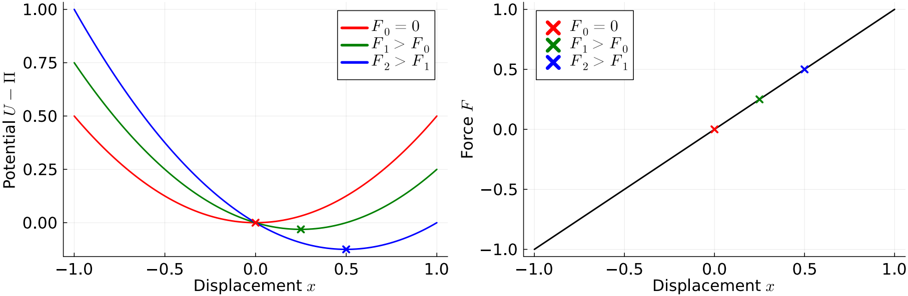


# 3.1. Snap-Through Buckling: Equilibrium Visualization

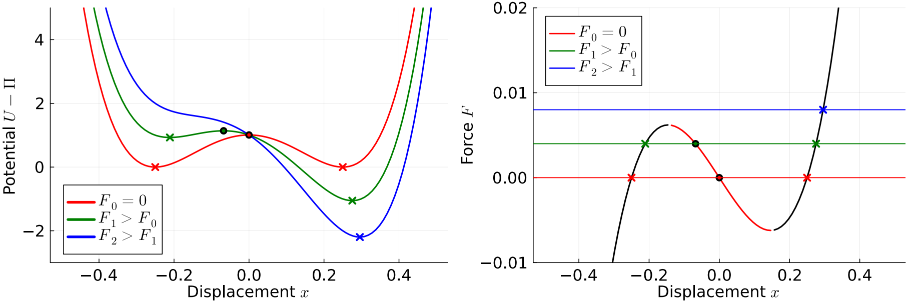
Energy Perspectives




***Linear System:*** $U - \Pi = \frac{k}{2}x^2 - Fx$

# 3.1. Snap-Through Buckling: Equilibrium Visualization

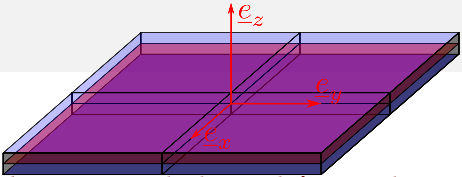
Energy Perspectives




***Snap-Through Problem:*** $U - \Pi = k\left(\sqrt{L^2 - v_0^2} - \sqrt{L^2 - v^2}\right)^2 - Fx$

# 4.1. Plate Buckling

Governing Equations




- Kichhoff-Love Plate Theory.
- **Kinematic Assumptions**: Lines along section-thickness deform as lines and stay perpendicular to the neutral axis.
- Governing equations written in the form

$$\frac{Et^3}{12(1-\nu^2)}(w_{,xxxx} + w_{,yyyy} + 2w_{,xxyy}) - (N_{xx}w_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}) = 0$$

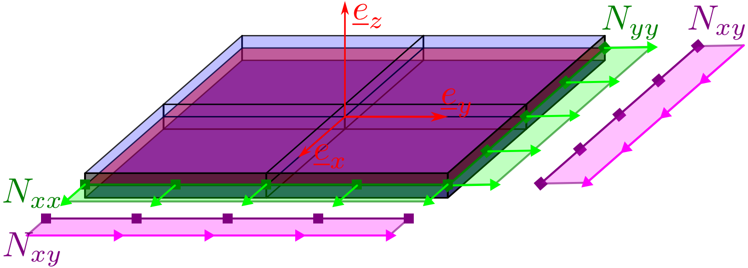
$$\boxed{D\nabla^4 w - (N_{xx}w_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}) = 0}$$

- This is all that is needed to conduct buckling analysis - the procedure is identical as above!
- Before this, however, let us develop intuition on the different reaction force components and their kinematic relationships.

# 4.2. Reaction-Kinematics Relationships

Plate Buckling

**Normal Reactions**

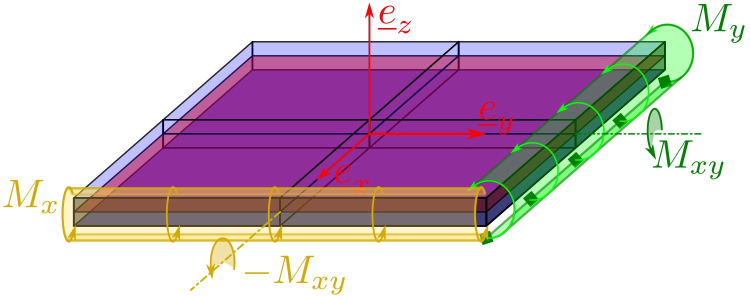


# 4.2. Reaction-Kinematics Relationships

Plate Buckling

**Normal Reactions**

**Moment Reactions**



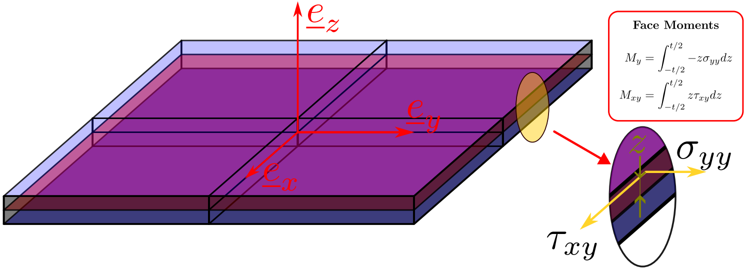
# 4.2. Reaction-Kinematics Relationships

Plate Buckling

**Normal Reactions**

**Moment Reactions**

## Stress-Moment Relationships




**Face Moments**

$$M_y = \int_{-t/2}^{t/2} -z\sigma_{yy} dz$$

$$M_{xy} = \int_{-t/2}^{t/2} z\tau_{xy} dz$$

# 4.2. Reaction-Kinematics Relationships

Plate Buckling

**Normal Reactions**

**Moment Reactions**

**Equilibrium Equations** (Shear Force-Moment Relationships)

$$\left.\begin{aligned} \sigma_{xx,x} + \tau_{xy,y} + \tau_{xz,z} &= 0 \\ \tau_{xy,x} + \sigma_{yy,y} + \tau_{yz,z} &= 0 \\ \tau_{xz,x} + \tau_{yz,z} + \sigma_{zz,z} &= 0 \end{aligned}\right\} \implies \begin{cases} Q_x &= M_{x,x} + M_{xy,y} \\ Q_y &= -M_{y,y} + M_{xy,x} \\ 0 &= Q_{x,x} + Q_{y,y}. \end{cases}$$

**Note:**

- Although the shear strains $\gamma_{xz}$ & $\gamma_{yz}$ are assumed zero by the Kirchhoff kinematic assumptions, and thereby, the stresses $\tau_{xz}$ & $\tau_{yz}$ are also zero, **the shear forces can not be zero for equilibrium!!**
- They are defined as $Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} dz, \quad Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz.$

# 4.2. Reaction-Kinematics Relationships

Plate Buckling

- With this background, we are ready to write the following:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_x \\ -M_y \\ M_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \left( \begin{bmatrix} t & 0 \\ 0 & -\frac{t^3}{12} \end{bmatrix} \otimes \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \right) \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \\ w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$

- A moment-free boundary condition (simply supported edge) would imply simply setting the second derivatives ($w_{,xx}, w_{,yy}, w_{,xy}$) to zero at the edge.

# 4.3. Thin Plates Under Uniaxial Compression

Plate Buckling

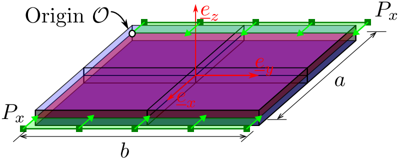



*Plate under uniaxial compression*

### Ansatz (Simply Supported Case)

$$w(x,y) = \sum_{m,n} W_{mn} \sin\left(m\frac{\pi x}{a}\right) \sin\left(n\frac{\pi y}{b}\right)$$

**Boundary Conditions:**

$w = 0,\ M_x, M_y = 0 \quad \text{on} \quad \Gamma$

### Governing Equations

$$D\nabla^4 w + P w_{,xx} = 0$$

$$\implies P_{cr,nm} = \frac{\pi^2 D}{b^2}\left(\frac{m}{a/b} + n^2\frac{a/b}{m}\right)^2$$

(n=1 always for minimum critical load)

$$\implies P_{cr,m} = \frac{\pi^2 D}{b^2}\left(\frac{m}{a/b} + \frac{a/b}{m}\right)^2$$

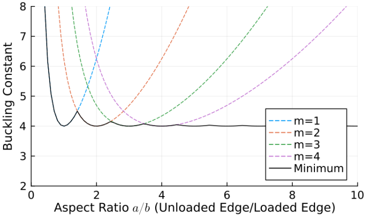
$$P_{cr} = \frac{\pi^2 D}{b^2} \underbrace{\min_{m\in\mathbb{Z}^+}\left(\frac{m}{a/b} + \frac{a/b}{m}\right)^2}_{k_{cr}(a/b)}$$

# 4.3. Thin Plates Under Uniaxial Compression

Plate Buckling

**Buckling Constant**

$$k_{cr}(r) = \min_{m\in\mathbb{Z}^+} \left(\frac{m}{r} + \frac{r}{m}\right)^2$$

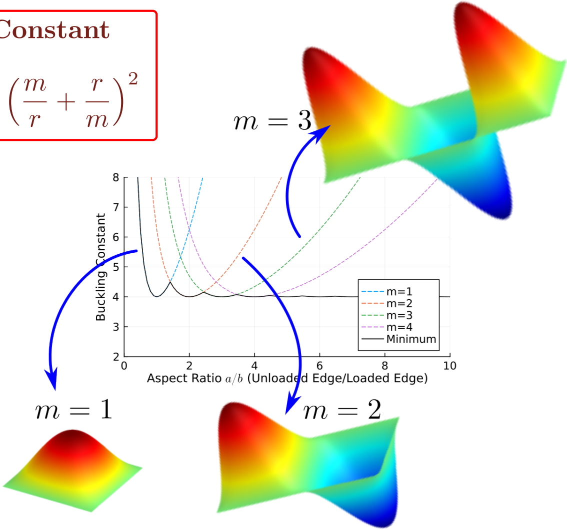


# 4.3. Thin Plates Under Uniaxial Compression

Plate Buckling

**Buckling Constant**

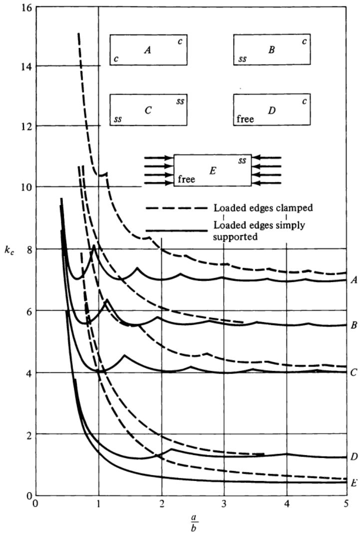
$$k_{cr}(r) = \min_{m \in \mathbb{Z}^+} \left( \frac{m}{r} + \frac{r}{m} \right)^2$$



# 4.3. Other Boundary Conditions

Thin Plates Under Uniaxial Compression

- It is possible to conduct the same analysis for other (combinations) of boundary conditions.
- The analysis is slightly more tedious (due to the Ansatz not being as simple any more), **but possible along the same lines**.
- The critical plot comes out as shown in your textbook.




(Figure 3.9 from Brush and Almroth 1975)

# 4.3. Other Boundary Conditions

Thin Plates Under Uniaxial Compression

- It is possible to conduct the same analysis for other (combinations) of boundary conditions.
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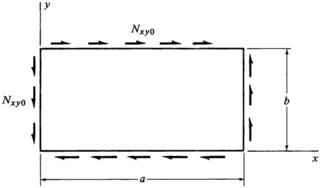
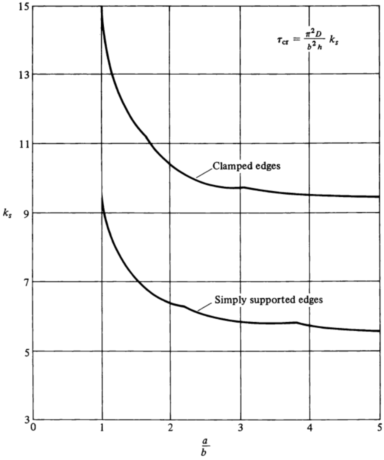
- The critical plot comes out as shown in your textbook.

**The same works for shear buckling too!**




*(Fig. 3.10 from Brush and Almroth 1975)*




*(Figure 3.11 from Brush and Almroth 1975)*

# References I

[1] Don Orr Brush and Bo O. Almroth. Buckling of Bars, Plates, and Shells, McGraw-Hill, 1975. ISBN: 978-0-07-008593-0 (cit. on pp. 2, 32, 57, 58).

[2] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on p. 2).

[3] Richard Wiebe et al. "On Snap-Through Buckling". In: *52nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*. Denver, Colorado: American Institute of Aeronautics and Astronautics, Apr. 2011. ISBN: 978-1-60086-951-8. DOI: 10.2514/6.2011-2083. (Visited on 02/18/2025) (cit. on p. 39).

# 6. Class Discussions (Outside of Slides)

- Ball on a hill. 2D, 3D cases.
- Assumptions behind compression of a bar.

# 6.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

- Let us use the energy approach to study the post-buckling behavior of a beam.
- We've developed some intuition that buckling blows up the displacement levels. Let us revise our kinematic description to capture this.
- The (simplified) approach we will follow is as follows:
  1. **Write out nonlinear kinematics**, identify normal force $N = \int_{\mathcal{A}} \sigma_{ax} dA$ and moment $M = \int_{\mathcal{A}} -y\sigma_{ax} dA$.
  2. **Assume transverse deformation field** $v = V \sin\left(\frac{\pi x}{\ell}\right)$
  3. **Assume axial tip deflection** $u_T$ **and derive axial deformation field.**
  4. **Express work done in terms of scalars** $V$ **and** $u_T$**.** $\rightarrow$ **Extremize**.
  5. **Plot force deflection curves, analyze stability**.

# 6.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

## Geometrically Nonlinear Kinematics

- The deformation field is written as $u_x = u - yv', \quad u_y = v$. Consider the deformation of a line from $(x, y)$ to $(x + \Delta x, y)$:

$$(x, y) \to (x + u - yv', y + v),$$
$$(x + \Delta x, y) \to (x + \Delta x + u - yv' + (u' - yv'')\Delta x, y + v + v'\Delta x),$$
$$\Delta S = \Delta x, \quad \Delta s^2 = \Delta x^2((1 + u' - yv'')^2 + {v'}^2).$$

- We write the axial strain as

$$\epsilon_{ax} = \frac{1}{2}\frac{\Delta s^2 - \Delta S^2}{\Delta S^2} = (u' - yv'') + \frac{1}{2}\left((u' - yv'')^2 + {v'}^2\right)$$
$$\boxed{\epsilon_{ax} \approx (u' - yv'') + \frac{{v'}^2}{2}}.$$

- The final assumption is sometimes referred to as Von Karman strain assumptions.

# 6.1. Post-Buckling Behavior (Out of Syllabus)

Class Discussions (Outside of Slides)

- Nearly nothing changes in the equilibrium equations. We first write out the area-normal stresses and moments:

$$N = \int_{\mathcal{A}} E\epsilon_{ax} dA = EA(u' + \frac{{v'}^2}{2}), \quad M = \int_{\mathcal{A}} -yE\epsilon_{ax} dA = EIv''.$$

- The axial force balance reads:

$$N' = EA\frac{d}{dx}\left(u' + \frac{{v'}^2}{2}\right) = 0, \quad u(x)|_{x=0} = 0, \quad u|_{x=\ell} = u_T.$$

# 6.1. Post-Buckling Behavior (Out of Syllabus): Axial Problem

Class Discussions (Outside of Slides)

- We next **impose the transverse deformation field** $v(x) = V \sin\left(\frac{\pi x}{\ell}\right)$ on the axial problem. Solving this, we get

$$u(x) = -\frac{\pi V^2}{8\ell} \sin\left(\frac{2\pi x}{\ell}\right) + C_1 x + C_2.$$

- Boundary conditioned are imposed by setting $C_1 = \frac{u_T}{\ell}$ and $C_2 = 0$.
- The parameterized axial deformation field, therefore, is

$$\boxed{u(x; V, u_T) = \frac{u_T}{\ell} x - \frac{\pi V^2}{8\ell} \sin\left(\frac{2\pi x}{\ell}\right)}.$$

- Note that we have not said anything about $V$ or $u_T$ so far.

# 6.1. Post-Buckling Behavior (Out of Syllabus): Strain Energy Density

Class Discussions (Outside of Slides)

- The strain energy density (per unit length) is written as,

$$\mathcal{V} = \int_{\mathcal{A}} \frac{E\epsilon_{ax}^2}{2} dA = \frac{E}{2}\int_{\mathcal{A}} (u' - yv'' + \frac{v'^2}{2})^2 dx$$
$$= \frac{EA}{2}\left(u' + \frac{v'^2}{2}\right)^2 + \frac{EI}{2}v''^2 \approx \frac{EI}{2}v''^2 + \frac{EA}{2}\frac{v'^4}{4}.$$

- Note that we have assumed $u_T \to 0$, i.e., providing negligible influence on the overall potential energy.
- Substituting the assumed deformation field $v = V\sin(\frac{\pi x}{\ell})$ and integrating over $(0, \ell)$ we have,

$$\mathcal{V}_{tot} = \int_0^\ell \mathcal{V}(x)dx = \frac{\pi^4 EI}{4\ell^3}V^2 + \frac{3\Pi^4 EA}{64\ell^3}V^4$$
$$= \frac{\pi^2 P_{cr}}{4\ell}V^2 + \frac{3\pi^2 AP_{cr}}{64I\ell}V^4.$$

# 6.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)

- The work done by an axial compressive load $P$ is given by

$$\Pi = \int_0^\ell \int_{\mathcal{A}} \frac{P}{A} \varepsilon_{ax} dA dx = \int_0^\ell \int_{\mathcal{A}} \frac{P}{A} (u' - yv'' + \frac{{v'}^2}{2}) dA dx$$

$$= P \int_0^\ell u' dx + \frac{P}{2} \int_0^\ell {v'}^2 dx$$

$$\boxed{\Pi = P u_T + \frac{\pi^2 P}{4\ell} V^2}.$$

- So the total work scalar ($W = \Pi - \mathcal{V}_{tot}$) is given as (we ignore $u_T$ here)

$$W(V) = \frac{\pi^2}{4\ell}(P - P_{cr})V^2 - \frac{3\pi^2 A}{64 I \ell} P_{cr} V^4.$$

# 6.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)

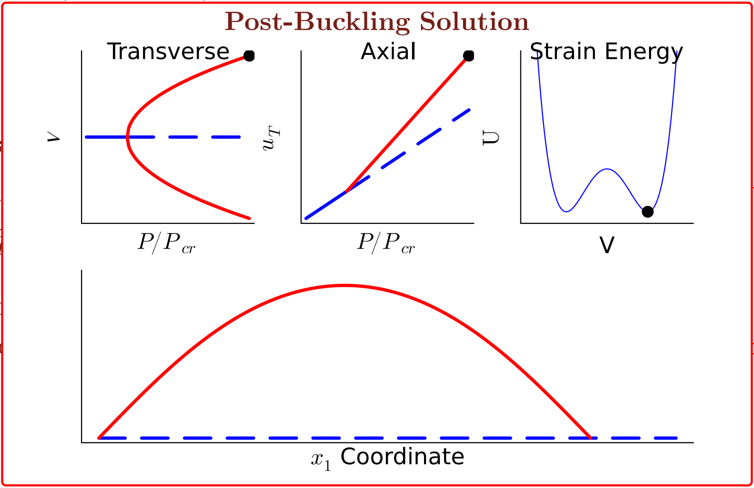
- Stationarizing the work we get,

$$\frac{dW}{dV} = \frac{\pi^2 P_{cr}}{2\ell} V \left( \left( \frac{P}{P_{cr}} - 1 \right) - \frac{3A}{8I} V^2 \right) \implies V = 0, \pm\sqrt{\frac{8I}{3A}\left(\frac{P}{P_{cr}} - 1\right)}.$$

  Note that the non-trivial solution is only active for $P >= P_{cr}$.
- We can next estimate $u_T$ easily by applying the boundary conditions.

# 6.1. Post-Buckling Behavior (Out of Syllabus): Work Stationarity

Class Discussions (Outside of Slides)




- St...

  d... $-1\Big).$

  No...
- We... ns.